by $a-\lambda e$. Then I consists of joint topological divisors of zero, so by Theorem 2 there is a $\varphi \in \mathscr{L}(A)$ with $I \subset \operatorname{Ker}\varphi$. In particular, $|\hat{a}(\varphi)| = |\lambda| = 1$ and it follows that $\varphi \in U$. This proves that $U \cap \mathscr{L}(A)$ is nonempty.

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A moment theory approach to the Riesz theorem on the conjugate function with general measures

by

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Abstract. A measure $\mu \geqslant 0$ belongs to the class $\mathfrak{R}_M$ if it satisfies the Riesz inequality

$$\int_0^{2\pi} |\check{f}(t)|^2 d\mu \leqslant M \int_0^{2\pi} |f(t)|^2 d\mu, \quad \forall f \in L^2(\mu),$$

where $\check{f}$ is (essentially) the conjugate functions of f, with fixed constant M.

Applying a moment theory approach we introduce (and give explicit formulae for) the canonical extremal (simple) elements for $\mathfrak{R}_M$, which prove to be given by $R(t)\,dt$, with $R(t)$ certain rational functions, making up a determinant set for $\mathfrak{R}_M$. These particular measures are for the class $\mathfrak{R}_M$ what the Dirac measures are for the class of all measures. Among the possible applications of this parallel construction is the analogue for the $\mathfrak{R}_M$-simple measures of Bochner's theorem of decomposition on Dirac measures.

1. Introduction. We say that a measure $\mu \geqslant 0$ satisfies the Riesz inequality in L^p if

$$(1.1)\qquad \int |\tilde{F}(t)|^p d\mu \leqslant M \int |F(t)|^p d\mu, \quad \forall F \in L^p(\mu),$$

where $\tilde{F}$ is the conjugate function (or the Hilbert transform) of F, and μ is defined in $(0, 2\pi)$ (or in $\boldsymbol{R}^n$). Here we consider the simplest case when $p = 2$ and μ acts in the unit circle; the essential part of this exposition can be extended to $L^p(0, 2\pi)$ and $L^p(\boldsymbol{R})$ if p is even. The generalization for all p and $\boldsymbol{R}^n$ will be considered elsewhere.

Let $\mathscr{E}_N$ be the set of all complex trigonometric polynomials of the form

$$(1.2)\qquad F(t) = \sum_{-N}^{N} a_n e_n(t), \quad e_n(t) = e^{int},$$

$\mathscr{E} = \bigcup_{N=0}^{\infty} \mathscr{E}_N$, and for each F of the form (1.2) let us set

$$(1.2a)\qquad \check{F}(t) = \sum_{n=0}^{N} a_n e_n(t),$$

so that $2i\breve{F} = ia_0 + iF - \tilde{F}$, where $\tilde{F}$ is the conjugate function of F, and therefore (1.1) is equivalent to

$$\int |\breve{F}|^p d\mu \leqslant M \int |F|^p d\mu, \quad F \in \mathscr{E}, \tag{1.1a}$$

with a different constant M.

Let $\mathscr{M}$ be the set of all measures $\mu \geqslant 0$ defined in $(0, 2\pi)$ satisfying

$$\mu(1) = \int_0^{2\pi} d\mu = 1 \quad (\text{where } \mu(F) = \int_0^{2\pi} F\, d\mu). \tag{1.3}$$

$\mathscr{M}$ is a convex compact set in the weak topology ($\mu_a \to \mu$ weakly if $\mu_a(F) \to \mu(F)$ for all $F \in \mathscr{E}$). For $\mu \in \mathscr{M}$, we define $\varphi_N(\mu) = \big(\mu(e_{-N}), \ldots, \mu(e_N)\big)$.

If M is a fixed constant, we shall write $\mu \in \mathfrak{R}_{MN}$ if $\mu \in \mathscr{M}$ and

$$\int |\breve{F}|^2 d\mu \leqslant M \int |F|^2 d\mu \quad \text{holds for all } |F|^2 \in \mathscr{E}_N, \tag{1.4}$$

and set $\mathfrak{R}_M = \bigcap \mathfrak{R}_{MN}$, $\mathfrak{R} = \bigcup \mathfrak{R}_M$. We shall see that if $M \leqslant 1$ then $\mathfrak{R}_M$ contains at most one measure, $d\mu = dt$; so we consider only $M > 1$ and focus $\mathfrak{R}$ through the study of $\varphi_N(\mathfrak{R}_{MN})$.

Helson and Szegö [4], extending a previous result of Gaposhkin, proved that $\mu \in \mathfrak{R}$ iff $\mu \in \mathscr{M}$ and

$$d\mu = w(t)\, dt, \quad \text{with} \quad w = e^{u+\tilde{v}}, \ u, v \in L^\infty, \ \|v\|_\infty < \pi/2. \tag{1.5}$$

Hunt, Muckenhoupt and Wheeden [5] proved that $\mu \in \mathfrak{R}$ iff $d\mu = w(t)\, dt \in \mathscr{M}$ and

$$\Big(\int_I w(t)\, dt\Big)\Big(\int_I w(t)^{-1} dt\Big) \leqslant c|I|^2, \quad \text{for all intervals } I. \tag{1.6}$$

Condition (1.6) is of great simplicity and its use opened the way for the generalization of the Riesz inequality to $L^p(\boldsymbol{R}^n)$; furthermore, it appears in different problems, revealing their deep connections (cf. [2]).

