STUDIA MATHEMATICA, T. LII. (1975)

On Taylor functional calculus

by

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Abstract. In the definition of the Taylor functional calculus ([4], § 4), the more elaborate space $\mathscr{B}$ may be replaced by the usual C^∞-space.

Introduction. Lately, the spectral theory in several variables has made great progress. A remarkable contribution to this theory is that of J. L. Taylor, which has obtained the natural definition of the spectrum [3] and has defined the corresponding functional analytic calculus [4]. His functional calculus has the advantage not only of being more abundant as the anterior (being defined for analytic functions in the neighbourhoods of a smaller spectrum) but also its definition is more accessible (for instance, he does not use the geometric integration theory).

However, because his construction needs a certain result on exactness and it is not known if this result is valid for the usual C^∞-space, Taylor is obliged to develop a supplementary apparatus; he defines a special function space (the space of all continuous functions, C^∞-differentiable in the distribution sense, with respect only to $\bar{z}_1, \ldots, \bar{z}_n$) and proves an exactness theorem for this space ([4], § 2).

It is the object of this paper to simplify more Taylor's construction showing that his more elaborate space may be replaced by the usual C^∞-space. We shall in fact remark that, if one does not extrapolate, the single result on exactness for the C^∞-space which would be necessary, is our Theorem 1. With this result at hand we can continue, without modification, the Taylor construction ([4], § 1 and § 3).

Our Theorem is a by-product of the exactness theorem for analytic functions ([3], Theorem 2.2) and of the Dolbeault – Grothendieck lemma ([2], Theorem 2.3.3 and Theorem 2.7.8).

1. We shall here use the notations from [3] and [4]. Let X be a complex Banach space and let V be an arbitrary open set in C^n (the space of n complex variables). We shall denote by $\mathscr{U}(V, X)$ the space of all X-valued analytic functions on V, and by $C^\infty(V, X)$ the space of all X-valued C^∞- differentiable functions on V. If $\sigma = (s_1, \ldots, s_n)$ is a n-tuple of indeterminates and Y is one of the spaces $\mathscr{U}(V,X)$ or $C^\infty(V,X)$, we shall denote by $\Lambda^p[\sigma, Y]$ the exterior forms of p degree in s having coefficients in Y. For an n-tuple $a=(a_1,\ldots,a_n)$ of linear continuous operators on X, we denote by $\mathrm{sp}(a,X)$ its spectrum and by $r(a,X)$, its resolvent set (that is, $r(a,X)=C^n\setminus \mathrm{sp}(a,X)$). Finally, by $\alpha(z)$ we shall denote the operator defined on the forms in s as the left exterior multiplication by $(z_1-a_1)s_1+\ldots+(z_n-a_n)s_n$.

2. We can now state:

THEOREM 1. *If X is a complex Banach space and $a=(a_1,\ldots,a_n)$ denotes an n-tuple of (linear continuous) operators on X, then for every open set $G\subset r(a,X)$ and any i, $0\leqslant i\leqslant 2n$, we have $H^i\big(C^\infty(G,X),\alpha\oplus\bar\partial\big)=0$.*

In order to prove this theorem we need a few auxiliary results.

LEMMA 1. *Let V_1 and V_2 be two open sets in C^n so that $V_1\cap V_2\neq\emptyset$. Then for any $f\in C^\infty(V_1\cap V_2,X)$, there exists $f_j\in C^\infty(V_j,X)$ $(j=1,2)$, such that $f=f_1-f_2$ on $V_1\cap V_2$.*

The proof uses the partition of unity and is identical to that from [1], Lemma 1.1 and Remark 1.1, so it will be omited.

LEMMA 2. *Assume that $H^{q-1}\big(C^\infty(U,X),\alpha\oplus\bar\partial\big)=0$ $(q\geqslant 1)$ for any open set $U\subset r(a,X)$. If V_1 and V_2 are two open sets in $r(a,X)$ and $\psi\in\Lambda^q[\sigma\cup d\bar z, C^\infty(V_1\cup V_2,X)]$ verifies $\psi=(\alpha\oplus\bar\partial)\varphi_1$ on V_1 and $\psi=(\alpha\oplus\bar\partial)\varphi_2$ on V_2, then there exists a form φ such that $\psi=(\alpha\oplus\bar\partial)\varphi$ on $V_1\cup V_2$.*

Proof. The case $V_1\cap V_2=\emptyset$ is obvious. Otherwise, we have $(\alpha\oplus\bar\partial)[\varphi_1-\varphi_2]=0$ on $V_1\cap V_2$, hence by assumption, there exists a form χ such that $\varphi_1-\varphi_2=(\alpha\oplus\bar\partial)\chi$ on $V_1\cap V_2$. According to Lemma 1 we can write χ in the form $\chi=\chi_2-\chi_1$ where χ_i is defined on V_i. Therefore $\varphi_1-\varphi_2=(\alpha\oplus\bar\partial)[\chi_2-\chi_1]$, from which $\varphi_1+(\alpha\oplus\bar\partial)\chi_1=\varphi_2+(\alpha\oplus\bar\partial)\chi_2$ on $V_1\cap V_2$; this allows us to define φ by

$$\varphi=\begin{cases}\varphi_1+(\alpha\oplus\bar\partial)\chi_1 & \text{on } V_1,\\ \varphi_2+(\alpha\oplus\bar\partial)\chi_2 & \text{on } V_2\end{cases}$$

and the proof is complete.

