for sufficiently large b. This implies that for some t_0 and $0 \leqslant f \in L_\infty(Y)$ with $\|f\|_\infty > 0$, $T_{t_0} 1_A \geqslant f$. Corollary 2 implies that

$$\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* 1_A(x)\,dt \geqslant \lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* f(x)\,dt \not\equiv 0,$$

since $\left\langle 1, \frac{1}{b}\int_0^b T_t^* f(x)\,dt \right\rangle = \langle 1, f\rangle \neq 0$ for any $b > 0$.

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) follow from Fatou's lemma, and (iii) $\Rightarrow$ (iv) is obvious.

(v) $\Rightarrow$ (i): It may be readily seen from Lemma 3 and [4], Theorem 5.3, that there exists a function $f \in L_1(Y)$ with $f > 0$ on Y and $T_t f \geqslant f$ for any $t > 0$. Let

$$g = \text{strong-}\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t f\,dt \quad \text{and} \quad E = X - \operatorname{supp} g.$$

It is clear that $T_t g = g$ for all $t > 0$ and $E \subset Z$. Since $T_t^* 1_E \in L_\infty(E)$ for all $t > 0$, $\int_0^b T_t^* 1_E(x)\,dt = 0$ on $X - E$ for any $b > 0$. This together with the fact that $s = 0$ on Z implies that

$$\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* 1_E(x)\,dt = 0 \text{ a.e.}$$

(v) implies $m(E) = 0$, and the proof is complete.

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DEPARTMENT OF MATHEMATICS, JOSAI UNIVERSITY
SAKADO, SAITAMA, JAPAN

Received October 10, 1973 (741)

STUDIA MATHEMATICA, T. LIII. (1975)

Local eigenvectors for group representations

by

ELLIOT C. GOOTMAN (Athens, Ga.)

Abstract. We prove that every unitary representation V of a group G has a local eigenvector (i.e., a common eigenvector for all $V(g)$, g ranging over a neighborhood of the identity) if and only if G_0, the connected component of the identity, is compact and abelian. It follows as a simple corollary that for G_0 compact and abelian, cocycle representations of G also have local eigenvectors. The proof uses Mackey's little group method.

Let V be a unitary or cocycle representation of a locally compact group G on a Hilbert space $\mathscr{H}$. A non-zero vector x in $\mathscr{H}$ is a local eigenvector (respectively, local fixed point) for V if x is a common eigenvector (respectively, fixed point) for all the unitary operators $V(g)$, as g ranges over some neighborhood of the identity e in G. It is known that all unitary representations of G have local fixed points if and only if G is totally disconnected. We extend this result by proving that all unitary representations of G have local eigenvectors if and only if G_0, the connected component of the identity, is compact and abelian. The proof uses a preliminary lemma that in fact all cocycle representations of totally disconnected groups have local eigenvectors, and as a simple corollary to the main theorem we show that indeed so do all cocycle representations of group G with G_0 compact and abelian. These results have application in determining the structure space of certain C^*-algebras associated with transformation groups ([3], Theorem 4.4).

As the proof of the main theorem involves application of Mackey's little group method, we assume that all groups are second countable and all Hilbert spaces are separable. All unitary representations are continuous and all cocycle representations are Borel. A cocycle representation with cocycle a will be called simply an *a-representation*. For terminology and basic results on cocycle representations we refer the reader to [1], Chapter I, Section 4. Throughout the paper we shall use without further explicit mention the simple observations that a local eigenvector for a unitary or cocycle representation V is a common eigenvector for all $V(k)$ as k ranges over some open subgroup containing G_0, and that if a, the cocycle of V, is cohomologous on an open subgroup K to a cocycle a' of K it suffices, in order to prove the existence of local eigenvectors, to replace V by the

α'-representation of K given by $(\chi V)(k) = \chi(k)V(k)$, where χ is such that

$$\alpha'(k_1, k_2) = \frac{\chi(k_1)\chi(k_2)}{\chi(k_1\cdot k_2)}\alpha(k_1, k_2).$$