As it stands, result (1.5) gives an explicit construction of all $\mu \in \mathfrak{R}$ and tells us "how many" they are, but it is difficult to check whether a given μ satisfies (1.5). On the other hand, (1.6) is easy to be checked if the measure is given in the form $d\mu = w(t)\, dt$, but neither is practical if μ is given as a functional $\mu(F)$. However, this is quite commonly the case, and, moreover, it is frequent to have μ given only by its *moments* or Fourier coefficients, that is, only by its values for $F = e_n$, $n = 0, \pm 1, \pm 2, \ldots$ This led us to try a moment theory approach for further study of $\mathfrak{R}$. The usefulness of such an approach comes to view in the treatment of the following question:

— is the class $\mathfrak{R}$ determined by a set whose elements are of simple nature?

This is a standard question in several classical theories, where a certain class $\mathfrak{A}$ is described by (i) by showing that every element of it is the limit of finite combinations of "simple" elements (which are of a special nature, such as characters, spherical functions, etc.), and (ii) by giving explicit formulae for those simple elements. In most known instances, where $\mathfrak{A}$ is a convex compact set (so that by Krein–Milman theorem every element of $\mathfrak{A}$ is the limit of convex combinations of extremal points), the special elements are essentially the extremal points of $\mathfrak{A}$.

In the case $\mathfrak{A} = \mathscr{M}$, the extremal points of $\mathscr{M}$ are the Dirac measures δ_ξ ($\delta_\xi(F) = F(\xi)$ if $F \in \mathscr{E}$) which are of simple nature indeed. However, there is still another natural type of "simple" elements in $\mathscr{M}$. We say that $\mu \in \mathscr{M}$ is of *finite order N in $\mathscr{M}$* (or *canonical of order N in $\mathscr{M}$*) if there exists a finite integer n such that the following condition holds:

$$\nu \in \mathscr{M} \quad \text{and} \quad \nu(F) = \mu(F) \quad \text{for all } F \in \mathscr{E}_n \quad \text{implies} \quad \nu \equiv \mu, \tag{1.7}$$

and N is the least such n, or equivalently, $\varphi_N(\mu) \in$ boundary $\varphi_N(\mathscr{M})$.

Fortunately, seeking the "simple" elements of $\mathscr{M}$ we need not to make a choice between the extremal points and those elements of finite order, since the extremal points are known to be just the measures of finite order 1 in $\mathscr{M}$ (cf. Section 4 below). So, if $\mathscr{M}_e$ is the set of the extremal measures and $\mathscr{M}_c$ is the set of the canonical measures, then $\mathscr{M}_e \cap \mathscr{M}_c = \mathscr{M}_e$. Thus, $\mathscr{M}_e \cap \mathscr{M}_c$ is the set of the "simple" objects of $\mathscr{M}$, and the convex combinations of these simple elements from a dense set in $\mathscr{M}$.

By analogy, in the case $\mathfrak{A} = \mathfrak{R}$ it would be natural to call simple those measures μ such that $\varphi_N(\mu)$ is both extremal and a boundary point of $\varphi_N(\mathfrak{R})$. However, since $\mathfrak{R}$ is convex but not compact (in the weak topology), we have to deal with $\mathfrak{R}_M$ — which is convex and compact for each fixed M — and more precisely with $\varphi_N(\mathfrak{R}_{MN}) = \mathfrak{B}_{MN}$.

We are then led to the following

Problem. Give explicit formulae for the simple elements of $\mathfrak{B}_{MN}$ and prove that it is determined by their convex combinations.

Remark. Let us observe that the proofs of (1.5) and (1.6) apparently do not furnish the exact value of M in (1.4), that is, (1.5) and (1.6) do not characterize the classes $\mathfrak{R}_M$, for a given M. A positive answer to the above problem gives a characterization of $\mathfrak{R}_M$ and therefore an information about the constant in the Riesz inequality.

The moment theory approach furnishes a positive answer to the problem, of the following type. To each finite sequence $\{m_n\}_{-N}^{N}$ we associate certain determinants and two trigonometric polynomials P_N, Q_N, and form the measure $d\mu = \left|\dfrac{P_N}{Q_N}\right|^2 dt = R(t)\, dt$, $R(t)$ a rational function; the simple elements of $\mathfrak{R}_{MN}$ are just all such measures corresponding to sequences $\{m_n\}$ whose associated determinants satisfy a fixed condition. Thus the simple measures of $\mathfrak{R}_{MN}$ are absolutely continuous with a density of a very special type ((1.5) only tells us that every $\mu\in\mathfrak{R}$ is absolutely continuous). In Sections 2, 3 and 4 we give the exact definitions and statements and indicate how to construct such sequences $\{m_n\}$, which tells us "how large" is $\mathfrak{R}_{MN}$.

Though our $\mathfrak{R}_M$-moment theory is parallel to that of $\mathscr{M}$ there are two striking differences: (i) while the simple elements of $\mathscr{M}$ are discrete measures, those of $\mathfrak{R}_{MN}$ are absolutely continuous measures with rational densities; (ii) while the simple elements of $\mathscr{M}$ are of order 1, those of $\mathfrak{R}_{MN}$ may have arbitrarily high orders. These differences are due to the fact that though both the measures of $\mathscr{M}$ and of $\mathfrak{R}_M$ are in 1-1 correspondence with non-negative forms $\sum k_{jn}\xi_j\overline{\xi_n}$, we have $k_{jn}=\mu(e_{j-n})$ if $\mu\in\mathscr{M}$, but $k_{jn}=\varepsilon_{jn}\mu(e_{j-n})$ if $\mu\in\mathfrak{R}_M$, with ε_{jn} taking the two values M and $M-1$. We also need a "lifting" operation unnecessary in the classical case.

