[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 167 (1972), pp. 207–226.
[7] M. Rosenblum, *Summability of Fourier Series in $L^p(d\mu)$*, Trans. Amer. Math. Soc. 105 (1962), pp. 32–42.
[8] A. Zygmund, *Trigonometric Series*, Vol. II, New York 1959.

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Self-decomposable probability measures on Banach spaces

by

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Abstract. Self-decomposable probability measures (laws) on a real, separable Banach space E are defined and identified as the limit laws of certain normed sums of independent, uniformly infinitesimal, E-valued random variables. It is shown that self-decomposable measures are infinitely divisible, and a characterization of such measures in terms of their Lévy–Khinchine representations is given on the spaces for which such a representation is known to exist. Finally, a representation theorem due to K. Urbanik for certain measures associated with self-decomposable probability measures on finite-dimensional spaces is generalized to separable Banach spaces.

In § 1 we introduce the notion of a self-decomposable probability measure and obtain a necessary and sufficient condition for a self-decomposable law to be stable in terms of its "component". In § 2 we first show the class of self-decomposable measures on a real, separable Banach space can be identified with the class L ([2], p. 145) on the space. It is then shown that a self-decomposable measure and its "components" are infinitely divisible. This result is of interest since it is not known whether the limit laws of uniformly infinitesimal triangular arrays of random variables with values in a separable Banach space are always infinitely divisible (see [9]). § 3 is devoted to characterizing self-decomposable probability measures on certain Orlicz sequence spaces in terms of their Lévy–Khinchine representations as given in [7]. The paper ends with the extension to the present context of the work of K. Urbanik ([13], [14]) on the representation of self-decomposable probability measures in § 4.

1. Notation and preliminaries. We shall denote by E a real separable Banach space and by R and R^+ the space of real numbers and strictly positive real numbers, respectively, with the usual topology. E^* will

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denote the (topological) dual of E, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between E and E^*. The elements of E and E^* will be denoted by $x, y, z, \ldots$ and of R by $a, b, c, \ldots$ For a probability measure μ on the Borel field $\mathscr{B}(E)$ of E, the *characteristic functional* (ch. f.) μ, denoted by $\hat{\mu}$, is the function on E^* defined by

$$\hat{\mu}(y) = \int_E e^{i\langle x, y\rangle} d\mu(x).$$

It is well known ([4], p. 37) that for a real separable Banach space the function $\hat{\mu}$ uniquely determines the measure μ on E. It is not difficult to see, using the fact that every singleton is compact in the sense of Definition 1.1 (b) below on a separable space, that $\hat{\mu}$ is continuous in the weak-* topology on bounded subsets of E^*. For two probability measures μ and ν on E, we shall denote by $\mu*\nu$ the convolution of μ and ν ([12], p. 56). For any measure μ on E and $a \in R$, $T_a\mu$ is defined to be the measure on E given by $T_a\mu(B) = \mu(a^{-1}B)$ for every $B \in \mathscr{B}(E)$; and for $a = 0$ we define $T_a\mu = \delta_0$, where for $x \in E$ δ_x is the unit point mass at x. We shall call δ_x the probability measure *degenerate at* x.

We need the following definitions.

DEFINITION 1.1. (a) A sequence $\{\mu_n\}$ of probability measure on E is said to *converge weakly* to a probability measure μ on E, denoted $\mu_n \Rightarrow \mu$, if for every bounded, continuous, real-valued function f on E, $\int_E f\,d\mu_n \to \int_E f\,d\mu$.

(b) A sequence $\{\mu_n\}$ of probability measures on E is said to be *compact* if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E$ such that $\mu_n(K_\varepsilon) > 1-\varepsilon$ for every n.

Remark 1.2. This notion of compactness is equivalent to conditional compactness in the topology of weak convergence ([12], p. 47) since E is separable.

The following theorem will be used repeatedly, and for further reference we state it here.

THEOREM 1.3 ([12], p. 58). *Let $\{\lambda_n\}, \{\mu_n\}, \{\nu_n\}$ be three sequences of probability measures on E such that $\lambda_n = \mu_n * \nu_n$ for each n. If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are compact, then so is the sequence $\{\nu_n\}$.*

LEMMA 1.4. *Let $\{\mu_n\}$ and μ be probability measures on E and let $\{a_n\}$ and a be elements of R. If $\mu_n \Rightarrow \mu$ and $a_n \to a$, then $T_{a_n}\mu_n \Rightarrow T_a\mu$.*

Proof. The lemma follows immediately from [1], p. 34.

PROPOSITION 1.5. *Let μ be a probability measure on E. If there exists a number $c > 0$ and a nondegenerate probability measure μ_c such that $\mu = T_c\mu * \mu_c$, then $c < 1$.*

Proof. Note that

$$\hat{\mu}(y) = \hat{\mu}(cy)\hat{\mu}_c(y), \qquad y \in E^*.$$

Fix $y \in E^*$, $y \neq 0$, and define a probability measure ν on R by $\nu(B) = \mu(y^{-1}(B))$ for all $B \in \mathscr{B}(R)$, and let ν_c be defined analogously from μ_c. Then

$$\hat{\nu}(t) = \hat{\mu}(ty) = \hat{\mu}(cty)\hat{\mu}_c(ty) = \hat{\nu}(ct)\hat{\nu}_c(t), \qquad t \in R.$$

Moreover, y can be chosen so that ν_c is nondegenerate. The fact that $c < 1$ now follows from [11], p. 322.

Now we are ready to define self-decomposable probability measures on E, following Loéve [11], p. 322.

DEFINITION 1.6. A probability measure μ on E is said to be *self-decomposable* if for each $0 < c < 1$ there exists a probability measure μ_c on E such that

$$\mu = T_c\mu * \mu_c. \tag{1}$$

The measure μ_c is called the *component* of μ with respect to c.

