On the Beurling-Ahlfors functional inequality

by

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Abstract. A real-valued function u on an interval I is *uniformly interior*, if it satisfies the *Beurling-Ahlfors functional inequality*

$$\frac{1}{\varrho} \leqslant \frac{u(x)-u\big((x+y)/2\big)}{u\big((x+y)/2\big)-u(y)} \leqslant \varrho, \quad x, y \in I,\ x \neq y,$$

for some constant $\varrho \geqslant 1$. The uniformly interior functions form a subclass of the strictly interior functions defined by Á. Császár. Slightly sharper measurability results than those known for strictly interior functions are obtained for uniformly interior functions. Further any uniformly interior function u: $\boldsymbol{R} \to \boldsymbol{R}$ is shown to have a factorization $u = \hat{u} \circ \varphi$, where φ is a Hamel function and $\hat{u}$: $\boldsymbol{R} \to \boldsymbol{R}$ is quasisymmetric, i.e. a strictly increasing and continuous solution of the Beurling–Ahlfors functional inequality.

We denote the closed interval $[\min(a, b), \max(a, b)]$ of the real line $\boldsymbol{R}$ by $[a, b]$, similarly the open interval $]\min(a, b), \max(a, b)[$ is denoted by $]a, b[$.

1. Let $I \subset \boldsymbol{R}$ be an interval. A continuous, strictly increasing function u: $I \to \boldsymbol{R}$ which for some constant $\varrho \geqslant 1$ satisfies the *Beurling–Ahlfors fnuctional inequality*

$$\frac{1}{\varrho} \leqslant \frac{u(x)-u\big((x+y)/2\big)}{u\big((x+y)/2\big)-u(y)} \leqslant \varrho \tag{1}$$

for all $x, y \in I$, $x \neq y$, is called a *ϱ-quasisymmetric function*; the quasisymmetric functions have an important role in the theory of quasiconformal mappings in the plane (Beurling–Ahlfors [1], cf. also Lehto–Virtanen [7], II.7). If we write $\lambda_1 = 1/(\varrho+1)$, $\lambda_2 = \varrho/(\varrho+1)$ for a fixed $\varrho \geqslant 1$, the inequality (1) is equivalent to the condition:

$$u\big((x+y)/2\big) \in [\lambda_1 u(x) + \lambda_2 u(y),\ \lambda_2 u(x) + \lambda_1 u(y)] \tag{2}$$

for all $x, y \in I$. Slightly generalizing the definition we shall call in the following *any continuous, monotone function* u: $I \to \boldsymbol{R}$ satisfying (2) a ϱ-quasisymmetric function.

However, if the function u is not required to be either continuous or monotone, the functional inequality (2) can admit highly discontinuous functions as solutions. Any *Hamel function*, i.e. a solution $u\colon \boldsymbol{R}\to\boldsymbol{R}$ of the *Cauchy functional equation*

$$u(x+y) = u(x)+u(y)\,, \tag{3}$$

obviously satisfies (2) for $\varrho = 1$. In [2] Á. Császár defined the class of *strictly interior functions* $u\colon I\to\boldsymbol{R}$ on an interval $I\subset\boldsymbol{R}$ by requiring that u satisfies

$$u\big((x+y)/2\big)\in\,]u(x),u(y)[\qquad\text{or}\qquad u\big((x+y)/2\big)=u(x)=u(y) \tag{4}$$

for all $x, y\in I$ (cf. also Császár [3], Marcus [8], [9], Deák [4]). Thus functions $u\colon I\to\boldsymbol{R}$ satisfying (2) for a fixed $\varrho\geqslant 1$ might be called *uniformly interior functions* on the interval $I\subset\boldsymbol{R}$.

2. The real line $\boldsymbol{R}$ is a vector space over the field of rational numbers $\boldsymbol{Q}$. We call a $\boldsymbol{Q}$-affine subspace M of $\boldsymbol{R}$, $\dim_{\boldsymbol{Q}}M>0$, a *rational manifold*; a rational manifold L is a *rational line* if $\dim_{\boldsymbol{Q}}L=1$. Every rational manifold inherits a metric and an order from $\boldsymbol{R}$. If I is a *non-degenerate interval* of the real line $\boldsymbol{R}$ and M is a rational manifold, the intersection $H=I\cap M$ is called a *rational interval*, respectively $G=I\cap L$ is a *rational line interval* if L is a rational line.

Let H be a rational interval. A function $u\colon H\to\boldsymbol{R}$ is called a *ϱ-function* if there is a fixed $\varrho\geqslant 1$ such that u satisfies (2) for all $x, y\in H$. It is easy to see that the following estimate holds for any ϱ-function $u\colon H\to\boldsymbol{R}$:

$$\lambda_1^n|u(x+t)-u(x)|\leqslant|u(x+2^{-n}t)-u(x)|\leqslant\lambda_2^n|u(x+t)-u(x)|\,, \tag{5}$$

where $x, x+t\in H$, $n\in\boldsymbol{N}$.