The object of this paper is mainly to indicate the existence of a connection between moment theory and singular integrals, and to point out the interest of considering the quasi Toeplitz forms associated with the Riesz measures. In Section 5 we outline the basic questions for further development that will be treated elsewhere.

2. Positive functionals and quasi Toeplitz forms associated with $\mathfrak{R}_M$. As in Section 1 let $\mathscr{M}$ denote the set of all measures $\mu\geqslant 0$, $\mu(1)=1$, and $\mathfrak{R}_M$ the subset of $\mathscr{M}$ of measures satisfying (1.4) for all N and a fixed M. $\mathscr{E}=\bigcup_{N=0}^{\infty}\mathscr{E}_N$ is the set of all trigonometric polynomials $F(t)=\sum_{-N}^{N}a_ne_n(t)$.

For each $\mu\in\mathscr{M}$ we set

$$m_n=\int_0^{2\pi}e^{-int}d\mu \tag{2.1}$$

so that

$$\mu(e_n)=m_{-n}=\overline{m_n}; \tag{2.1a}$$

$\{m_n\}_{-\infty}^{\infty}$ is the sequence of the Fourier coefficients or the trigonometric moments of μ. Let us denote by $\mathscr{W}$ the set of all sequences $\boldsymbol{m}=\{m_n\}_{-\infty}^{\infty}$ such that there exists $\mu\in\mathscr{M}$ with $m_n=\mu(e_{-n})$ for all n; and by $\mathscr{V}_M$ the set of all sequences $\boldsymbol{m}=\{m_n\}_{-\infty}^{\infty}$ such that $m_n=\mu(e_{-n})$ with $\mu\in\mathfrak{R}_M$.

As it is well known, the measures $\mu\in\mathscr{M}$ are in $1-1$ correspondence with the positive linear functionals $\mu(F)$ defined in $\mathscr{E}$ with $\mu(1)=1$. In other words, if

$$\mathscr{C}=\{H\in\mathscr{E}:\ H(t)\geqslant 0 \text{ for all } t\in(0,2\pi)\}, \tag{2.2}$$

then $\mathscr{C}$ is a cone in the vector space $\mathscr{E}$ and $\mu\in\mathscr{M}$ iff $\mu(F)$ is linear in $\mathscr{E}$ and positive with respect to the cone $\mathscr{C}$: $\mu(H)\geqslant 0$ for $H\in\mathscr{E}$. By the classical Féjer–Riesz theorem (cf. [3]) the polynomial $H(t)=\sum_{-N}^{N}a_ne_n(t)$ belongs to $\mathscr{C}$ iff

$$H(t)=|Q(t)|^2\quad\text{where}\quad Q(t)=\sum_0^N\xi_ne_n(t), \tag{2.2a}$$

so that

$$H(t)=\sum_{j,k=0}^{N}\xi_j\overline{\xi_k}e_{j-k}(t), \tag{2.2b}$$

and

$$\mu(H)=\sum_{j,k=0}^{N}m_{k-j}\xi_j\overline{\xi_k},\qquad m_j=\mu(e_{-j}), \tag{2.2c}$$

so that $\mu\in\mathscr{M}$ iff

$$T_N(\xi;\,\boldsymbol{m})=\sum_{j,k=0}^{\infty}m_{k-j}\xi_j\overline{\xi_k}\geqslant 0,\qquad m_j=\mu(e_{-j}), \tag{2.2d}$$

for every sequence $\xi_0,\ldots,\xi_N$, $N=0,1,\ldots$

Thus, to each finite or infinite sequence $\boldsymbol{m}=\{m_n\}$, $m_{-n}=\overline{m_n}$, is associated the (finite or infinite) *Toeplitz form*

$$T(\xi;m)=\sum_{j,k\geqslant 0}m_{k-j}\xi_j\overline{\xi_k} \tag{2.2e}$$

and $\boldsymbol{m}$ belongs to $\mathscr{W}$ iff $m_0=1$ and its associated Toeplitz form is non negative (i.e.: the expression (2.2e) is $\geqslant 0$ for any sequence $\xi=\{\xi_j\}_{-\infty}^{\infty}$ such that $\xi_j\neq 0$ for j in a finite set).

Now μ satisfies inequality (1.4) iff $\mu(M|F|^2-|\breve{F}|^2)\geqslant 0$ for every $F\in\mathscr{E}$. This suggests the introduction of the sets

$$\mathscr{K}_M^0=\{F\in\mathscr{E}:\ F=M|Q|^2-|\breve{Q}|^2\} \tag{2.3}$$

and

$$\mathscr{K}_M=\{\text{all finite sums of elements of }\mathscr{K}_M^0\}. \tag{2.3a}$$

Clearly $\mathscr{K}_M$ is a cone: $H\in\mathscr{K}_M$, $\lambda\geqslant 0$ imply $\lambda H\in\mathscr{K}_M$, and $H_1,H_2\in\mathscr{K}_M$ imply $H_1+H_2\in\mathscr{K}_M$. This cone defines an order $\prec$ in $\mathscr{E}$:

$$G\prec H \text{ iff } H-G\in\mathscr{K}_M;\qquad G\in\mathscr{K}_M \text{ iff } G\succ 0. \tag{2.3b}$$

Observe that $H(t)\geqslant 0$ for all t implies $H\succ 0$, that is $H\in\mathscr{C}$ implies $H\in\mathscr{K}_M$, as seen from (2.2b) where $\breve{Q}=Q$, so that

$$H=\frac{1}{M-1}(M|Q|^2-|\breve{Q}|^2).$$

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If $\mu(F)$ is a linear functional in $\mathscr{E}$ which is positive with respect to $\prec$ ($\mu(F) \geqslant 0$ if $F \succ 0$), then it is positive in the ordinary order $\leqslant$, and μ is a non negative measure; moreover $\mu \in \mathfrak{R}_M$. *Therefore* $\mu \in \mathfrak{R}_M$ *iff* $\mu(F)$ *is a linear functional in* $\mathscr{E}$, *non negative with respect to* $\mathscr{K}_M$ (*or to the order* $\prec$), $\mu(1) = 1$.