PROPOSITION 1.7. *If μ is self-decomposable on E, then $\hat{\mu}(y) \neq 0$ for all $y \in E^*$.*

Proof. Since μ is self-decomposable, for each $c \in (0,1)$

$$\hat{\mu}(y) = \hat{\mu}(cy)\hat{\mu}_c(y), \qquad y \in E^*.$$

For $y \in E^*$, $y \neq 0$, let ν and ν_c be defined from y, μ and μ_c as in the proof of Proposition 1.5. Then ν is self-decomposable on R, so by [11], p. 322, $\hat{\nu}(t) \neq 0$ for all $t \in R$. Taking $t = 1$ gives $\hat{\mu}(y) \neq 0$.

DEFINITION 1.8 ([8], p. 136). A probability measure μ on E is said to be *stable* if for each $a, b \in R^+$ there exist $c \in R^+$ and $x \in E$ such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x.$$

PROPOSITION 1.9. *Let μ be a nondegenerate, self-decomposable probability measure on E, and suppose that for each $c \in (0,1)$ the component μ_c of μ is given by*

$$\mu_c = \delta_x * T_{(1-c^\lambda)^{1/\lambda}}\mu \tag{2}$$

for some $x \in E$ and $0 < \lambda \leqslant 2$. Then μ is stable. Conversely, if μ is a stable probability measure on E, then μ is self-decomposable and for each $0 < c < 1$ the component of μ is given by (2).

Proof. For a, $b \in R^+$ and $0 < \lambda \leqslant 2$, let

$$[a, b]_\lambda = (a^\lambda + b^\lambda)^{1/\lambda}.$$

Suppose μ is self-decomposable with component given by (2) for each c. For $a, b \in R^+$, (1) with $c = a[a,b]_\lambda^{-1}$ gives

$$\mu = T_{a[a,b]_\lambda^{-1}}\mu * T_{b[a,b]_\lambda^{-1}}\mu * \delta_x.$$

Consequently,

$$T_a\mu * T_b\mu = T_{[a,b]_\lambda}\mu * \delta_{-[a,b]_\lambda x}, \quad a, b \in R^+. \tag{3}$$

Hence, μ is stable.

Conversely, if μ is stable, then there exist $x \in E$ and $0 < \lambda \leqslant 2$ such that (3) holds ([8], Lemma 2.6). Take $b = (1-c^\lambda)^{1/\lambda}$, where $0 < c < 1$. Then (3) gives

$$T_c\mu * T_b\mu = \mu * \delta_{-x}.$$

It follows that μ is self-decomposable and has component for each c of the form (2).

2. Self-decomposable laws and limit laws. In this section we show that the class of self-decomposable measures coincides with certain limit laws of sums of independent, Banach-space-valued random variables and consists of infinitely divisible measures.

DEFINITION 2.1. A collection μ_{nj}, $j = 1, 2, \ldots, k_n$; $n = 1, 2, \ldots$ of measures on E is called *uniformly infinitesimal* if for every neighborhood of 0 in E,

$$\lim_{n\to\infty} \inf_{1\leqslant j\leqslant k_n} \mu_{nj}(U) = 1.$$

PROPOSITION 2.2. *Let $\{\mu_{nj}\}$ be measures indexed as in Definition 2.1, and consider the following conditions.*

(a) *$\{\mu_{nj}\}$ is a uniformly infinitesimal collection.*

(b) *For each choice of j_n, $1 \leqslant j_n \leqslant k_n$, $n = 1, 2, \ldots$, $\mu_{nj_n} \Rightarrow \delta_0$.*

(c) *For all $a > 0$,*

$$\lim_{n\to\infty} \sup_{1\leqslant j\leqslant k_n} \sup_{\|y\|\leqslant a} |\hat{\mu}_{nj}(y) - 1| = 0.$$

Conditions (a) *and* (b) *are equivalent and imply* (c). *If the collection $\{\mu_{nj}\}$ is compact, then* (c) *implies* (a) *and* (b).

Proof. It is easy to see that (a) and (b) are equivalent, since one can easily verify that a sequence $\{\nu_n\}$ of measures on E converges weakly to δ_0 if and only if $\nu_n(U) \to 1$ for every neighborhood U of 0. The remainder of the assertions follows from [12], p. 171.

DEFINITION 2.3. Denote by $\mathcal{N}_1(E)$ [resp., $\mathcal{N}_2(E)$] the class of probability measures μ on E with the property that there exist sequences $\{x_n\} \subset E$, $\{b_n\} \subset R^+$ and $\{\mu_n\}$ of probability measures on E such that

(a) $\delta_{x_n} * \prod_{k=1}^{n} T_{b_k}\mu_k \Rightarrow \mu$

(of course, the product here represents convolution) and

(b¹) for all $y \in E^*$,

$$\lim_{n\to\infty} \sup_{1\leqslant k\leqslant n} \sup_{0\leqslant a\leqslant 1} |\hat{\mu}_k(b_n a y) - 1| = 0$$

[resp.,

(b²) the collection $T_{b_n}\mu_k$, $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$, is uniformly infinitesimal].

LEMMA 2.4. *If $\mu \in \mathcal{N}_1(E)$, then $\hat{\mu}(y) \neq 0$ for all $y \in E^*$. If μ is nondegenerate, then $b_n \to 0$ and $b_n/b_{n+1} \to 1$ as $n \to \infty$.*

Proof. Since $\mu \in \mathcal{N}_1(E)$, for any $y \in E^*$ $\nu \in \mathcal{N}_1(R)$, where ν is defined from μ and y as in the proof of Proposition 1.5. Hence by [2], p. 147, $\hat{\nu}$ never vanishes on R. Therefore $\hat{\mu}(y) \neq 0$ for any $y \in E^*$. If μ is nondegenerate, then so is ν for suitably chosen y. The second assertion then follows from [2], p. 146.

THEOREM 2.5. *The following are equivalent for a probability measure μ on E.*

(a) $\mu \in \mathcal{N}_1(E)$.

