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On non-separable Banach spaces with a symmetric basis

by

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Abstract. In this paper equivalent norms in non-separable Banach spaces with a symmetric basis are considered. The results obtained indicate the impossibility of a natural extension to the non-separable case of certain theorems valid for separable Banach spaces with an unconditional basis.

1. Introduction. James (cf. e. g. [6], p. 152) proved that any non-reflexive Banach space with an unconditional basis contains a subspace isomorphic either to c_0 or l_1. Later on Bessaga and Pełczyński (cf. e. g. [6], p. 155) extended that result to non-reflexive subspaces of spaces with an unconditional basis. Lindenstrauss and Tzafriri [3] studied the Orlicz spaces l_M and proved that any l_M contains a subspace isomorphic to l_p for some $p \geqslant 1$. Lindenstrauss [5] showed that every separable space with an unconditional basis can be isomorphically embedded in some separable space with a symmetric basis.

In this paper we deal with the question of the existence of equivalent norms which are uniformly convex or uniformly smooth in every direction, in Banach spaces with a symmetric basis. It turns out that this question is closely related to that of the existence, in such spaces, of subspaces isomorphic to $c_0(\Gamma)$ or $l_1(\Gamma)$ for some uncountable Γ. As a corollary it results that the above-mentioned theorems of James [6], Lindenstrauss–Tzafriri [3] and Lindenstrauss [5] admit no natural extension to the non-separable case.

In particular, we shall show that a non-separable Banach space X with a symmetric basis admits an equivalent norm which is uniformly convex in every direction (resp. uniformly smooth in every direction) iff X is not isomorphic to $c_0(\Gamma)$ (resp. to $l_1(\Gamma)$) for some uncountable set Γ.

2. Definitions and notations. Let X be a Banach space and let Γ be an abstract set. A function $w(\gamma)$ defined on Γ with values in X is said to be *unconditionally summable* to $x \in X$ if for any $\varepsilon > 0$ there exists a finite set $B \subset \Gamma$ such that for every finite set $A \subset \Gamma$ with $A \supset B$ we have

$$\Big\| \sum_{\gamma \in A} w(\gamma) - x \Big\| < \varepsilon .$$

The element x will be written as $x = \sum_{\gamma\in\Gamma} w(\gamma)$ and the series $\sum_{\gamma\in\Gamma} w(\gamma)$ will be said to *converge to x unconditionally*.

A function $u(\gamma)$ defined on Γ with values in X is called an *unconditional basis for X* if for any $x\in X$ there exists a unique real-valued function $\varphi_x(\gamma)$ defined on Γ and such that the series $\sum_{\gamma\in\Gamma}\varphi_x(\gamma)u(\gamma)$ converges to x unconditionally. In the sequel we shall write rather $\{u_\gamma\}_{\gamma\in\Gamma}$ instead of $u(\gamma)$. The symbol X^* denotes the *conjugate space* of X and $\{u_\gamma^*\}_{\gamma\in\Gamma}$ is the system in X^* *conjugate to the basis* $\{u_\gamma\}_{\gamma\in\Gamma}$, i. e. $u_\gamma^*(u_\beta) = 0$ for $\gamma\neq\beta$, $u_\gamma^*(u_\gamma) = 1$.

Bases $\{u_\gamma\}_{\gamma\in\Gamma}$ and $\{v_\gamma\}_{\gamma\in\Gamma}$ in space X, resp. Y, are called *equivalent* if there exists a bounded linear operator $T\colon X\to Y$ with a bounded inverse and such that $Tu_\gamma = v_\gamma$ for all $\gamma\in\Gamma$. Note that this is the case if and only if the series $\sum_{\gamma\in\Gamma} a(\gamma)u_\gamma$ and $\sum_{\gamma\in\Gamma} a(\gamma)v_\gamma$ are simultaneously convergent or divergent, for any real-valued function $a(\gamma)$ defined on Γ.