And from (2.3) it is clear that $\mu(F)$ is positive with respect to $\mathscr{K}_M$ iff $\mu(H) \geqslant 0$ whenever $H \in \mathscr{K}_M^0$.

If $Q(t) = \sum_{-N}^{N} \xi_n e_n(t) \in \mathscr{K}_M^0$ then

$$M|Q(t)|^2 - |\breve{Q}(t)|^2 = M \sum_{j,k=-N}^{N} \xi_j \bar{\xi}_k e_{j-k}(t) - \sum_{j,k=0}^{N} \xi_j \bar{\xi}_k e_{j-k}(t) \tag{2.3c}$$

$$= \sum_{j,k=-N}^{N} \varepsilon_{jk} \xi_j \bar{\xi}_k e_{j-k}(t),$$

where

$$\varepsilon_{jk} = \begin{cases} M-1 & \text{if both } j, k \text{ are} \geqslant 0, \\ M & \text{otherwise.} \end{cases} \tag{2.3d}$$

So that $\mu \in \mathfrak{R}_M$ iff $\mu(1) = 1$ and

$$T_N^M(\xi; \boldsymbol{m}) = \sum_{j,k=-N}^{N} \varepsilon_{jk} m_{k-j} \xi_j \bar{\xi}_k \geqslant 0, \tag{2.3e}$$

for every sequence $\xi_{-N}, \ldots, \xi_N$, $N = 0, 1, \ldots$.

To each sequence $\boldsymbol{m} = \{m_n\}$, $m_{-n} = \overline{m_n}$, we associate now a (finite or infinite) *quasi Toeplitz from*

$$T^M(\xi; \boldsymbol{m}) = \sum_{j,k} \varepsilon_{jk} m_{k-j} \xi_j \bar{\xi}_k. \tag{2.3f}$$

Then, $\boldsymbol{m} \in \mathscr{V}_M$ iff $m_0 = 1$ and the form (2.3f) is non negative for every sequence $\xi = \{\xi_n\}_{\infty}^{-\infty}$ such that $\xi_n \neq 0$ only for n in a finite set, or equivalently, if all the finite forms (2.3e) are non negative for $N = 0, 1, \ldots$

We say that $\mu(F)$ is strictly positive with respect to the cone $\mathscr{K}_M$ (or to the order $\prec$) if $H \succ 0$, $H \neq 0$, imply $\mu(H) > 0$. Let $\mathfrak{R}_M^+$ denote the set of all measures $\mu \in \mathfrak{R}_M$ such that $\mu(F)$ is strictly positive. Similarly, let $\mathscr{V}_M^+$ be the set of all $\boldsymbol{m} \in \mathscr{V}_M$ which are positive definite. Then the above remarks may be summarized as follows.

PROPOSITION 2.1. *There is a 1-1 correspondence between the measures* $\mu \in \mathfrak{R}_M$, *the functionals* $\mu(F)$, $\mu(e_0) = 1$, *which are non negative with respect to the cone* $\mathscr{K}_M$, *and the non negative infinite quasi Toeplitz forms with* $m_0 = 1$. *Moreover,* $\mu \in \mathfrak{R}_M^+$ *iff the associated quasi Toeplitz form is positive definite.*

Since in the case of $\mathscr{M}$ the forms have indexes ranging from 0 to ∞ (because of (2.2a)), while in the case of $\mathfrak{R}_M$ they range from $-\infty$ to ∞, to work with the partial forms we have to adopt the following ordering of the integers:

$$0, -1, 1, -2, 2, \ldots, -n, n, \ldots \tag{2.4}$$

and consequently we shall order the double sequence $\{e_n\}_{-\infty}^{\infty}$ as

$$e_0, e_{-1}, e_1, e_{-2}, e_2, \ldots \tag{2.4a}$$

and similarly $\xi = \{\xi_n\}$ and $\boldsymbol{m} = \{m_n\}$. Then the infinite quasi Toeplitz form $T^M(\xi; \boldsymbol{m})$ is non negative iff so are all the finite forms

$$T_{J_N}^M(\xi; \boldsymbol{m}) = \sum_{\substack{j \in J_N \\ k \in J_N}} \varepsilon_{jk} m_{k-j} \xi_j \bar{\xi}_k \tag{2.3g}$$

where J_N is any of the sets of the sequence

$$J_0 = \{0\} \subset J_1 = \{0, -1\} \subset J_2 = \{0, -1, 1\} \subset J_3 = \{0, -1, 1, -2\} \subset \ldots \tag{2.4b}$$

A sequence of sets of indexes related to $\{J_N\}$ that will be used in section 3 is $\{J'_N\}$ formed by

$$J'_N = \begin{cases} J_{N-1} \cup \left\{\dfrac{N+1}{2}\right\} & \text{if } N \text{ is odd,} \\ J_{N-1} \cup \left\{-\dfrac{N+2}{2}\right\} & \text{if } N \text{ is even.} \end{cases} \tag{2.4c}$$

(Observe that J_N consists of the first $N+1$ integers of the ordering (2.4) and that J'_N is obtained from J_N by substituting the last term by the following one in (2.4).)