(b) $\mu \in \mathcal{N}_2(E)$.

(c) *μ is self-decomposable.*

Proof. The implication "(b) implies (a)" follows from Proposition 2.2. If μ is degenerate, then choosing $x_n = 0$, $b_n = 1/n$ and $\mu_n = \mu$ for every n shows that $\mu \in \mathcal{N}_1(E)$. So we can assume that μ is nondegenerate.

(c) implies (b). Suppose μ is self-decomposable, and define probability measures $\nu_k = T_k\mu_{c_k}$, $k = 2, 3, \ldots$, where $c_k = (k-1)/k$ and μ_c is the component of μ. For $k = 1$, define $\nu_1 = \mu$. Then

$$\hat{\nu}_k(y) = \frac{\hat{\mu}(ky)}{\hat{\mu}((k-1)y)}, \quad k = 1, 2, \ldots$$

Hence $\prod_{k=1}^{n} T_{1/n}\nu_k = \mu$. So take $x_n = 0$, $b_n = 1/n$ for $n = 1, 2, \ldots$ and for each $a > 0$ note that

$$\sup_{1\leqslant k\leqslant n} \sup_{\|y\|\leqslant a} \left| \frac{\hat{\mu}\left(\frac{k}{n}y\right) - \hat{\mu}\left(\frac{k-1}{n}y\right)}{\hat{\mu}\left(\frac{k-1}{n}y\right)} \right|^2 \leqslant \frac{2\sup\limits_{\|y\|\leqslant a} |1 - \hat{\mu}(y/n)|}{\inf\limits_{1\leqslant k\leqslant n} \inf\limits_{\|y\|\leqslant a} \left|\hat{\mu}\left(\frac{k-1}{n}y\right)\right|^2}$$

(see, for example, [11], p. 195). Since $\hat{\mu}$ is weak-* continuous on bounded sets, we conclude from Lemma 2.4 that

$$\lim_{n\to\infty}\inf_{1\leqslant k\leqslant n}\inf_{\|y\|\leqslant a}\left|\hat{\mu}\left(\frac{k-1}{n}y\right)\right| \geqslant \inf_{\|y\|\leqslant a}|\hat{\mu}(y)| > 0.$$

Thus, since $\left\|\frac{y}{n}\right\|\to 0$ uniformly for $\|y\|\leqslant a$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\sup_{1\leqslant k\leqslant n}\sup_{\|y\|\leqslant a}\left|\hat{\nu}_k\left(\frac{y}{n}\right)-1\right| = 0.$$

Hence, by Proposition 2.2, it suffices to show the collection $T_{1/n}\nu_k$, $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$, is compact. To see this, first note that (1), Lemma 1.4 and Theorem 1.3 imply that the sequence $\{\mu_{c_k}\}$ is compact, so given any $\varepsilon > 0$ there is a compact set K in E such that $\mu_{c_k}(K) > 1-\varepsilon$ for all k. Let $\tilde{K} = \bigcup_{0\leqslant a\leqslant 1} aK$. Then $\tilde{K}$ is also compact, and we have

$$T_a\mu_{c_k}(\tilde{K}) > 1-\varepsilon$$

for all $0\leqslant a\leqslant 1$ and all k. Since

$$T_{1/n}\nu_k = T_{k/n}\mu_{c_k},$$

the collection $\{T_{1/n}\nu_k\}$ is compact.

(a) implies (c). Let $\mu\in\mathcal{N}_1(E)$ and let $\{x_n\}$, $\{b_n\}$ and $\{\mu_n\}$ satisfy (a) and (b¹) of Definition 2.3. Then by Lemma 2.4, $b_n\to 0$ and $b_n/b_{n+1}\to 1$. Consequently, given a $c\in(0,1)$, we can correspond to every integer n an integer $m_n < n$ such that $b_n/b_{m_n}\to c$, $m_n\to\infty$ and $n-m_n\to\infty$ as $n\to\infty$ ([11], p. 323). Let $a_n = b_n/b_{m_n}$, $n = 1, 2, \ldots$ and note that

$$\nu_n = T_{a_n}\nu_{m_n}*\lambda_{m_n},$$

where λ_{m_n} is the probability measure given by

$$\delta_{x_n-a_nx_{m_n}}*\prod_{k=m_n+1}^{n}T_{b_n}\mu_k.$$

Since $\nu_n\Rightarrow\mu$, Lemma 1.4 gives

$$T_{a_n}\nu_{m_n}\Rightarrow T_c\mu;$$

and hence by Theorem 1.3, $\{\lambda_{m_n}\}$ is compact. In view of Lemma 2.4,

$$\hat{\lambda}_{m_n}(y)\to\frac{\hat{\mu}(y)}{\hat{\mu}(cy)}\quad\text{for all } y\in E^*;$$

and hence by [12], p. 153,

$$\nu_{m_n}\Rightarrow\mu_c, \tag{4}$$

where μ_c is given by

$$\hat{\mu}_c(y) = \hat{\mu}(y)/\hat{\mu}(cy),\qquad y\in E^*.$$

Thus $\mu = T_c\mu*\mu_c$. This completes the proof of the theorem.

From this theorem we conclude that the class L of probability measures and the class of self-decomposable probability measures on a real separable Banach space are the same.