PROPOSITION 1. *A ϱ-function $u\colon H\to\boldsymbol{R}$ is strictly monotone on any rational line interval $G\subset H$, or constant on G.*

Proof. We may suppose that $G=I\cap\boldsymbol{Q}\subset H$ for some non-degenerate interval $I\subset\boldsymbol{R}$. Using (2) repeatedly we see that u is either strictly monotone on $I\cap p\boldsymbol{Z}\subset G$, or constant on $I\cap p\boldsymbol{Z}$ for any p belonging to the multiplicative group $\boldsymbol{Q}^*$ of the rational field $\boldsymbol{Q}$. But for any $p, q\in\boldsymbol{Q}^*$ the union $p\boldsymbol{Z}\cup q\boldsymbol{Z}$ is contained in $r\boldsymbol{Z}$ for some $r\in\boldsymbol{Q}^*$; thus we see that u must either be strictly monotone on the whole rational line interval $G=I\cap\boldsymbol{Q}$, or constant on G.

PROPOSITION 2. *If a ϱ-function $u\colon H\to\boldsymbol{R}$ is monotone on the rational interval H, it has a unique ϱ-quasisymmetric extension $\hat u\colon \bar H\to\boldsymbol{R}$ to the closed interval $\bar H\subset\boldsymbol{R}$.*

Proof. As u is monotone, it is bounded on $H\cap J$ for every compact interval J contained in the interior of the closure $\bar H$, so that by (2) u must be bounded on every intersection $H\cap K$ of H with a compact interval K.

Now it follows from (5) that u is locally uniformly continuous on H. Thus u has a unique continuous extension $\hat u\colon\bar H\to\boldsymbol{R}$, which also must be monotone and satisfy (2) on the interval $\bar H$.

THEOREM 1. *A uniformly interior function $u\colon I\to\boldsymbol{R}$ which is bounded from above (below) on a non-degenerate subinterval $J\subset I$ is quasisymmetric.*

Proof. As a uniformly interior function u is a ϱ-function for some $\varrho\geqslant 1$. If now u is bounded from above on a non-degenerate compact subinterval $K\subset I$, it follows from (2) that u must be bounded also from below on K. From (5) it follows then further that

$$M=\sup_{x\in J}u(x)\,,\qquad m=\inf_{x\in J}u(x)$$

are finite for any interval $J=[x_1,x_2]$, $x_1,x_2\in I$. Let us choose a point $x\in J$ such that $u(x)>M-\lambda_1\varepsilon$, $\varepsilon>0$. Then either $y=2x-x_1$ or $y=2x-x_2$ belongs to the interval J, and $u(y)\leqslant M$. Thus by (2) either $u(x_1)>M-\varepsilon$ or $u(x_2)>M-\varepsilon$. As $\varepsilon>0$ was arbitrary, we have shown that $M=\max\big(u(x_1),u(x_2)\big)$. Similarly $m=\min\big(u(x_1),u(x_2)\big)$, so that u is a monotone ϱ-function on the interval I. The conclusion follows now from Proposition 2.

COROLLARY 1.1. *A uniformly interior function $u\colon I\to\boldsymbol{R}$ which is bounded from above (below) on a Lebesgue measurable subset $E\subset I$ of positive measure is quasisymmetric.*

Proof. If $u\leqslant M$ on a measurable set $E\subset I$, $m(E)>0$, it follows from (2) that on $F\subset I$,

$$F=(E+E)/2=\{(x+y)/2|\ x,y\in E\}\,,$$

we have $u\leqslant M$, too. But by the theorem of Steinhaus F contains a non-void open interval.

COROLLARY 1.2. *Any Lebesgue measurable uniformly interior function $u\colon I\to\boldsymbol{R}$ is quasisymmetric.*

Corollary 1.2 follows from the corresponding measurability result for interior functions, too. Actually, the above properties of uniformly interior functions are quite similar to the respective properties of Hamel functions, cf. Kestelman [6].