LEMMA. *Every cocycle representation V of a totally disconnected group G on a Hilbert space has a local eigenvector.*

Proof. It follows from Theorem 1.1 of [2], the theorem in [7], Chapter 2.3, and the above remarks that we need only consider an α-representation W of a compact open subgroup K of G, where α is a continuous cocycle. W decomposes as a direct sum of finite-dimensional α-representations and it clearly suffices to verify that one of these finite-dimensional components has a local eigenvector. By the theorems in [2], top of page 10, and [7], Chapter 2.3, we are reduced to considering a unitary representation of a compact open subgroup L of K. As L is totally disconnected, its unitary representations have local eigenvectors.

THEOREM. *Every unitary representation of G on a Hilbert space has a local eigenvector if and only if G_0 is compact and abelian.*

Proof. ($\Rightarrow$) It will suffice to assume only that the left-regular representation V of G on $L^2(G)$, and all its subrepresentations, have local eigenvectors. As the closed linear span of the set of all local eigenvectors is clearly G-invariant, it is therefore all of $L^2(G)$.

Furthermore, since two eigenvectors for a unitary operator $V(g)$ with distinct eigenvalues are orthogonal, it follows that $L^2(G)$ decomposes as a direct sum of mutually orthogonal subspaces, each invariant under $V|_{G_0}$, the restriction of V to G_0, and on each of which $V(g) = \chi(g)I$ for $g \in G_0$ and some character χ on G_0. Thus $V(g_1g_2) = V(g_1)V(g_2) = V(g_2)V(g_1) = V(g_2g_1)$ for all $g_1, g_2 \in G_0$ and since V separates points of G ([4], page 130), G_0 is abelian. That G_0 is also compact follows from the fact that there is a common eigenvector for the restriction of V to some open subgroup K. $L^2(G) = \sum_{G\diagdown K}\oplus L^2(K\bar{g})$ and as each subspace is invariant under K, some $L^2(K\bar{g})$, $\bar{g} \in G\diagdown K$, has a common eigenvector for $V|_K$. $V|_K$ on $L^2(K\bar{g})$ is unitarily equivalent, however, to $V|_K$ on $L^2(K)$, so the left-regular representation of K has a one-dimensional invariant subspace, K is compact ([5], Corollary to Theorem 8.2) and so is $G_0 \subseteq K$.

($\Leftarrow$) Let V be a unitary representation of G on $\mathscr{H}$. The totally disconnected group G/G_0 has a compact open subgroup whose inverse image (under the natural map $G \to G/G_0$) is a compact open subgroup K of G containing G_0. $V|_K$ decomposes as a direct sum of finite-dimensional irreducible subrepresentations and we prove that each of these subrepresentations has a local eigenvector by using Mackey's little group method to analyze irreducible representations of K in terms of the closed normal type I subgroup G_0 and certain cocycle representations of subgroups of K/G_0. Accordingly, let π be an irreducible representation of K. As K is compact and the dual group $\hat{G}_0$ of G_0 is discrete, orbits of K in $\hat{G}_0$ are finite and the isotropy subgroup H_χ of an element $\chi \in \hat{G}_0$ is open, with $K \supseteq H_\chi \supseteq G_0$. Thus ([6], Theorem 8.1) π is induced from an irreducible representation σ of H_χ, for some $\chi \in \hat{G}_0$, and ([6], Theorem 8.3) σ is the tensor product of a certain cocycle representation τ of H_χ/G_0 and an extension of χ to a cocycle representation $\tilde{\chi}$ of H_χ. As $\tilde{\chi}$ is one-dimensional and H_χ/G_0 is totally disconnected it follows from our lemma that σ has a local eigenvector x. The function f: $K \to$ the Hilbert space of σ, defined by $f(k) = \sigma(k)^{-1}x$ for $k \in H_\chi$, $f(k) = 0$ for $k \notin H_\chi$, is then a local eigenvector for the induced representation π (recall that H_χ is open in K) and we are done.