THEOREM 2.6. *Let E be a real separable Banach space and μ be a self-decomposable probability measure on E. Then μ and for each c $(0 < c < 1)$ the component μ_c are infinitely divisible (i.d.).*

Proof. Let $0 < c < 1$. From (1) it follows by iteration that for each n,

$$\mu = \mu_c*T_c\mu_c*T_{c^2}\mu_c*\ldots*T_{c^{n-1}}\mu_c*T_{c^n}\mu = \lambda_{n,c}*T_{c^n}\mu$$

where

$$\lambda_{n,c} = \mu_c*T_c\mu_c*T_{c^2}\mu_c*\ldots*T_{c^{n-1}}\mu_c.$$

Since $c^n\to 0$ as $n\to\infty$, it follows from Lemma 1.4 that

$$T_{c^n}\mu\Rightarrow\delta_0.$$

Consequently by Theorem 1.3 and [12], p. 153,

$$\lambda_{n,c}\Rightarrow\mu\quad\text{as}\quad n\to\infty. \tag{5}$$

Let m be a positive integer, and for $n = 1, 2, \ldots$ let

$$\nu_{n,c} = \mu_c*T_{c^m}\mu_c*T_{c^{2m}}\mu_c*\ldots*T_{c^{(n-1)m}}\mu_c.$$

Then

$$\nu_{n,c}*T_c\nu_{n,c}*T_{c^2}\nu_{n,c}*\ldots*T_{c^{m-1}}\nu_{n,c} = \mu_c*T_c\mu_c*\ldots*T_{c^{nm-1}}\mu_c = \lambda_{nm,c}, \tag{6}$$

and the right hand side in (6) converges weakly to μ as $n\to\infty$ by (5). Consequently by [12], p. 59, $\{\nu_{n,c}\}$ is shift-compact, i.e., there exists a sequence $\{x_{n,c}\}\subset E$ such that $\{\nu_{n,c}*\delta_{x_{n,c}}\}$ is compact. Hence, by passing to a subsequence if necessary, we may assume that $\nu_{n,c}*\delta_{x_{n,c}}\Rightarrow\Lambda_{m,c}$, a probability measure on E, as $n\to\infty$. From (5) and (6) it follows that

$$\Lambda_{m,c}*T_c\Lambda_{m,c}*T_{c^2}\Lambda_{m,c}*\ldots*T_{c^{m-1}}\Lambda_{m,c}*\delta_{y_{mc}} = \mu, \tag{7}$$

where

$$y_{m,c} = -\frac{1-c^m}{1-c}\lim_{n\to\infty}x_{n,c}.$$

Let $\{c_k\}$ be a sequence in $(0,1)$ converging to 1. Since (7) holds with $c = c_k$ for all k, $\{\Lambda_{m,c_k}\}$ is shift-compact, and we can argue as above and apply Lemma 1.4 to conclude that there is a probability measure Λ_m

on E and an element $y_m \in E$ such that $\Lambda_m^m * \delta_{y_m} = \mu$. Since m was an arbitrary positive integer, μ is i.d.

To prove that μ_c is infinitely divisible for each $c \in (0, 1)$, consider a Hilbert space H containing E such that $\{B \cap E: B \in \mathscr{B}(H)\} = \mathscr{B}(E)$ ([6], p. 355). We can regard μ and μ_c as measures on H. Then (1) holds on H, so μ_c is i.d. by Theorem 2.5, (4) and [12], p. 199. Hence for each m there exists a probability measure ν_m on H such that

$$\nu_m^m = \mu_c \quad \text{on} \quad H. \tag{8}$$

The proof will be complete once we show that ν_m is concentrated on E.

Since μ is infinitely divisible on E, there exists a probability measure λ_m on E such that

$$\lambda_m^m = \mu \tag{9}$$

on E. We can regard λ_m as a measure on H, and then (9) holds on H as well. By (1), (8) and (9),

$$\lambda_m^m = T_c \lambda_m^m * \nu_m^m$$

on H. Hence

$$[\hat{\lambda}_m(y)]^m = [\hat{\lambda}_m(cy)]^m [\hat{\nu}_m(y)]^m = [\hat{\lambda}_m(cy)\hat{\nu}_m(y)]^m, \qquad y \in H. \tag{10}$$

By Proposition 1.7 and the fact that both sides of (10) are continuous and equal to 1 at $y = 0$, it follows that

$$\hat{\lambda}_m(y) = \hat{\lambda}_m(cy)\hat{\nu}_m(y), \qquad y \in H.$$

Consequently, $\lambda_m = T_c \lambda_m * \nu_m$ on H. Since λ_m and $T_c \lambda_m$ are concentrated on E, it follows that ν_m is also concentrated on E. This completes the proof of the theorem.

Remark 2.7. Theorems 2.5 and 2.6 generalize the classical results about self-decomposable laws on R ([11], p. 323, Theorem 23.3A and Corollary, [2], pp. 145-149). We note that these theorems are very easy to handle in the finite-dimensional case because of the availability of the powerful Lévy continuity theorem. However, in the case of an infinite-dimensional Banach space, no complete analogue of the Lévy continuity theorem is available and hence the methods used here are founded on the properties of characteristic functionals. The fact that μ and μ_c are i.d. will enable us below to use the uniqueness of the Lévy–Khinchine representation [7], [15] to obtain a condition relating the Lévy measures for μ and μ_c on certain Orlicz spaces.

3. The Lévy-Khinchine representation on certain Orlicz spaces. The Lévy–Khinchine representation for the characteristic functionals of probability measures on certain Orlicz sequence spaces has been studied by J. Kuelbs and V. Mandrekar [7]. In this section we shall obtain necessary and sufficient conditions for an i.d. probability measure μ on such an Orlicz space to be self-decomposable in terms of the representing measure for μ.

Before examining measures on more general spaces, we shall first characterize self-decomposable probability measures on a Hilbert space. Let H be a real, separable Hilbert space, which we shall also identify with H^* in the usual way. Recall that a complex-valued function φ on H is the ch.f. of and i.d. probability measure on H if and only if it can be represented in the form

$$\varphi(y) = \exp\Big[i\langle x_0, y\rangle - \frac{1}{2}\langle Dy, y\rangle + \int_H K(x, y)\, dM(x)\Big], \qquad y \in H, \tag{11}$$

where $x_0 \in H$,

$$K(x, y) = e^{i\langle x, y\rangle} - 1 - \frac{i\langle x, y\rangle}{1 + \|x\|^2}, \tag{12}$$

D is an S-operator, M is a σ-finite measure on H finite on the complement of every neighborhood of zero in H, and $\int_{\|x\| \leqslant 1} \|x\|^2 dM(x) < \infty$ ([15], p. 227; [12], pp. 181-182). Moreover, the representation (11) is unique.