An unconditional basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is called *symmetric* if for any two sequences $\{\alpha_i\}_{i=1}^\infty$ and $\{\beta_i\}_{i=1}^\infty$ in Γ the bases $\{u_{\alpha_i}\}_{i=1}^\infty$ and $\{u_{\beta_i}\}_{i=1}^\infty$ are equivalent.

By $c_0(\Gamma)$ we denote the *space of all real-valued functions $x(\gamma)$* defined on Γ and such that for any $\varepsilon>0$ the set $\{\gamma\colon |x(\gamma)|>\varepsilon\}$ is finite; $\|x\| = \max_{\gamma\in\Gamma}|x(\gamma)|$. An equivalent norm has been defined by Day [1]:

$$D(x) = \sup\Big[\sum_{i=1}^{m} 2^{-i}x^2(\alpha_i)\Big]^{1/2},$$

where the supremum is taken with respect to all finite subsets $\{\alpha_i\}_{i=1}^m\subset\Gamma$. For $x\in c_0(\Gamma)$, let $\sigma(x)$ denote the sequence $\{\gamma_i\}_{i=1}^N\subset\Gamma$ (N an integer or infinity) such that $|x(\gamma_i)|\geqslant|x(\gamma_{i+1})|>0$, $i=1,2,\ldots$, and $x(\gamma)=0$ for $\gamma\notin\sigma(x)$. It can easily be seen that

$$D(x) = \Big[\sum_{i=1}^{N} 2^{-i}x^2(\gamma_i)\Big]^{1/2}.$$

A continuous convex function $M(t)$ on $[0,\infty)$ is called an *Orlicz function* if $M(0) = 0$ and $M(t)>0$ for all $t>0$. With every Orlicz function $M(t)$ we can associate an *Orlicz space* $l_M(\Gamma)$ consisting of all real-valued functions $x(\gamma)$ such that for some $t>0$ the function $M(|x(\gamma)|/t)$ is unconditionally summable in γ;

$$\|x\| = \inf\Big\{t>0;\ \sum_{\gamma\in\Gamma} M(|x(\gamma)|/t)\leqslant 1\Big\}.$$

If $M(t) = t^p$, $p\geqslant 1$, we write $l_p(\Gamma)$ instead of $l_{t^p}(\Gamma)$. If Γ is countable, we write c_0 and l_p for $c_0(\Gamma)$ and $l_p(\Gamma)$.

Let $\chi_\gamma(\beta)$ be the function on Γ defined by $\chi_\gamma(\beta) = 0$ for $\beta\neq\gamma$, $\chi_\gamma(\gamma) = 1$. The function $e(\gamma) = \chi_\gamma$ can be regarded as a function defined in Γ with values in $c_0(\Gamma)$ or $l_M(\Gamma)$. Observe that $e(\gamma)$ is a symmetric basis in $c_0(\Gamma)$ and $l_M(\Gamma)$ if $\overline{\lim_{t\to 0}} M(2t)/M(t)<\infty$. The basis $\{e_\gamma\}_{\gamma\in\Gamma}$ is called the *natural basis* for $c_0(\Gamma)$ and $l_M(\Gamma)$. $\{e_\gamma^*\}_{\gamma\in\Gamma}$ will denote the system *conjugate* to $\{e_\gamma\}_{\gamma\in\Gamma}$.

The norm of a Banach space X is called *uniformly convex in every direction* if the conditions:

$$\|x_n\| = \|y_n\| = 1,\quad x_n - y_n = \lambda_n z,\quad \lim_{n\to\infty}\|x_n+y_n\| = 2,\quad x_n, y_n, z\in X$$

imply

$$\lim_{n\to\infty}|\lambda_n|\,\|z\| = 0.$$

The norm of X^*, the conjugate space of X, is called *weakly* uniformly convex* if the conditions:

$$\|f_n\| = \|g_n\| = 1,\quad \lim_{n\to\infty}\|f_n+g_n\| = 2,\quad f_n, g_n\in X^*$$

imply

$$\lim_{n\to\infty}\big(f_n(x)-g_n(x)\big) = 0$$

for all $x\in X$.