3. Suppose that we are given a *fixed non-constant uniformly interior function* $v\colon\boldsymbol{R}\to\boldsymbol{R}$ such that $v(0)=0$. We define a set P by

$$P=\{x\in\boldsymbol{R}|\ v(x)\geqslant 0\}\,. \tag{6}$$

As $v(0)=0$, it follows from (2) that for every $x\in\boldsymbol{R}$ either $v(x)\geqslant 0$ or $v(-x)\geqslant 0$, so that

$$P\cup -P=\boldsymbol{R}\,, \tag{7}$$

and if $v(x) > 0$, we must have $v(-x) < 0$. Further (2) implies that $(x+y)/2 \in P$ if $x, y \in P$, and as $v(0) = 0$, it follows that also $x+y \in P$, and thus

$$P+P \subset P. \tag{8}$$

As $P \subset \boldsymbol{R}$ has the properties (7) and (8), we can define a *group preorder* $\preccurlyeq$ on the additive group of $\boldsymbol{R}$ with P as the *set of positive elements* by setting $x \preccurlyeq y$ if $y-x \in P$. The intersection $S = P \cap -P = \{x \in \boldsymbol{R} |\ v(x) = 0\}$ is a $\boldsymbol{Q}$-linear subspace of $\boldsymbol{R}$, and $\preccurlyeq$ defines an *order* on the *quotient space* $\boldsymbol{R}/S$, i.e. for two elements $\xi, \eta \in \boldsymbol{R}/S$ the simultaneous inequalities $\xi \preccurlyeq \eta$ and $\eta \preccurlyeq \xi$ imply $\xi = \eta$. We denote the *quotient mapping* by π, and write $\xi \prec \eta$ if $\xi \preccurlyeq \eta$ and $\xi \neq \eta$, $\xi, \eta \in \boldsymbol{R}/S$.

Proposition 3. *The order $\preccurlyeq$ defined by $\pi(P)$ on the quotient space $\boldsymbol{R}/S$ is archimedean. Furthermore, the function v can be factorized through $\boldsymbol{R}/S$, $v = v_0 \circ \pi$, $v_0 = \boldsymbol{R}/S \to \boldsymbol{R}$.*

Proof. If ξ and η are two strictly positive elements of $\boldsymbol{R}/S$, $0 \prec \xi$, $0 \prec \eta$, there exist $x, y \in \boldsymbol{R}$ such that $\xi = \pi(x)$, $\eta = \pi(y)$, $0 < v(x)$, $0 < v(y)$. Because $v(0) = 0$, we have $v(-y) < 0$. It follows further from (5) that there exists a positive integer $n_0 \in \boldsymbol{N}$ such that $-\lambda_2 v(-y) < \lambda_1 v(n_0 x)$. But then $0 < v(n_0 x - y)$ by (2), and as $\pi(n_0 x) = n_0 \xi$, we have $\eta \prec n_0 \xi$. Hence the order $\preccurlyeq$ on $\boldsymbol{R}/S$ is archimedean.

As S is a $\boldsymbol{Q}$-linear subspace of $\boldsymbol{R}$, it follows from (2) that $|v(x+2^k t)| \leqslant \lambda_2 |v(2x)|$ for all $x \in \boldsymbol{R}$, $t \in S$, $k \in \boldsymbol{Z}$. Now (5) implies that

$$|v(x+t) - v(x)| \leqslant \lambda_2^n \big(|v(x)| + \lambda_2 |v(2x)|\big)$$

for all $x \in \boldsymbol{R}$, $t \in S$, $n \in \boldsymbol{N}$. Thus we have $v(x+t) = v(x)$ for all $x \in \boldsymbol{R}$, $t \in S$, so that the function v can be factorized through the quotient mapping π, $v = v_0 \circ \pi$.

Theorem 2. *Every non-constant uniformly interior function u: $\boldsymbol{R} \to \boldsymbol{R}$ has a representation as*

$$u = \hat{u} \circ \varphi, \tag{9}$$

where $\hat{u}$: $\boldsymbol{R} \to \boldsymbol{R}$ is a strictly increasing quasisymmetric function and φ: $\boldsymbol{R} \to \boldsymbol{R}$ a Hamel function. The representation is essentially unique, $\hat{u}' \circ \varphi' = \hat{u} \circ \varphi$ if and only if

$$\hat{u}(x) = \hat{u}'(ax), \qquad \varphi' = a\varphi$$

for a strictly positive $a \in \boldsymbol{R}$.