The following proposition is just our theorem for polydiscs and its proof is similar to that from [1], Proposition 2.1.

PROPOSITION 1. *For every open polydisc $D\subset r(a,X)$, we have $H\big(C^\infty(D,X),\alpha\oplus\bar\partial\big)=0$, $0\leqslant i\leqslant 2n$.*

Proof. Let us first remark that according to Theorem 2.2 from [3] (see also Definition 1.1), we have $H^i\big(\mathscr{U}(D,X),\alpha\big)=0$, $0\leqslant i\leqslant n$, for any open polydisc $D\subset r(a,X)$. Let now $\psi\in\Lambda^q[\sigma\cup d\bar z, C^\infty(D,X)]$ such that $(\alpha\oplus\bar\partial)\psi=0$. If $q=0$ then ψ is a simple function and the equality $(\alpha\oplus\bar\partial)\psi=0$ implies $\alpha\psi=0$ and $\bar\partial\psi=0$, so that ψ is an analytic function which satisfies $\alpha\psi=0$, hence $\psi=0$ (since $H^0\big(\mathscr{U}(D,X),\alpha\big)=0$). If $0<q\leqslant n$, we can write ψ in the form

$$\psi=\psi_{0,q}+\psi_{1,q-1}+\ldots+\psi_{q-1,1}+\psi_{q,0},$$

where the first index denotes the degree with respect to s and the second, the degree with respect to $d\bar z$. Then the equality $(\alpha\oplus\bar\partial)\psi=0$ gives us $\bar\partial\psi_{0,q}=0$, $\alpha\psi_{0,q}+\bar\partial\psi_{1,q-1}=0,\ldots,\alpha\psi_{q-1,1}+\bar\partial\psi_{q,0}=0$ and $\alpha\psi_{q,0}=0$. Since $\bar\partial\psi_{0,q}=0$, by Theorem 2.3.3 and Theorem 2.7.8 from [2], there exists a form $\varphi_{0,q-1}$ such that $\psi_{0,q}=\bar\partial\varphi_{0,q-1}$; replacing then $\psi_{0,q}$ by $\bar\partial\varphi_{0,q-1}$ in $\alpha\psi_{0,q}+\bar\partial\psi_{1,q-1}=0$, we obtain $\alpha\bar\partial\varphi_{0,q-1}+\bar\partial\psi_{1,q-1}=0$, therefore $\bar\partial[\psi_{1,q-1}-\alpha\varphi_{0,q-1}]=0$; this allows us to apply again the quoted theorems and we obtain another form $\varphi_{1,q-2}$ verifying $\psi_{1,q-1}-\alpha\varphi_{0,q-1}=\bar\partial\varphi_{1,q-2}$. Arguing in the same manner, at the penultimate step we shall obtain a form $\varphi^*_{q-1,0}$ so that $\bar\partial[\psi_{q,0}-\alpha\varphi^*_{q-1,0}]=0$; but the form $\psi_{q,0}-\alpha\varphi^*_{q-1,0}$ is only in s, hence this equality just means that it has analytic coefficients; on the other hand we have $\alpha[\psi_{q,0}-\alpha\varphi^*_{q-1,0}]=0$, therefore using that $H^q\big(\mathscr{U}(D,X),\alpha\big)=0$, there exists a form φ having analytic coefficients such that $\psi_{q,0}-\alpha\varphi^*_{q-1,0}=\alpha\varphi$ Denoting $\varphi^*_{q-1,0}+\varphi$ by $\varphi_{q-1,0}$, we conclude that

$$\psi=(\alpha\oplus\bar\partial)[\varphi_{0,q-1}+\varphi_{1,q-2}+\ldots+\varphi_{q-2,1}+\varphi_{q-1,0}].$$

Finally, if $n<q\leqslant 2n$, there appears a difference only in writing: $\psi=\psi_{q-n,n}+\psi_{q-n+1,n-1}+\ldots+\psi_{n-1,q-n+1}+\psi_{n,q-n}$, and we can continue as before.