The norm of a Banach space X is called [*uniformly*] *smooth* (in every direction) if for any $x, y\in X$ with $\|x\| = \|y\| = 1$

$$\lim_{\tau\to\infty}(\|x+\tau y\|+\|x-\tau y\|-2) = 0$$

[uniformly in x].

For Banach spaces X, Y, $X\times Y$ will denote their product with the norm $\|(x,y)\| = (\|x\|^2+\|y\|^2)^{1/2}$.

3. In this section we construct an equivalent norm and investigate its properties.

PROPOSITION 1. *Let T be a linear operator mapping a Banach space X into $c_0(\Gamma)$ with some Γ in such a way that for any $\varepsilon>0$ there exists an integer $k = k(\varepsilon)$ such that for all $x\in X$ $\|x\|\leqslant 1$, the set $\{\gamma\colon |e_\gamma^*(Tx)|\geqslant\varepsilon\}$ contains at most k elements. Then if*

$$|||x_n||| = |||y_n||| = 1,\quad \lim_{n\to\infty}|||x_n+y_n||| = 2, \tag{1}$$

then for any $\gamma\in\Gamma$ we have

$$\lim_{n\to\infty} e_\gamma^*(Tx_n - Ty_n) = 0,$$

where $|||x||| = \big(\|x\|^2+D^2(Tx)\big)^{1/2}$ and $\{e_\gamma^*\}_{\gamma\in\Gamma}$ is the conjugate system to the natural basis of $c_0(\Gamma)$.

The proof will be proceded by a lemma concerning the norm $D(x)$ in $c_0(\Gamma)$.

LEMMA 1. *Let $x \in c_0(\Gamma)$, $\sigma(x) = \{\gamma_i\}_{i=1}^N$ and $|x(\alpha)| > \sqrt{2}\,|x(\gamma_j)|$. Then*

$$D^2(x) \geqslant D^2(x-\xi e_\alpha)+2^{-j}\xi^2,$$

where $\xi = x(\alpha)$.

Proof. Let $\alpha = \gamma_m$. Clearly, $m<j$. Thus

$$\begin{aligned} D^2(x) &= \sum_{i=1}^{m-1} 2^{-i}x^2(\gamma_i)+\sum_{i=m+1}^{N} 2^{-i}\big(x^2(\gamma_m)+x^2(\gamma_i)\big) \\ &\geqslant \sum_{i=1}^{m-1} 2^{-i}x^2(\gamma_i)+\sum_{i=m+1}^{j-1} 2^{-i}\big(x^2(\gamma_m)+x^2(\gamma_i)\big)+ \\ &\quad +\sum_{i=j}^{N}2^{-i}\big(x^2(\gamma_m)+x^2(\gamma_i)\big)+2^{-j}x^2(\gamma_m) \\ &\geqslant D^2(x-\xi e_{\gamma_m})+2^{-j}x(\gamma_m). \ \blacksquare \end{aligned}$$

Proof of Proposition 1. Observe that

$$\begin{aligned} &2(|||x_n|||^2+|||y_n|||^2)-|||x_n+y_n|||^2 \\ &= [2(\|x_n\|^2+\|y_n\|^2)-\|x_n+y_n\|^2]+[2(D^2(Tx_n)+D^2(Ty_n))-D^2(Tx_n+Ty_n)]. \end{aligned}$$

Since the expressions in the square brackets are non-negative, thus by (1)

(2) $$\lim_{n\to\infty}\big[2\big(D^2(Tx_n)+D^2(Ty_n)\big)-D^2(Tx_n+Ty_n)\big] = 0.$$

Suppose that the assertion of the proposition is false. Then without loss of generality we may assume that there exist $\alpha \in \Gamma$ and $\delta > 0$ such that

(3) $$|e_\alpha^*(Tx_n-Ty_n)| \geqslant \delta, \quad n = 1, 2, \ldots$$

We shall show that

(4) $$\overline{\lim_{n\to\infty}}|e_\alpha^*(Tx_n+Ty_n)| > 0.$$

Suppose the contrary, i.e.