Proof. We may apply the preceding proposition to the uniformly interior function $v = u - u(0)$. As the order $\preccurlyeq$ on $\boldsymbol{R}/S$ is archimedean, the *ordered group* $(\boldsymbol{R}/S, \preccurlyeq)$ is isomorphic to a subgroup of the *ordered group* $(\boldsymbol{R}, \leqslant)$ by the theorem of Hölder, so that we have an *order preserving $\boldsymbol{Q}$-linear monomorphism* φ_0: $\boldsymbol{R}/S \to \boldsymbol{R}$. Let us denote $\varphi = \varphi_0 \circ \pi$. By assumption the function v is not constant, thus $\boldsymbol{R}/S \neq 0$ and $\varphi_0(\boldsymbol{R}/S) = \varphi(\boldsymbol{R}) \subset \boldsymbol{R}$ is a rational manifold. Using the factorization $v = v_0 \circ \pi$ we define a *ϱ-function* v_1 on $\varphi(\boldsymbol{R})$ by $v_1 = v_0 \circ \varphi_0^{-1}$. As φ_0 is order preserving, we have for $x \in \varphi(\boldsymbol{R})$ that $0 < v_1(x)$ if and only if $0 < x$, so that by Proposition 1 the function v_1 is strictly increasing on every rational line $L \subset \varphi(\boldsymbol{R})$ and thus on the whole rational manifold $\varphi(\boldsymbol{R})$. Proposition 2 implies now the existence of a unique, strictly increasing ϱ-quasisymmetric extension $\hat{v}$: $\boldsymbol{R} \to \boldsymbol{R}$ of v_1. If we write $\hat{u} = \hat{v} + u(0)$, then $u = \hat{u} \circ \varphi$ is the required representation.

Let $u = \hat{u}' \circ \varphi'$ be another representation of the same type. As $\varphi'(S) = 0$, we have a factorization $\varphi' = \varphi_0' \circ \pi$. The $\boldsymbol{Q}$-linear mapping $\psi = \varphi_0' \circ \varphi_0^{-1} = \varphi_0(\boldsymbol{R}/S) \to \varphi_0'(\boldsymbol{R}/S)$ is strictly increasing on $\varphi_0(\boldsymbol{R}/S)$, so that is has a continuous, strictly increasing extension $\hat{\psi}$: $\boldsymbol{R} \to \boldsymbol{R}$ by Proposition 2. Thus $\hat{\psi}$ must be of the form $\hat{\psi}(x) = ax$ for some $a > 0$, $a \in \boldsymbol{R}$.

The last two results are not valid for arbitrary strictly interior functions, as can easily be shown by a counterexample.

4. We have not used the axiom of choice in the above considerations. On the other hand, Hamel's construction of discontinuous solutions of the Cauchy functional equation (3) is essentially based on the axiom of choice (cf. Hamel [5]). That this is really necessary follows from the recent results of Robert M. Solovay. In [10] he shows that if the axiom of choice is not included in the Zermelo–Fraenkel axioms of set theory, we can instead assume without any further contradiction that all subsets of the real line $\boldsymbol{R}$ are Lebesgue measurable. Together with Corollary 1.2 this would imply the continuity of all solutions of the Cauchy functional equation (3), as well as the continuity of all uniformly interior functions.

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Concerning unicoherence of continua

by

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Abstract. In this paper we investigate the unicoherence of a continuum M, knowing that the elements of a certain decomposition $\mathcal{G}$ of M are unicoherent. We confine ourselves to considering only upper semi-continuous monotone decompositions. In this case, if the decomposition space $M/\mathcal{G}$ is a dendroid and for each subcontinuum K of M each element G of $\mathcal{G}$ the intersection $K \cap G$ is a continuum, then M is unicoherent.

If M is a hereditarily decomposable continuum which is irreducible about a finite set and $\mathcal{G}$ is an admissible decomposition of M, the suppositions may reduce to the single one that the elements of $\mathcal{G}$ are unicoherent.

We obtain analogous assertions concerning the hereditary unicoherence of continua if the elements of $\mathcal{G}$ are such.

In this paper a continuum means a compact connected metric space.

Let M be a continuum. A family $\mathcal{G}$ of closed disjoint subsets of M covering M is said to be a decomposition of M.

The decomposition $\mathcal{G}$ of M is said to be *monotone* if its elements are continua.

The decomposition $\mathcal{G}$ of M is said to be *upper semi-continuous* if for each open subset U of M containing some element G of $\mathcal{G}$ there exists an open subset V of M such that $G \subset V \subset U$ and V is the union of the elements of $\mathcal{G}$ intersecting it. For equivalent definitions of this concept see [3], pp. 183–185, or [8], p. 122.

Let I be a continuum irreducible from a to b. Suppose that one can define a non-trivial upper semi-continuous monotone decomposition $\mathcal{G}$ of I such that each element of $\mathcal{G}$ not containing a and b separates I. It is shown in [7] that in this case there exists a unique decomposition which is minimal with respect to the above properties. Its elements are called *layers* of I.

The upper semi-continuous monotone decomposition $\mathcal{G}$ of the continuum M is said to be *admissible* if, for each irreducible continuum $I \subset M$ and for each layer T of I, there exists an element G of $\mathcal{G}$ containing T (compare [2], p. 115).

A dendroid means an arcwise connected and hereditarily unicoherent continuum.