As a consequence of the above theorem, we have the following

COROLLARY. *Let α be a cocycle on G. Every α-representation of G on a Hilbert space has a local eigenvector if and only if G_0 is compact and abelian.*

Proof. ($\Rightarrow$) This part of the proof is similar to that of the theorem and as there, it suffices to assume only that the left-regular α-representation V of G on $L^2(G)$ (see [2], p. 12 for a definition) and all its subrepresentations have local eigenvectors. As before, this implies that $V(g_1)V(g_2) = V(g_2)V(g_1)$ for $g_1, g_2 \in G_0$. From this fact, however, we can deduce only that $V(g_1g_2)$ is a scalar multiple of $V(g_2g_1)$, but the method of proof in [4], p. 130, yields $g_1g_2 = g_2g_1$ in this case also, so G_0 is abelian. Again as before, the left regular α-representation of an open subgroup K on $L^2(K)$ has a common eigenvector. But then α is a coboundary on K, the left-regular α-representation is unitarily equivalent to the left-regular unitary representation, and we are done.

($\Leftarrow$) Every cocycle on a compact connected abelian group is a coboundary ([9], § 29, [8], Theorem 2.1 and Proposition 2.1) and the group extension of T by G_0, defined by the cocycle α, is isomorphic to the direct product group $T \times G_0$, which is clearly compact and abelian (see [1], Chapter I, Section 4). Let E denote the group extension of T by G, defined by the cocycle α. It follows from the continuity of the homomorphisms $T \to E$ and $E \to G$ that E_0 is a subgroup of the extension of T by G_0, so E_0 is compact and abelian. In fact, E_0 is isomorphic to the extension of T by G_0 since this latter group is connected. By our theorem, every unitary representation of E, and hence in particular every α-representation of G, has a local eigenvector.

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UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602

Received October, 16, 1973 (747)

STUDIA MATHEMATICA, T. LIII. (1975)

On commutators of singular integrals*

by

CALIXTO P. CALDERÓN* (Minneapolis, Minn.)

Abstract. The commutator v.p. $\int_{-\infty}^{\infty} \frac{F(x)-F(y)}{(x-y)^2} g(y)\,dy = T(F', g)$ as well as its n-dimensional generalizations are treated through this paper. The previous known results stated that if $g \in L^p$, $F'(x) \in L^q$, with $1/p+1/q<1$, then $\|T(F', g)\|_r < C_{pq}\|F'\|_q\|g\|_p$; $1/r = 1/p+1/q$. Here it is presented the following novelty: that the restriction $1/p+1/q<1$ is not any longer necessary. We face the cases $1/p+1/q>1$, obtaining as expected the same inequality in this situation.

0. Introduction. The purpose of this paper is to extend and generalize the results proved in [1] and [3].

We shall be concerned with singular integrals of the type

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{F(x)-F(y)}{(x-y)^2} g(y)\,dy = T(F', g) \tag{0.1}$$

and their n-dimensional generalizations. Here, $F(x)$ stands for a function having a derivative in the distributions sense in the class L^q; g stands for a measurable function belonging to a class L^p.

If $\frac{1}{p}+\frac{1}{q}<1$, $1<p<\infty$, $1<q<\infty$ and r is given by $\frac{1}{r} = \frac{1}{q}+\frac{1}{p}$; then $T(F', g)$ exists in L^r-norm and pointwise a.e., see [1] and [3].

Through this paper the condition $1/p+1/q<1$ is relaxed to the following one:

$$1<q\leqslant\infty, \quad 1<p\leqslant\infty, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \quad r\neq\infty, \tag{0.2}$$

The condition $r\neq\infty$ means that p and q cannot be infinity simultaneously.

Under the condition (0.2) we show that the operator defined in (0.1) exists as a principal value in the metric L^r (notice that r can be less than

* The author was partially supported by NSF Grant No. GP-15832.