(5) $$\lim_{n\to\infty} e_\alpha^*(Tx_n+Ty_n) = 0.$$

Choose n_0 such that

(6) $$|e_\alpha^*(Tx_n+Ty_n)| < \tfrac{1}{2}\delta \quad \text{for} \quad n > n_0.$$

From (3) and (6) follows

$$|e_\alpha^*(Tx_n)| > \tfrac{1}{4}\delta, \quad |e_\alpha^*(Ty_n)| > \tfrac{1}{4}\delta \quad \text{for} \quad n > n_0.$$

Hence, by Lemma 1,

(7) $$D^2(Tx_n) > D^2\big(Tx_n-e_\alpha^*(Tx_n)e_\alpha\big)+2^{-k-5}\delta^2 \quad \text{for} \quad n > n_0,$$

(8) $$D^2(Ty_n) \geqslant D^2\big(Ty_n-e_\alpha^*(Ty_n)e_\alpha\big)+2^{-k-5}\delta^2 \quad \text{for} \quad n > n_0,$$

where $k = k(\delta/4\sqrt{2})$.

It follows from the triangle inequality, the definition of $D(x)$ and from (1) that

(9) $$\begin{aligned} D^2(Tx_n+Ty_n) &\leqslant D^2\big(Tx_n+Ty_n-e_\alpha^*(Tx_n+Ty_n)e_\alpha\big)+ \\ &\quad +(2\sqrt{2}+1)\|T\|\cdot|e_\alpha^*(Tx_n+Ty_n)|. \end{aligned}$$

We have, by (7), (8), (9),

$$\begin{aligned} &2\big(D^2(Tx_n)+D^2(Ty_n)\big)-D^2(Tx_n+Ty_n) \\ &\quad > 2^{-k-3}\delta^2-(2\sqrt{2}+1)\|T\|\cdot|e_\alpha^*(Tx_n+Ty_n)| \quad \text{for} \quad n > n_0. \end{aligned}$$

But this together with (5) contradicts (2), so (4) is proved. Without affecting the generality we may thus assume that there exists $\varepsilon > 0$ such that for all $n = 1, 2, \ldots$

(10) $$|e_\alpha^*(Tx_n+Ty_n)| > \varepsilon.$$

Let $\sigma(Tx_n+Ty_n) = \{\gamma_{i,n}\}_{i=1}^{N_n}$. By the definition of $D(x)$ we have

$$\begin{aligned} &2\big(D^3(Tx_n)+D^2(Ty_n)\big)-D^2(Tx_n+Ty_n) \\ &\geqslant \Big[D^2(Tx_n)-\sum_{i=1}^{N_n}2^{-i}\big(e_{\gamma_{i,n}}^*(Tx_n)\big)^2\Big]+\Big[D^2(Ty_n)-\sum_{i=1}^{N_n}2^{-i}\big(e_{\gamma_{i,n}}^*(Ty_n)\big)^2\Big]+ \\ &\quad +\Big[\sum_{i=1}^{N_n}2^{-i}\big(e_{\gamma_{i,n}}^*(Tx_n-Ty_n)\big)^2\Big]. \end{aligned}$$

Since the expressions in the square brackets are non -negative,we have, by (2),

$$\lim_{n\to\infty}\sum_{i=1}^{N_n}2^{-i}\big(e_{\gamma_{i,n}}^*(Tx_n-Ty_n)\big)^2 = 0.$$

On the other hand, it follows from (10) that $\alpha \in \{\gamma_{i,n}\}_{i=1}^{k}$, where $k = k(\varepsilon/2))$ for all $n = 1, 2, \ldots$, hence, again by (10),

$$\sum_{i=1}^{N_n}2^{-i}\big(e_{\gamma_{i,n}}^*(Tx_n-Ty_n)\big)^2 > 2^{-k}\delta^2, \quad n = 1, 2, \ldots$$

This contradiction concludes the proof of Proposition 1.