THEOREM 3.1. *A complex-valued function φ on the real, separable Hilbert space H is the ch.f. of a self-decomposable probability measure on H if and only if φ can be represented in the form* (11) *and for each $c \in (0, 1)$ the measure M which appears in* (11) *satisfies $M = T_c M + M_c$ for some* (*nonnegative*) *measure M_c on H.*

Proof. Suppose φ is the ch.f. of a self-decomposable probability measure on H. Then for each $c \in (0, 1)$ there exists a ch.f. φ_c on H such that

$$\varphi(y) = \varphi(cy)\varphi_c(y). \tag{13}$$

By Theorem 2.6 φ and φ_c are the ch.f.'s of i.d. measures. Fix $c \in (0, 1)$, and let the representations (11) for φ and φ_c be given in terms of elements x_0 and x_c of H, S-operators D and D_c and measures M and M_c, respectively. Note that

$$K(x, cy) = e^{i\langle cx, y\rangle} - 1 - \frac{i\langle cx, y\rangle}{1 + \|cx\|^2} + \frac{i\langle cx, y\rangle}{1 + \|cx\|^2} - \frac{i\langle x, cy\rangle}{1 + \|x\|^2}$$

$$= K(cx, y) + \frac{i\langle cx, y\rangle \|x\|^2(1 - c^2)}{(1 + \|x\|^2)(1 + c^2\|x\|^2)}$$

and that for all $y \in H$,

$$\int \frac{\langle x, y\rangle \|x\|^2}{(1+\|x\|^2)(1+c^2\|x\|^2)}\, dM(x) \leqslant \|y\| \int_{\|x\|\leqslant 1} \|x\|^2 dM(x) + \|y\| \int_{\|x\|>1} dM(x) < \infty, \tag{14}$$

so the left-hand side of (14) has the form $\langle \bar{x}, y\rangle$ for some $\bar{x} \in H$. Then (11) gives

$$\varphi(cy) = \exp\Big[ic\langle x_0 + (1-c^2)\bar{x}, y\rangle - \frac{1}{2}c^2\langle Dy, y\rangle + \int_H K(x, y)\, dT_c M(x)\Big]. \tag{15}$$

It follows from (13) that the representation (11) for φ can be given in terms of the element

$$cx_0 + c(1-c^2)\bar{x} + x_c,$$

the S-operator $c^2D + D_c$ and the measure $T_cM + M_c$. By the uniqueness of (11) we have $M = T_cM + M_c$.

To prove the sufficiency, let μ be the probability measure on H whose ch.f. is φ and for $c \in (0, 1)$ let μ_c have ch.f. φ_c represented as in (11) in terms of the point

$$(1-c)x_0 - c(1-c^2)\bar{x},$$

where $\bar{x}$ is determined from (14) as above, by the S-operator $(1-c^2)D$, and by the measure M_c. Then (15) implies (13) holds, so $\mu = T_c\mu * \mu_c$. This completes the proof of the theorem.

Remarks 3.2. (1) The proof of Theorem 3.1 shows that if μ is self-decomposable on H with Lévy–Khinchine measure M, then the component μ_c has Lévy–Khinchine measure $M_c = M - T_cM$.

(2) The main result of Jajte in [5] can be obtained as a corollary of the above theorem in the following manner.

Let μ be a stable probability measure on H. Then μ is self-decomposable on H by Proposition 1.9, and it follows from (2) that for each $c \in (0, 1)$,

$$M_c = T_{(1-c^\lambda)^{1/\lambda}} M.$$

Hence Theorem 3.1 gives $M = T_cM + T_{(1-c^\lambda)^{1/\lambda}}M$. Now, by the same argument as in [8], Lemma 3.6, we get $T_cM = c^\lambda M$ for every $c \in (0, 1)$, whence for all $c \in R^+$.

We now turn to the consideration of self-decomposable probability measures on Orlicz sequence spaces. For the construction and properties of Orlicz spaces in general and Orlicz sequence spaces in particular we refer the reader to [16] and to [3] and [10], respectively.

Definition 3.3. The function a which we shall now fix for the remainder of this section is to have the following properties.

(a) a is defined on $[0, \infty)$ into $[0, \infty)$.

(b) $a(0) = 0$ and $a(R^+) \subset R^+$.

(c) a is convex and strictly increasing on $[0, \infty)$.

(d) $a(2s) \leqslant Ma(s)$ for all $s \in R^+$ and some positive constant M.

(e) $\int_R a(u^2)\, d\nu(u) \leqslant Ca\big[\int_R u^2 d\nu(u)\big]$ for all Gaussian measures ν on R with mean zero, where C is some constant.

Let a_c be the function complementary to a in the sense of Young ([16], p. 77), let $\gamma(t) = a(t^2)$, $t \in [0, \infty)$, and let γ_c be the complementary function to γ. Denote by S_a, S_{a_c}, S_γ and S_{γ_c} the Orlicz sequence spaces corresponding to the functions a, a_c, γ, γ_c, respectively, and by Σ_{a_c} the subspace of S_{a_c} of all sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{i=1}^\infty a_c(rx_i) < \infty$ for all $r > 0$. By Theorem 3.1 we may assume $S_\gamma \neq l_2$. For each λ in the positive cone of Σ_{a_c} whose norm is less than or equal to one-half, define $\|x\|_\lambda^2 = \sum_1^\infty \lambda_i x_i^2$. The space of sequences with the property that $\|x\|_\lambda < \infty$ will be denoted by H_λ. Obviously $S_\gamma \subset H_\lambda$ by Young's inequality ([16], p. 77). In fact, S_γ and all its Borel measurable subsets are measurable subsets of H_λ ([7], Lemma 7.1). As done in [7], if $S_\gamma = l_2$ we assume $a(t) = t$, $t \in R^+$, and define $\|x\|_\lambda = \|x\|_\gamma$.