COROLLARY. *If $H^{q-1}\big(C^\infty(U,X),\alpha\oplus\bar\partial\big)=0$ $(q\geqslant 1)$, for any open set $U\subset r(a,X)$, then for every compact set $K\subset r(a,X)$ and any form ψ of q degree verifying $(\alpha\oplus\bar\partial)\psi=0$ in a neighbourhood of K, there exists a form φ of $q-1$ degree verifying $\psi=(\alpha\oplus\bar\partial)\varphi$ in a neighbourhood of K.*

The proof of Theorem 1. Let us start with $q=0$. It is easy to observe that if $f\in C^\infty(G,X)$ and $(\alpha\oplus\bar\partial)f=0$, then $f=0$ on any open polydisc contained in G, hence $f=0$. We shall now assume that $H^{q-1}\big(C^\infty(U,X),\alpha\oplus\bar\partial\big)=0$ for any open set $U\subset r(a,X)$ and we shall prove that $H^q\big(C^\infty(G,X),\alpha\oplus\bar\partial\big)=0$. Let $\{K_j\}_{j=1}^\infty$ be an increasing sequence of compact sets such that $\bigcup_{j=1}^\infty K_j=G$. If $\psi\in\Lambda^q[\sigma\cup d\bar z, C^\infty(G,X)]$ and $(\alpha\oplus\bar\partial)\psi=0$ we shall show that it is possible to define a sequence of forms $\{\varphi_j\}_{j=1}^\infty$, $\varphi_j\in\Lambda^{q-1}[\sigma\cup d\bar z, C^\infty(G,X)]$ so that $\varphi_{j+1}=\varphi_j$ in a neighbourhood of K_j and $\psi=(\alpha\oplus\bar\partial)\varphi_j$ in a neighbourhood of K_j. A simple application of the preceding corollary gives us φ_1. Let us suppose that $\varphi_1,\ldots,\varphi_j$ have already been defined and let us define φ_{j+1}. Again by Corollary, there exists φ^*_{j+1} such that $\psi=(\alpha\oplus\bar\partial)\varphi^*_{j+1}$ in a neighbourhood of K_{j+1}, and we can suppose it to be defined on G. Of course, it can occur that it is not suitable; then we shall modify it according to the following: we have $(\alpha\oplus\bar\partial)[\varphi^*_{j+1}-\varphi_j]=0$ in neighborhood of K_j, consequently, by assumption, there exists χ such that $\varphi_{j+1}^* - \varphi_j = (\alpha \oplus \bar\partial)\chi$ in a neighbourhood of K_j (moreover, one may suppose that χ is defined on G). Now we can define $\varphi_{j+1} = \varphi_{j+1}^* - (\alpha \oplus \bar\partial)\chi$. This completes our inductive argument.

With this sequence at hand, it is clear that $\varphi = \lim_{j\to\infty} \varphi_j$ exists and that $\psi = (\alpha \oplus \bar\partial)\varphi$, so that Theorem 1 is proved.

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Received September 27, 1973 (749)

STUDIA MATHEMATICA, T. LII. (1975)

A multiplier theorem for Jacobi expansions

by

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Abstract. Multiplier operators on Jacobi expansions of functions in L^p, $1 \leqslant p < \infty$, are studied by realizing these operators as a sequence of kernels of singular integral type. It then follows from the Calderón–Zygmund Theory that such operators must be of strong type (p, p) for $1 < p < \infty$ and weak type $(1, 1)$.

1. Introduction. In this paper we utilize a theory developed in an earlier paper [4] to prove new and interesting multiplier theorems for Jacobi expansions. The basic idea is to represent the multiplier transformation M as a limit of convolution operators with kernels that have the properties of singular integral kernels. It then follows from the Calderón–Zygmund Theory that the operator M must be of strong type (p, p) and weak type $(1,1)$. This is a particular application of the idea of "spaces of homogeneous type" devised by Professors Coifman and Weiss in [3]. An exact statement of the theorem is given in § 3.

The key to the representation of M is finding an approximate identity with the desired properties. Here, as in the earlier paper, we use the Poisson kernel. There are many technical difficulties in these calculations, and many of the lemmas look quite different. One reason for this is the lack of symmetry in the polynomial $P_n^{(\alpha,\beta)}(x)$ which introduces more cases that must be handled. Another reason is the complicated expression for the Poisson kernel.

It is well known that any multiplier theorem for Jacobi polynomials will have important consequences in group theory. When $\alpha = \beta = (n-1)/2$, we obtain a multiplier theorem for the zonal spherical harmonics on the unit sphere Σ_n. When $\alpha = (n-1)/2$, $\beta = 0$, a multiplier theorem follows for the zonal spherical functions on the complex n-dimensional projective space. There are theorems of this sort for all of the compact rank -1 symmetric spaces. See Muckenhoupt and Stein [6], p. 22, Bonami and Clerc [2], § 7.

We mention here two other applications of our multiplier theorem, both of which will be developed elsewhere.

* Both authors were supported by AFOSR Grant No. 71-2047.