4. In this section we apply Proposition 1 to find necessary and sufficient conditions for the existence of an equivalent norm, uniformly convex (smooth) in every direction, in non-separable Banach spaces with a symmetric basis. As is shown in [2], in any separable Banach space an equivalent norm, uniformly convex in every direction, can be introduced. It is also proved in [2] that in any separable Banach space an equivalent norm can be defined such that the norm of the conjugate space will be weakly* uniformly convex. Shmulyan [7] proved that the norm of a Banach space X is uniformly smooth in every direction if and only if the norm of X^* is weakly* uniformly convex. Thus it follows from [2] and [7] that in any separable Banach space there exists an equivalent norm uniformly smooth in every direction.

It is well known that for any unconditional basis $\{u_\gamma\}_{\gamma\in\Gamma}$ there exists a positive constant $c = c(\{u_\gamma\})$ such that for any finite system $\{\gamma_i\}_{i=1}^m \subset \Gamma$ and any system $\{a_i\}_{i=1}^m$ of real numbers we have

$$\Big\|\sum_{i=1}^m a_i u_{\gamma_i}\Big\| \geqslant c \max_{|\varepsilon_i|\leqslant 1}\Big\|\sum_{i=1}^m \varepsilon_i a_i u_{\gamma_i}\Big\|. \tag{11}$$

It can be also shown, by the definition of an unconditional basis, that there exists a positive constant $d = d(\{u_\gamma\})$ such that for any finite systems $\{a_i\}_{i=1}^m, \{\beta_i\}_{i=1}^m \subset \Gamma$ and any system $\{a_i\}_{i=1}^m$ of real numbers we have

$$\Big\|\sum_{i=1}^m a_i u_{a_i}\Big\| \geqslant d\Big\|\sum_{i=1}^m a_i u_{\beta_i}\Big\|. \tag{12}$$

Lemma 2. *Let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a symmetric basis in a Banach space X. Then either for any $\varepsilon > 0$ there exists an integer k such that for all $f\in X^*$ the sets $\{\gamma\colon |f(u_\gamma)| > \varepsilon\|f\|\}$ contain at most k elements, or the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of $l_1(\Gamma)$.*

Proof. Suppose that for some $\varepsilon > 0$ there are sequences $\{f_n\}_{n=1}^\infty \subset X^*$ and $\{\gamma_i\}_{i=1}^\infty \subset \Gamma$ such that

$$\|f_n\| = 1, \quad |f_n(u_{\gamma_i})| > \varepsilon; \quad i = i_n+1,\, i_n+2\,\ldots,\, i_{n+1}; \quad i_{n+1}-i_n = n,\ n = 1, 2, \ldots \tag{13}$$

Then for any finite system $\{\beta_i\}_{i=1}^n \subset \Gamma$ and any system $\{a_i\}_{i=1}^n$ of real numbers we have

$$\Big\|\sum_{i=1}^n a_i u_{\beta_i}\Big\| \geqslant d\Big\|\sum_{i=i_n}^{i_{n+1}} a_{i+1-i_n} u_{\gamma_{i+1}}\Big\| \geqslant \varepsilon c d\sum_{i=1}^n |a_i|. \tag{14}$$

It follows from (12) and (14) that the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of the space $l_1(\Gamma)$.

Lemma 3. *Let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a symmetric basis in a Banach space X and let $\{u_\gamma^*\}_{\gamma\in\Gamma} \subset X^*$ be the conjugate system to $\{u_\gamma\}_{\gamma\in\Gamma}$. Then either for any $\varepsilon > 0$ there exists an integer k such that for all $x\in X$ the sets $\{\gamma\colon |u_\gamma^*(x)| > \varepsilon\|x\|\}$ contain at most k elements, or the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of $c_0(\Gamma)$.*