With the notation

$$k + J_N = \{j \in \boldsymbol{Z} \colon j = n + k, n \in J_N\}, \tag{2.4d}$$

we have $J'_N = -(-1)^N + J_N$.

We associate with each $\boldsymbol{m}$ the sequence of the determinants of the form $T^M(\xi; \boldsymbol{m})$

$$\Delta_0 = \Delta_{J_0} = M-1, \quad \Delta_1 = \Delta_{J_1} = \begin{vmatrix} M-1 & Mm_{-1} \\ Mm_1 & M \end{vmatrix}, \tag{2.5}$$

$$\Delta_2 = \Delta_{J_2} = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ (M-1)m_{-1} & Mm_{-2} & M-1 \end{vmatrix},$$

$$\Delta_3 = \Delta_{J_3} = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 & Mm_{-2} \\ Mm_1 & M & Mm_2 & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & M-1 & Mm_{-3} \\ Mm_2 & Mm_1 & Mm_3 & M \end{vmatrix}, \ldots,$$

in general,

$$\Delta_N = \Delta_{J_N} = |a_{pq}|_{p,q=1}^{N+1},$$

where

$$(2.5a) \qquad a_{pq} = \begin{cases} Mm_{k-j} & \text{if} \quad p = 2j,\ q = 2k, \\ (M-1)m_{j-k} & \text{if} \quad p = 2j+1,\ q = 2k+1, \\ Mm_{-j-k} & \text{if} \quad p = 2j,\ q = 2k+1, \\ Mm_{j+k} & \text{if} \quad p = 2j+1,\ q = 2k \end{cases} \qquad (j, k \in J_N).$$

More generally, for every finite set Λ of integers we have the finite form

$$(2.6) \qquad T_\Lambda^M(\xi; \boldsymbol{m}) = \sum_{\substack{j\in\Lambda \\ k\in\Lambda}} \varepsilon_{jk} m_{k-j} \xi_j \bar{\xi}_k$$

and the corresponding determinant Δ_Λ. These determinants (in which j and k vary in the same set Λ) are called the partial determinants of $\boldsymbol{m}$: among them are $\Delta_0, \Delta_1, \ldots$

From the classical theorem on positive hermitian forms, we have then

PROPOSITION 2.2. (a) *$\boldsymbol{m} \in \mathscr{V}_M^+$ iff $m_0 = 1$ and all its associated determinants Δ_N are positive for $N = 0, 1, 2, \ldots$*

(b) *$\boldsymbol{m} \in \mathscr{V}_M$ iff all its partial determinants Δ_Λ are non negative.*

From Propositions 2.1 and 2.2 we get

COROLLARY 2.3. *If $\mu \in \mathfrak{R}_M$ then $M-1 \geqslant 0$ and $|\mu(e_{-n})| = |m_n| \leqslant \sqrt{\frac{M-1}{M}}$ for $n \neq 0$. Hence for $M < 1$ there are no measures $\mu \in \mathfrak{R}_M$ and for $M = 1$ there is only one measure $\mu \in \mathfrak{R}_1$, namely $d\mu = dt$, with $m_n = 0$ for all $n \neq 0$.*

Thus, from now on we are justified in assuming $M > 1$.

Let us consider now the measures $\mu \in \mathfrak{R}_M$ which are not in $\mathfrak{R}_M^+$, that is, $\mu(H) = 0$ for some $H \in \mathscr{K}_M$, $H \neq 0$. Such an H is of the form $H = \sum_n (M|Q_n(t)|^2 - |\breve{Q}_n(t)|^2)$ and since $\mu(M|Q_n|^2 - |\breve{Q}_n|^2) \geqslant 0$ for every n, this implies that the last expression is equal 0 for every n. That is, H can always be taken from $\mathscr{K}_M^0$. Observe that in the case of $\mathscr{M}$, if $\mu \in \mathscr{M}$ is not strictly positive, there exists $Q \in \mathscr{E}$ such that $\mu(|Q|^2) = 0 = \int_0^{2\pi} |Q(t)|^2 d\mu$,

which implies that μ is concentrated in the finite set $\{t_1, \ldots, t_k\}$ where e^{it_j}, $j = 1, \ldots, k$, are the zeros of the polynomial $Q(e^{it})$, located in the unit circle. Observe also that if $\mu \in \mathscr{M}$ is finitely discrete, e.g. $\mu = \delta$ the Dirac measure, so that $m_n = \mu(e_{-n}) = 1$ for all n, $\mu \sim \sum_{-\infty}^{\infty} m_n e_n(t)$, then $\mu^0 = \sum_{-\infty}^{-1} m_n e_n(t)$ is not a measure but only a distribution.