It was shown in [7], Theorem 7.2, that a probability measure μ on S_γ is infinitely divisible if and only if for each λ in the positive cone of Σ_{a_c} we can write

$$\hat{\mu}(y) = \exp\Big[i\langle x_0, y\rangle - \tfrac{1}{2}T(y, y) + \int_U K_\lambda(x, y)\, dM(x) + \int_{S_\gamma \setminus U} K_\gamma(x, y)\, dM(x)\Big], \quad y \in S_\gamma^* = S_{\gamma_c}, \tag{16}$$

where $x_0 \in S_\gamma$, T is an a-operator on S_{γ_c} ([7], p. 118),

$$U = \Big\{x \in S_\gamma\colon \sum_{i=1}^\infty \gamma(|x_i|) \leqslant 1\Big\},$$

K_λ and K_γ are given by (12) with the norms being respectively the λ-norm and γ-norm, and M is a measure on S_γ finite on the complement of every neighborhood of 0 in S_γ and satisfying

$$\sum_{i=1}^\infty a\Big(\int_U x_i^2\, dM(x)\Big) < \infty.$$

The representation (16) is unique for any given λ. (Notice that if $S_\gamma = l_2$ (16) reduces to (11) and the conditions for M given here as identical to those given prior to Theorem 3.1.)

Remark 3.4. In [7] it was assumed that the function α_c also satisfies the "Δ_2-condition" (d). However, by considering only elements λ in the positive cone of Σ_{α_c}, not of S_{α_c}, and using results appearing in [3], [10] we can avoid this added restriction, thus apparently enlarging the collection of Orlicz spaces S_γ for which a representation of the form (16) exists for all i.d. probability measures.

THEOREM 3.5. *A probability measure μ on S_γ is self-decomposable if and only if for each fixed λ as above $\hat{\mu}$ has the form* (16) *and for each $c \in (0, 1)$ the Lévy-Khinchine measure M appearing in* (16) *satisfies $M = T_c M + M_c$ for some measure M_c on S_γ. If μ is self-decomposable, then the component μ_c has Lévy-Khinchine measure $M_c = M - T_c M$.*

Proof. First recall that every measure on S_γ can be considered as a measure on H_λ because

$$\{B \cap S_\gamma \colon B \in \mathscr{B}(H_\lambda)\} = \mathscr{B}(S_\gamma)$$

([7], pp. 136-137). A probability measure μ on S_γ is i.d. [resp., self-decomposable] if and only if the extension of μ to H_λ is i.d. [resp., self-decomposable] (cf. the proof of Theorem 2.6). Moreover, if μ is i.d. on S_γ then the measures M which appear in (11) and (16) coincide. To see this, recall that we can write

$$\mu = \nu * \beta \tag{17}$$

where β is the Gaussian part of μ and

$$e(F_n) * \delta_{x_n} \Rightarrow \nu \tag{18}$$

for some increasing sequence $\{F_n\}$ of finite measures on S_γ and some $\{x_n\} \subset S_\gamma$ [15]. (Here $e(F_n)$ is the usual normalized exponential of F_n.) Then β is Gaussian as a probability measure on H_λ ([7], pp. 140-141). Now, (17) and (18) are valid on H_λ as well as on S_γ, since the embedding of S_γ in H_λ is continuous. It follows from the uniqueness of the representations (11) and (16) and from [7], Lemma 7.4, that in both cases the measures M must be given by $M = \lim_{n\to\infty} F_n$. Hence our theorem follows from Theorem 3.1.

4. The Urbanik representation.

We shall extend to the spaces S_γ defined in § 3 the representation given in [14] for the ch.f.'s of self-decomposable probability measures on a finite-dimensional space. Our development parallels that of [14], but some modifications are necessary to deal with the infinite-dimensional case. Before proceeding, we shall modify the representation (16) slightly by replacing the measure M with a finite measure as follows.

Let M be a measure on S_γ satisfying the conditions stated prior to Theorem 3.4 with reference to the representation (16), and for fixed λ as in § 3 define a measure N on S_γ by

$$N(B) = \int_B \frac{\|x\|_\lambda^2}{1+\|x\|_\lambda^2}\, dM(x), \qquad B \in \mathscr{B}(S_\gamma).$$

Then, since U contains a neighborhood of 0 in S_γ ([7], p. 143), N is a finite measure on S_γ. The following fact is obtained by direct computation.

PROPOSITION 4.1. *Let M and N be as above. Then M satisfies $M = T_c M + M_c$ for some measure M_c on S_γ if and only if there is a measure N_c on S_γ such that for all $B \in \mathscr{B}(S_\gamma)$,*

$$N_c(B) = N(B) - c^2 \int_{c^{-1}B} \frac{1+\|x\|_\lambda^2}{1+c^2\|x\|_\lambda^2}\, dN(x). \tag{19}$$

Theorem 3.4 and Proposition 4.1 imply that a probability measure μ on S_γ is self-decomposable if and only if it can be represented as in (16) — with the functions $K(x, y)$ suitably modified — in terms of an element $x_0 \in S_\gamma$, an a-operator T, and a measure N such that for each $c \in (0, 1)$ there is a measure N_c on S_γ given by (19). We proceed to examine, in a more general context, the class of all measures N with this property. Recall that the spaces S_γ are all separable conjugate spaces [3], [10].