Proof. Since $\{u_\gamma^*\}_{\gamma\in\Gamma}$ is a symmetric basis for its closed linear envelope $\operatorname{span}\{u_\gamma^*\}$, thus by Lemma 2 either there exists an integer k with the desired property, or there exists a positive constant b such that for any finite system $\{\gamma_i\}_{i=1}^n \subset \Gamma$ and any system $\{a_i\}_{i=1}^n$ of real numbers we have

$$\Big\|\sum_{i=1}^n a_0 u_{\gamma_i}^*\Big\| \geqslant b\sum_{i=1}^n |a_i|. \tag{15}$$

Now take an arbitrary finite subset $B \subset \Gamma$. We can find a finite subset $A \subset \Gamma$ and real numbers $\{a_a\}_{a\in A}$ such that

$$\Big\|\sum_{a\in A} a_a u_a^*\Big\| \leqslant 1,\ \Big\|\sum_{\beta\in B} u_\beta\Big\| \leqslant \sum_{a\in A} a_a u_a^*\Big(\sum_{\beta\in B} u_\beta\Big) + 1/b.$$

Hence, by (15), we get

$$\Big\|\sum_{\beta\in B} u_\beta\Big\| \leqslant 2/b. \tag{16}$$

It follows from (11) and (16) that the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of the space $c_0(\Gamma)$.

Proposition 2 (cf. [2]). *If Γ is uncountable then the space $c_0(\Gamma)$ admits no equivalent norm, uniformly convex in every direction.*

Proposition 3 (cf. [1]). *If Γ is uncountable then the space $l_1(\Gamma)$ admits no equivalent smooth norm.*

Theorem 1. *Let X be a non-separable Banach space with a symmetric basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then the space X admits an equivalent norm, uniformly convex in every direction, if and only if the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is not equivalent to the natural basis of the space $c_0(\Gamma)$.*

Proof. The "only if" part is an immediate consequence of Proposition 2.

The "if" part. We define a bounded linear operator T from X into $c_0(\Gamma)$: for $x\in X$ we put $Tx = y$ where $y(\gamma) = u_\gamma^*(x)$ for all $\gamma\in\Gamma$. The norm $|||\cdot|||$ is defined as in Proposition 1. Now let $|||x_n||| = |||y_n||| = 1$, $x_n - y_n = \lambda_n z$, $\lim_{n\to\infty} |||x_n+y_n||| = 2$ and let $\|z\| > 0$. Then there exists $a\in\Gamma$ such that $|u_a^*(z)| > 0$. Hence, by Lemma 3 and Proposition 1, $\lim_{n\to\infty}\lambda_n = 0$. ■

Corollary 1. *Let X be a Banach space with a symmetric basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then if X contains a subspace isomorphic to $c_0(\Delta)$ with Δ uncountable, then the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the natural basis of $c_0(\Gamma)$.*

Proof. It follows from Proposition 2 that X admits no equivalent norm, uniformly convex in every direction; the assertion thus results in view of Theorem 1.

THEOREM 2. *Let X be a non-separable Banach space with a symmetric basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then the space X admits an equivalent norm, uniformly smooth in every direction, if and only if the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is not equivalent to the natural basis of the space $l_1(\Gamma)$.*

Proof. The "only if" part is an immediate consequence of Proposition 3.

The "if" part. We define a bounded linear operator T from X^* into $c_0(\Gamma)$: for $f\in X^*$ we put $Tf = y$ where $y(\gamma) = f(u_\gamma)$ for all $\gamma\in\Gamma$. The equivalent norm $|||\cdot|||$ in X^* is define as in Proposition 1. For $x\in X$ put

$$|||x|||' = \sup\{|f(x)|:\ |||f|||\leqslant 1\};$$

this is an equivalent norm in X. Since the operator T is weak* continuous, the $|||\cdot|||$-unit ball is weak* compact and hence the space $(X^*, |||\cdot|||)$ is conjugate to $(X, |||\cdot|||')$.

We shall prove that the norm $|||\cdot|||'$ is uniformly smooth in every direction. According to Shmulyan's result [8] referred to above, it suffices to show that the norm $|||\cdot|||$ is weakly* uniformly convex.