Instead in the case of $\mu \in \mathfrak{R}_M$ we have the following

PROPOSITION 2.4. *Let $\mu \in \mathfrak{R}_M$ be such that $\mu(H) = 0$, where*

$$H(t) = M|Q(t)|^2 - |\breve{Q}(t)|^2, \qquad Q(t) = \sum_{-N}^{N} a_n e_n(t), \qquad H \neq 0;$$

then

(a) *it holds*

$$(2.7) \qquad \mu\big(Q(t) e_n(t)\big) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

$$(2.7a) \qquad \mu\Big(\big(MQ(t) - \breve{Q}(t)\big) e_n(t)\Big) = 0 \quad \text{for} \quad n = 0, -1, -2, \ldots$$

or equivalently,

$$(2.8) \qquad \sum_{k=-N}^{N} a_k m_{-k-n} = 0 \quad \text{for} \quad n \geqslant 1$$

$$(2.8a) \quad M \sum_{k=-N}^{-1} a_k m_{-k+n} + (M-1) \sum_{k=0}^{N} a_k m_{-k+n} = 0 \quad \text{for} \quad n \geqslant 0$$

where $m_n = \mu\big(e_{-n}(t)\big)$.

(b) *If $\mu \sim \sum_{-\infty}^{\infty} m_n e_n(t)$ and $\mu^0 \sim \sum_{-\infty}^{-1} m_n e_n(t)$, then both μ and μ^0 are absolutely continuous measures: $d\mu = w(t)\,dt$, $d\mu^0 = w^0(t)\,dt$, $w, w^0 \in L^1(dt)$, $w(t) \neq 0$ p.p. Moreover,*

$$(2.9) \qquad w^0(t) = \frac{P^0(t)}{Q(t)}, \quad \text{“degree” of } P^0 \leqslant \text{“degree” of } Q,$$

where $P^0(t) = \sum_{-N}^{N-1} b_n e_n(t)$ is given by $b_n = \sum_{k=n+1}^{N} a_k m_{-k+n}$,

$$(2.9a) \qquad w(t) = w^0(t) + \overline{w^0(t)} + 1, \quad \text{and } w \text{ is of the form } \left|\frac{P(t)}{Q(t)}\right|^2.$$

Proof. (a) Letting $Q_\varepsilon(t) = Q(t) + \varepsilon e_{-n}(t)$, if $n \geqslant 1$, we shall have

$$\breve{Q}_\varepsilon(t) = \breve{Q}(t),$$

$$M|Q_\varepsilon(t)|^2 - |\breve{Q}_\varepsilon(t)|^2 = M|Q(t)|^2 + \varepsilon M\overline{Q(t)}e_{-n}(t) + \varepsilon MQ(t)e_n(t) + M\varepsilon^2 - |\breve{Q}(t)|^2.$$

Since $f(\varepsilon) = \mu(M|Q_\varepsilon(t)|^2 - |\breve{Q}_\varepsilon(t)|^2) \geqslant 0$ for all real ε and $f(\varepsilon) = 0$ for $\varepsilon = 0$, we have $f'(0) = 0$, which yields

$$\mu\big(\bar{Q}(t)e_{-n}(t)\big) + \mu\big(Q(t)e_n(t)\big) = 0.$$

Taking $i\varepsilon$ instead of ε, we obtain similarly

$$\mu\big(\bar{Q}(t)e_{-n}(t)\big) - \mu\big(Q(t)e_n(t)\big) = 0,$$

so that

$$\mu\big(Q(t)e_n(t)\big) = 0 \quad \text{for} \quad n = 1, 2, \ldots$$

Instead if $n = 0, -1, -2, \ldots$, then $\breve{Q}_\varepsilon(t) = \breve{Q}(t) + \varepsilon e_{-n}(t)$ and

$$f'(0) = \mu\big((M\bar{Q}(t) - \bar{\breve{Q}}(t))e_{-n}\big) + \mu\big((MQ(t) - \breve{Q}(t))e_n\big) = 0$$

and we obtain

$$\mu\big((MQ - \breve{Q})e_n\big) = 0 \quad \text{for} \quad n = 0, -1, -2, \ldots$$

(b) Let ν be the linear functional in $\mathscr{E}$ defined by $\nu(e_n) = \mu(e_n) - \mu^0(e_n)$, that is, $\nu(e_n) = \mu(e_n)$ if $n \geqslant 0$ and $\nu(e_n) = 0$ if $n < 0$. Then for each $F \in \mathscr{E}$ we have

$$|\nu(F)| = |\mu(\breve{F})| \leqslant \int |\breve{F}|\,d\mu \leqslant \Big(\int |\breve{F}|^2\,d\mu\Big)^{1/2} \leqslant M^{1/2}\Big(\int |F|^2\,d\mu\Big)^{1/2} \leqslant M^{1/2}\|F\|_\infty.$$

This shows that $\nu(F)$ is continuous in the norm of the space $C(0, 2\pi)$ and therefore ν is a measure (= bounded linear functional in $C(0, 2\pi)$). Since $\nu(e_n) = 0$ for $n < 0$, by the classical theorem of F. and M. Riesz (cf. [6]), ν is absolutely continuous. Therefore as the Fourier development of μ^0 is equal to that of $-1+\bar{\nu}$, μ^0 is also absolutely continuous, $d\mu^0 = w^0(t)\,dt$, $w^0 \in L^1$, and so $d\mu = w(t)\,dt$, $w \in L^1$.

If we set $w^0(t)Q(t) = \sum_{-\infty}^{\infty} b_n e_n(t)$, then

$$b_0 = \sum_{k=1}^{N} a_k m_{-k},$$

$$b_1 = \sum_{k=2}^{N} a_k m_{-k+1},$$

$$\cdots\cdots\cdots\cdots$$

$$b_N = b_{N+1} = \ldots = 0.$$

Instead, if $n < -N$, then by (2.8),

$$b_n = \sum_{k=-N}^{\infty} a_k m_{-k-|n|} = 0.$$

Thus $b_n = 0$ for $|n| \geqslant N$ and $w^0(t)Q(t) = P^0(t)$ is a polynomial of degree $\leqslant N-1$. This proves (2.9). Since $w(t) \geqslant 0$, it follows from (2.9) and the Féjer–Riesz theorem that $w(t)$ is of the form (2.9a). Q.E.D.