DEFINITION 4.2. Let E be a separable, conjugate Banach space, and let U and S denote the closed unit ball and unit sphere of E, respectively. Let $[0, \infty]$ be the usual compactification of R^+, and set $K = U \times [0, \infty]$. If U is endowed with the relative weak-* topology of E, then K becomes a compact metric space. Define $h\colon E\setminus\{0\} \to K$ by $h(x) = (x/\|x\|, \|x\|)$. If U were given the relative norm topology of E, then h would be a homeomorphism of $E\setminus\{0\}$ onto $S \times R^+$. Thus, since it is well known that the Borel fields on E with respect to the norm topology and with respect to the weak-* topology concide, h and its inverse on $S \times R^+ \subset K$ are measurable. For $a \in [0, \infty]$ and $x = (w, r) \in K$, set $\|x\| = r$ and $ax = (w, ar)$. Let $\boldsymbol{N}(E)$ and $\boldsymbol{N}(K)$ denote the set of all finite Borel measures N on E and K, respectively, such that for each $c \in (0, 1)$ there is a measure N_c satisfying (19) with the λ-norm being replaced by the norm on E. Here the integrand on K is assumed to have its limiting value c^{-2} when $x \in U \times \{\infty\}$. Let $\boldsymbol{P}(E)$ and $\boldsymbol{P}(K)$ denote the set of probability measures in $\boldsymbol{N}(E)$ and $\boldsymbol{N}(K)$, respectively, and denote by $\boldsymbol{N}^0(E)$ the elements of $\boldsymbol{N}(E)$ concentrated on $E\setminus\{0\}$ and by $\boldsymbol{P}^0(E)$ the set $\boldsymbol{P}(E) \cap \boldsymbol{N}^0(E)$. Clearly all these sets of measures are convex. Recall also that the space of all probability measures on K is compact and metrizable in the topology of weak convergence ([12], pp. 45-46).

LEMMA 4.3. *The set $\boldsymbol{P}(K)$ is compact.*

Proof. It is only necessary to show $\boldsymbol{P}(K)$ is closed. For $N \in \boldsymbol{P}(K)$, $c \in (0, 1)$ and $f \in C(K)$, (19) gives

$$\int_K f\,dN_c = \int_K f\,dN - c^2 \int_K f(cx)\frac{1+\|x\|^2}{1+c^2\|x\|^2}\,dN(x). \tag{20}$$

Since the integrand in the latter integral of (20) is continuous on K, it follows that, if $\{N^{(n)}\} \subset \boldsymbol{P}(K)$ converges weakly to a probability measure N on K and if N_c is determined by (20), then

$$\int_K f\,dN_c^{(n)} \to \int_K f\,dN_c \quad \text{for all } f \in C(K);$$

whence N_c is a nonnegative measure, so $N \in \boldsymbol{P}(K)$.

LEMMA 4.4. *There is a one-to-one mapping of K onto the set $\boldsymbol{P}(K)^e$ of extreme points of $\boldsymbol{P}(K)$.*

Proof. For any Borel set B of U, the sets $B \times \{0\}$, $B \times \{\infty\}$ and $B \times R^+$ are invariant under multiplication by elements of R^+. Hence if $N \in \boldsymbol{N}(K)$ the restriction of N to any of these sets is again in $\boldsymbol{N}(K)$. It follows that every extreme point of $\boldsymbol{P}(K)$ must be either degenerate at x for some x with $\|x\| = 0$ or ∞ or concentrated on $\{x\} \times R^+$ for some $x \in U$. Moreover, a measure concentrated on $\{x\} \times R^+$ is in $\boldsymbol{P}(K)^e$ if and only if, considered as a measure on R, it is in $\boldsymbol{P}(R)^e$. In [13] Urbanik showed that $\boldsymbol{P}(R)^e$ consists of δ_0 and all measures λ_a given by

$$\lambda_a(B) = \frac{2}{\log(1+a^2)} \int_{B \cap I_a} \frac{|t|}{1+t^2}\,dt, \qquad B \in \mathscr{B}(R)$$

for $a \in R$, where $I_a = (0, a)$ or $(a, 0)$ according as $a > 0$ or $a < 0$. Thus, for $u \in K$, let $\lambda_u = \delta_u$ if $\|u\| = 0$ or ∞, and for $u = (w, r)$ with $r \in R^+$ define λ_u by

$$\lambda_u(B) = \frac{2}{\log(1+\|u\|^2)} \int_B \frac{\|x\|}{1+\|x\|^2}\,dm_u(x), \qquad B \in \mathscr{B}(K), \tag{21}$$

where m_u is Lebesgue measure on the interval $\{w\} \times I_r$. The correspondence $u \to \lambda_u$ is the desired mapping of K onto $\boldsymbol{P}(K)^e$.

LEMMA 4.5. *The mapping $u \to \lambda_u$ is a homeomorphism.*

Proof. It is to be shown that if $\{u_n\} \subset K$ and $u \in K$, then $u_n \to u$ if and only if $\lambda_{u_n} \Rightarrow \lambda_u$. Using the definition of the λ_u and considering individually the three cases (a) $\|u\| = 0$, (b) $\|u\| = \infty$ and (c) $\|u\| \in R^+$, this can be accomplished. The details will be omitted as they may be easily supplied by the reader.

Via the mapping h, let us consider the measures λ_u for $u \in S \times R^+$ as measures on E as well as on K.

THEOREM 4.6. *Let E be a separable, conjugate Banach space. A measure N on E is in $\boldsymbol{N}^0(E)$ if and only if there is a finite measure ω on E with $\omega(\{0\}) = 0$ such that*

$$\int_E f\,dN = \int_E \int_E f(x)\,d\lambda_u(x)\,d\omega(u) \tag{22}$$

for every bounded measurable function f on E.