Let $|||f_n||| = |||g_n||| = 1$ and $\lim\limits_{n\to\infty}|||f_n+g_n||| = 2$. Take an $x\in X$. Fix $\varepsilon > 0$. We can find a finite subset $B\subset\Gamma$ and real numbers $\{a_\beta\}_{\beta\in B}$ such that

$$\Big\|x-\sum_{\beta\in B}a_\beta u_\beta\Big\| < \tfrac{1}{2}\varepsilon. \tag{17}$$

In view of Lemma 2 and Proposition 1 there exists an integer n_0 such that for $n > n_0$ we have

$$\Big|f_n\Big(\sum_{\beta\in B}a_\beta u_\beta\Big)-g_n\Big(\sum_{\beta\in B}a_\beta u_\beta\Big)\Big| < \tfrac{1}{2}\varepsilon. \tag{18}$$

From (17) and (18) follows

$$|f_n(x)-g_n(x)| < \varepsilon \quad \text{for} \quad n > n_0. \ \blacksquare$$

COROLLARY 2. *Let X be a Banach space with a symmetric basis $\{u_\gamma\}_{\gamma\in\Gamma}$. Then if X contains a subspace isomorphic to $l_1(\Delta)$ with Δ uncountable, then the basis $\{u_\gamma\}_{\gamma\in\Gamma}$ is equivalent to the basis of $l_1(\Gamma)$.*

Proof. It follows from Proposition 3 that X admits no equivalent smooth norm; the assertion thus results in view of Theorem 2.

COROLLARY 3. *If Γ and Δ are infinite sets and $\Gamma\cup\Delta$ is uncountable, then the space $c_0(\Gamma)\times l_1(\Delta)$ cannot be isomorphically embedded in a space with a symmetric basis.*

Proof. Apply Corollary 1 and Corollary 2.

PROPOSITION 4. *Let $M(t)$ be an Orlicz function such that*

$$\lim_{t\to 0} M(t)/t = 0 \quad \text{and} \quad \lim_{t\to 0} tM'(t)/M(t) = 1.$$

Then for any set Γ the Orlicz space $l_M(\Gamma)$ does not contain any subspace isomorphic to $l_1(\Delta)$ for uncountable Δ while every infinite-dimensional subspace of $l_M(\Gamma)$ contains a subspace isomorphic to l_1.

Proof. Since the natural basis of $l_M(\Gamma)$ is not equivalent to the natural basis of $l_1(\Gamma)$, Corollary 2 implies that $l_M(\Gamma)$ does not contain any subspace isomorphic to $l_1(\Delta)$ for uncountable Δ. It follows from [3] and [4] that every infinite subspace of $l_M(\Gamma)$ contains a subspace isomorphic to l_1.

PROPOSITION 5. *There exists a Banach space U with a symmetric basis $\{u_\gamma\}_{\gamma\in\Gamma}$ such that U does not contain any subspace isomorphic to $c_0(\Delta)$ for uncountable Δ while every infinite-dimensional subspace of U contains a subspace isomorphic to c_0.*

Proof. Let $M(t)$ be an Orlicz function such that $\lim\limits_{t\to 0} tM'(t)/M(t) = \infty$ and $M(1) = 1$. Let U denote the subspace of $l_M(\Gamma)$ generated by characteristic functions $u_\gamma(\sigma)$ of all one point subsets of Γ. Note that $\{u_\gamma\}_{\gamma\in\Gamma}$ is a symmetric basis for U. Since $\{u_\gamma\}_{\gamma\in\Gamma}$ is non-equivalent to the unit vector basis of $c_0(\Gamma)$, it follows from Corollary 1 that U does not contain any subspace isomorphic to $c_0(\Delta)$ for uncountable Δ.