Corollary 2.5. *If $\mu(Q) = 0$ where $Q \in \mathscr{K}_M \cap \mathscr{E}_N$, then $\nu \in \mathfrak{R}_M$ and $\nu(F) = \mu(F)$ for $F \in \mathscr{E}_N$ imply $\nu \equiv \mu$.*

Proof. In this case $\nu(Q) = 0$, $Q \in \mathscr{K}_M$, and by Proposition 2.4 (b), $d\nu = w(t)\,dt$ where $w(t)$ is given by (2.9a).

The converse of (2.7) and (2.7a) are also true:

Proposition 2.6. (a) *If F is a continuous function, $\mu \in \mathfrak{R}_M$ and $\mu(Fe_n) = 0$ for $n \geqslant 1$, $\mu\big((MF - \breve{F})e_{-n}\big) = 0$ for $n \geqslant 0$, then*

$$\mu(M|F|^2 - |\breve{F}|^2) = 0.$$

(b) *If $Q = \sum_{k \in J_N} a_k e_k$, any linear functional and if $\mu(Qe_n) = 0$ only for $n \in J_N$, $n \geqslant 1$, and $\mu\big((MQ - \breve{Q})e_{-n}\big) = 0$ only for $n \in J_N$, $n \geqslant 0$, then*

$$\mu(M|Q|^2 - |\breve{Q}|^2) = 0.$$

Proof. (a) Observe that the proof of part (b) of Proposition 2.3 is based only on (2.7) and (2.7a). So these identities imply that $d\mu = w(t)\,dt$. As $MQ(t)w(t)$ is analytic by (2.7) and $\breve{Q}(t)w(t) - MQ(t)w(t)$ is antianalytic by (2.7a), $MQ(t)w(t)$ is the analytic part of $\breve{Q}(t)w(t)$:

(2.10) $$MQw = (\breve{Q}w)^{\check{}}.$$

Thus

(2.10a) $$iMQw - i(\breve{Q}w - MQw) = (\breve{Q}w)^{\sim}$$

(where $\tilde{F}$ is the conjugate function of F). Moreover, as

(2.10b) $$\tilde{Q} = 2i\breve{Q} - iQ; \qquad Q = (iQ - 2i\breve{Q})^{\sim}.$$

Since $(\tilde{F}, G) = -(F, \tilde{G})$, we have

$$\int (\breve{Q}w)^{\sim}\,\bar{Q} = -\int \breve{Q}w\tilde{Q},$$

and by (2.10) and (2.10b),

$$\int (2iMQw - i\breve{Q}w)\bar{Q} = -\int \breve{Q}w\overline{(2i\breve{Q} - iQ)} = \int \breve{Q}w(2i\bar{\breve{Q}} - i\bar{Q}),$$

so that $M\int Qw\bar{Q} = \int \breve{Q}w\bar{\breve{Q}}$, hence

$$\mu(M|Q|^2 - |\breve{Q}|^2) = 0.$$

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(b) Let $Q = \sum_{k\in J_N} a_k e_k = \big(\sum_{p\in J_N^-} + \sum_{k\in J_N^+}\big) a_k e_k$, where

$$J_N^- = \{n\in J_N\colon n<0\},\qquad J_N^+ = \{n\in J_N\colon n\geqslant 0\},$$

so that

$$M|Q|^2 - |\breve{Q}|^2 = \sum_{k\in J_N^-} MQ\overline{a_k}\,\overline{e_k} + \sum_{k\in J_N^+} (MQ-\breve{Q})\,\overline{a_k}\,\overline{e_k},$$

and

$$\mu(M|Q|^2-|\breve{Q}|^2) = \sum_{k\in J_N^-} M\overline{a_k}\,\mu(Q\overline{e_k}) + \sum_{k\in J_N^+} \overline{a_k}\,\mu\big((M\breve{Q}-Q)\overline{e_k}\big).$$

Since by hypothesis $\mu(Q\overline{e_k}) = 0$ for $k\in J_N^-$ and $\mu\big((MQ-\breve{Q})\overline{e_k}\big) = 0$ for $k\in J_N^+$, the conclusion follows at once. Q.E.D.

3. The reduced moment problem for $\mathfrak{R}_M$. Let us write $\{m_n\}_{-N}^N\in\mathscr{W}_N$ if there exists $\mu\in\mathscr{M}$ satisfying $\mu(e_{-n}) = m_n$ for $|n|\leqslant N$. The classical reduced moment problem studies necessary and sufficient conditions for a given finite sequence $\{m_n\}_{-N}^N$ to belong to $\mathscr{W}_N$ and the properties of the measures $\mu\in\mathscr{M}$ having $\{m_n\}_{-N}^N$ as their first $N+1$ moments.

Similarly we write $\{m_n\}_{-N}^N\in\mathscr{V}_M^N$ (or $\mathscr{V}_M^{N+}$) if there exists $\mu\in\mathfrak{R}_M$ (or $\mathfrak{R}_M^+$) such that $\mu(e_{-n}) = m_n$ for $|n|\leqslant N$, and consider the reduced moment problem for $\mathfrak{R}_M$.