Proof. To prove the necessity it suffices to consider $N \in \boldsymbol{P}^0(E)$. Consider such an N first as an element of $\boldsymbol{P}(K)$, via h. By a well-known corollary of the Krein-Milman Theorem there is a probability measure ω on $\boldsymbol{P}(K)^e$ which "represents" N in the sense that

$$\int_K f\,dN = \int_{\boldsymbol{P}(K)^e} \int_K f(x)\,d\lambda(x)\,d\omega(\lambda) \tag{23}$$

for every $f \in C(K)$. It is easy to see that (23) holds for all bounded measurable functions on K, and by Lemma 4.5 we may assume ω is defined on K. Moreover, it follows from (23) with $\boldsymbol{P}(K)^e$ replaced by K that N is concentrated on $S \times R^+$ if and only if ω is. Thus, since h is measurable and so is its inverse on $S \times R^+$, the desired representation (22) holds. The sufficiency is clear.

COROLLARY 4.7. *Let S_γ and $\lambda \in S_{a_c}$ be as in § 3 and let μ be a probability measure on S_γ. Then μ is self-decomposable if and only if $\hat{\mu}$ has the form*

$$\hat{\mu}(y) = \exp\left[i\langle x_0, y\rangle - \frac{1}{2}T(y, y) + \int_{S_\gamma} \frac{\Phi(x, y)}{\log(1+\|x\|_\lambda^2)}\,d\omega(x)\right], \qquad y \in S_{\gamma_c};$$

where x_0 and T are as in § 3; ω is a finite measure on S_γ with $\omega(\{0\}) = 0$; and

$$\Phi(x, y) = \int_0^{\langle x, y\rangle} \frac{e^{it}-1}{t}\,dt - i\langle x, y\rangle\left[\frac{\tan^{-1}\|\bar{x}\|_\lambda}{\|x\|_\lambda} + \frac{\tan^{-1}\|x\|_\gamma - \tan^{-1}\|\bar{x}\|_\gamma}{\|x\|_\gamma}\right],$$

$\bar{x}$ for $x \in E \setminus \{0\}$ being given by $\bar{x} = a_x x$ where U is as in § 3 and $a_x = \sup\{a:$ $0 < a \leqslant 1,\ ax \in U\}$.

Proof. By Theorem 3.5, Proposition 4.1 and Definition 4.2, μ is self-decomposable if and only if there are x_0 and T as in § 3 and a measure $N \in \boldsymbol{N}^0(S_\gamma)$ such that

$$\hat{\mu}(y) = \exp\left[i\langle x_0, y\rangle - \tfrac{1}{2}T(y, y) + \int_{S_\gamma} \tilde{K}(x, y)\,dN(x)\right], \qquad y \in S_{\gamma_c},$$

where

$$\tilde{K}(x, y) = K(x, y)\frac{1+\|x\|_\lambda^2}{\|x\|_\lambda^2},$$

$K(x, y)$ denoting $K_\lambda(x, y)$ or $K_\gamma(x, y)$ according as $x \in U$ or $x \notin U$. If N is considered as a measure on H_λ and ω on H_λ is twice the measure appearing in Theorem 4.6, then it is easy to see that ω is concentrated on S_γ and that the representation (22) holds for all $f \in L^1(N)$. If for each $y \in S_{\gamma_0}$ we set $f(x) = \tilde{K}(x, y)$ in (22) we obtain, after some computation, the asserted representation of $\hat{\mu}$.

Setting $S_\gamma = l_2 = H_\lambda$ in Corollary 4.7 we obtain the representation of self-decomposable probability measures on Hilbert space which is the exact analog of the finite-dimensional case.

COROLLARY 4.8. *Let μ be a probability measure on the real, separable Hilbert space H. Then μ is self-decomposable if and only if*

$$\hat{\mu}(y) = \exp\left[i\langle x_0, y\rangle - \tfrac{1}{2}\langle Dy, y\rangle + \int_H \frac{\Phi(x, y)}{\log(1+\|x\|^2)}\, d\omega(x)\right], \qquad y \in H,$$

where x_0 and D are as in § 3, ω is a finite measure on H with $\omega(\{0\}) = 0$, and

$$\Phi(x, y) = \int_0^{\langle x, y\rangle} \frac{e^{it}-1}{t}\, dt - \frac{i\langle x, y\rangle}{\|x\|}\tan^{-1}\|x\|.$$

References

[1] P. Billingsley, *Convergence of Probability Measures*, New York 1968.

[2] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Revised ed., transl. from Russian by K. L. Chung, Reading, Mass., 1968.

[3] Y. Gribanov, *On the theory of l_M-spaces* (Russian), Učen. Zap. Kazansk. Un-ta 117 (1957), pp. 62–65.

[4] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), pp. 35–48.

[5] R. Jajte, *On stable distributions in Hilbert space*, Studia Math. 30 (1964), pp. 63–71.

[6] J. Kuelbs, *Gaussian measures on a Banach Space*, J. Functional Anal. 5 (1970), pp. 354–367.

[7] J. Kuelbs and V. Mandrekar, *Harmonic analysis on F-spaces with a basis*, Trans. Amer. Math. Soc. 169 (1972), pp. 113–152.

[8] A. Kumar and V. Mandrekar, *Stable probability measures on Banach spaces*, Studia Math. 42 (1972), pp. 133–144.

[9] L. LeCam, *Remarques sur le théorème limit central dans les espaces localement convex*, Les Probabilités sur les Structures Algébriques, Colloques Internationaux du C. N. R. S., France, 1970, pp. 233–249.

[10] K. Lindberg, *On subspaces of Orlicz sequence spaces*, Studia Math. 45 (1973), pp. 119–146.

[11] M. Loève, *Probability Theory*, 3rd. ed., Princeton, N. J., 1963.

[12] K. R. Parthasarathy, *Probability Measures in Metric Spaces*, New York 1967.

[13] K. Urbanik, *A representation of self-decomposable distributions*, Bull. Acad. Polon. Sci. 16 (1968), pp. 196–204.

[14] — *Self-decomposable probability distributions on R^m*, Zast. Mat.–Applicationes Math. (Steinhaus Jubilee Volume) 10 (1969), pp. 91–97.

[15] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhyā 24 (1962), pp. 213–238.

[16] A. C. Zaanen, *Linear Analysis*, New York 1953.

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