Let X be an infinite-dimensional subspace of U. By [8], there exist sequences $\{a_i\}_{i=1}^\infty$, $\{i_n\}_{n=1}^\infty$, $\{\gamma_i\}_{i=1}^\infty$ such that the space generated by $x_n = \sum\limits_{i=i_n}^{i_{n+1}-1} a_i u_{\gamma_i}$, $\|x_n\| = 1$, $u = 1, 2, \ldots$, is isomorphic to a subspace of X. Let us set

$$M_n(t) = \sum_{i=i_n}^{i_{n+1}-1} M(|a_i|t).$$

Let us observe that $M_n(1) = 1$. Since $tM'(t)\leqslant M(2t)$ for $t\geqslant 0$, we have $M_n'(t)\leqslant 2$ for $0\leqslant t\leqslant 2^{-1}$. Hence without loss of generality (if necessary passing to a subsequence) one may assume that

$$|M_{n+1}(t)-M_n(t)|\leqslant 2^{-n-1} \quad \text{for} \quad n = 1, 2, \ldots \text{ and for } 0\leqslant t\leqslant 2^{-1}. \tag{19}$$

We pick a sequence $\{\tau_j\}_{j=1}^\infty$ so that $tM_n'(t)/M_n(t) > j+2$ for $0 < t\leqslant \tau_j$ $(j = 1, 2, \ldots)$. Then

$$M_n(\tau_j/2)/M_n(\tau_j) = \exp\Big[-\int_{\tau_j/2}^{\tau_j}\frac{M_n'(t)}{M_n(t)}\,dt\Big] < 2^{-j-2} \tag{20}$$

for $j = 1, 2, \ldots$

We shall show that there exist finite mutually disjoint sets of the indices A_j and positive numbers a_j and λ such that

(21) $$\sum_{n\in A_j} M_n(a_j) \geqslant 2^{-1} \quad \text{for} \quad j = 1, 2, \ldots,$$

(22) $$\sum_{j=1}^{k} \sum_{n\in A_j} M(a_j/\lambda) \leqslant 1-2^{-k} \quad \text{for} \quad k = 1, 2, \ldots$$

Let us consider two cases:

(*) $\lim\limits_{n} M_n(t) > 0$ for all $t\in(0, 2^{-1}]$. Let us put $\lambda = 2$, $A_1 = \{n_1\}$ where n_1 is an arbitrary positive integer and $a_1 = 1$. Suppose that for some $k > 1$ the sets $A_1, A_2, \ldots, A_{k-1}$ and positive numbers $a_1, a_2, \ldots, a_{k-1}$ have been defined to satisfy with $\lambda = 2$ the conditions (21) and (22). We pick n_k so that if $n \geqslant n_k$ then

$$n \notin \bigcup_{j=1}^{k-1} A_j$$

and

(23) $$M_{n_k}(\tau_k/2) > 2^{-n_k}.$$

We put $A_k = \{n_k, n_k+1, \ldots, n_k+[1/M_{n_k}(\tau_k)]\}$ and $a_k = \tau_k$. Then (19), (20) and (23) imply (21) for $j = k$ and $\sum\limits_{n\in A_k} M_n(a_k/2) < 2^{-k}$.

(**) $\lim\limits_{n\to\infty} M_n(t_0) = 0$ for some $t_0 > 0$. Then there exists an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of the indices such that

$$M_{n_j}(t_0) < 2^{-j} \quad \text{for} \quad j = 1, 2 \ \ldots$$

We put $\lambda = t_0^{-1}$ $a_j = 1$ and $A_j = \{n_j\}$ for $j = 1, 2, \ldots$

Finally, let

$$y_j = a_j \sum_{n\in A_j} x_n \quad (j = 1, 2, \ldots).$$

It follows from (21) and (22) that

$$\|y_j\| \geqslant 2^{-1}, \quad \Big\|\sum_{j=1}^{k} y_j\Big\| \leqslant \lambda \quad \text{for} \quad j = 1, 2, \ldots; \ k = 1, 2, \ldots$$

Hence the unconditional basis $\{y_j\}_{j=1}^{\infty}$ is equivalent to the natural basis in c_0.

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