If $\mu\in\mathfrak{R}_M$ then the $\mathscr{K}_M$-positive functional $\mu(F)$ is also a *real* linear functional: if $\mathscr{E}^R = \bigcup \mathscr{E}_N^R$ is the set of all real trigonometric polynomials $F = \sum a_n e_n$, $a_{-n} = \overline{a_n}$, then $\mu(F)$ is real for every $F\in\mathscr{E}^R$, and μ is determined by its restriction to $\mathscr{E}^R$.

To each sequence $\{m_n\}_{-N}^N$ with $m_{-n} = \overline{m_n}$ we associate the real linear functional $l(F)$ in $\mathscr{E}_N^R$ defined by $l(e_{-n}) = m_n$, $|n|\leqslant N$.

Then the reduced moment problem of whether $\{m_n\}_{-N}^N$ belongs to $\mathscr{V}_M^N$ (respectively to $\mathscr{W}_N$) is equivalent to the problem of extending l to a $\mathscr{K}_M$-positive (respectively, a $\mathscr{C}$-positive) real functional on $\mathscr{E}^R$. Combining this fact with the Hahn–Banach theorem we get the following condition:

Proposition 3.1. *$\{m_n\}_{-N}^N\in\mathscr{V}_M^N$ (respectively $\mathscr{W}_N$) iff $m_0 = 1$ and the corresponding real linear functional l, defined by $l(e_{-n}) = m_n$, $|n|\leqslant N$, is non negative in $\mathscr{E}_N^R$ with respect to $\mathscr{K}_M$ (respectively, to $\mathscr{C}$), i.e.:*

(3.1) $l(F)\geqslant 0$ *if* $F\succ 0$ *(respectively,* $F\geqslant 0$*) or, equivalently, if* $F\in\mathscr{K}_M\cap\mathscr{E}_N$ *(respectively* $F\in\mathscr{C}\cap\mathscr{E}_N$*).*

Proof. If $\mu\in\mathfrak{R}_M$ (or $\mu\in\mathscr{M}$) and $\mu(e_{-n}) = m_n$ for $|n|\leqslant N$, then $\mu = l$ on $\mathscr{E}_N^R$ and (3.1) holds.

Conversely, assume that (3.1) holds and let us prove that there exists $\mu\in\mathfrak{R}_M$ (or $\mu\in\mathscr{M}$) with $\mu = l$ in $\mathscr{E}_N^R$.

If $F\in\mathscr{E}^R$ and $|F(t)|\leqslant\lambda$ for all t, then $\lambda-F\geqslant 0$, that is, $\lambda-F\in\mathscr{C}$ and therefore also $\lambda\succ F$. So the function $1 = e_0$ is a unit of $\mathscr{E}^R$ in both orders $\geqslant$ and $\succ$. By a known theorem of Hahn–Banach type (cf. III of [1]) of extension of positive functionals, this implies that l may be extended to a positive linear functional $\mu(F)$ in $\mathscr{E}^R$. Q.E.D.

We say that the polynomial $Q\in\mathscr{E}_A$ if $Q = \sum_{n\in A} a_n e_n$. In the case of $\mathscr{M}$, $F\in\mathscr{C}\cap\mathscr{E}_N$ implies $F\geqslant 0$ and by the Féjer–Riesz theorem we obtain that

(3.2) $$F\in\mathscr{C}\cap\mathscr{E}_N \text{ iff } F = |Q|^2 \text{ where } Q\in\mathscr{E}_{[0,N]}.$$

Hence, the condition of Proposition 3.1 reduces to

(3.3) $$\{m_n\}_{-N}^N\in\mathscr{W}_N \text{ iff } l(F)\geqslant 0 \text{ for } F = |Q|^2 \text{ with } Q\in\mathscr{E}_{[0,N]}$$

or equivalently

(3.3a) $$\{m_n\}_{-N}^N\in\mathscr{W}_N \text{ iff the associated form } T_N(\xi;\boldsymbol{m}) = \sum_{j,k=0}^N m_{k-j}\,\xi_j\overline{\xi_k} \text{ is non negative for all } \xi_0,\dots,\xi_N.$$

The derivation of a reduction similar to (3.3) and (3.3a) for $\mathfrak{R}_M$ is more complicated due to the presence of the factors ε_{jk} in the quasi Toeplitz forms, and the fact that $F\in\mathscr{K}_M^0$ need not be non negative in the ordinary sense (so that the Féjer–Riesz theorem does not apply). More precisely, in the case of $\mathfrak{R}_M$ we have to take into account that

(i) while every element of $\mathscr{K}_M$ is a finite sum of elements of $\mathscr{K}_M^0$, not every $F\in\mathscr{K}_M\cap\mathscr{E}_N$ is a finite sum of elements of $\mathscr{K}_M^0\cap\mathscr{E}_N$. In fact, it may happen that $F, G\in\mathscr{K}_M$, $F\notin\mathscr{E}_N$, $G\notin\mathscr{E}_N$, but $F+G\in\mathscr{E}_N$.

(ii) an element $F\in\mathscr{K}_M^0\cap\mathscr{E}_N$ may be of the form $F = M|Q|^2-|\breve{Q}|^2$ with $Q\notin\mathscr{E}_{J_N}$. However, this is easily overcomed since

(3.5) $$l(F)\geqslant 0 \text{ for } F\in\mathscr{K}_M^0\cap\mathscr{E}_N \quad\text{iff}\quad l(F)\geqslant 0 \text{ for } F = M|Q|^2-|\breve{Q}|^2$$

with

(3.5a) $$Q\in\mathscr{E}_{n+J_N} \text{ for some } n,\ |n|\leqslant N.$$

Observe that $n+J_N$ is such that $\{j-k|\ j,k\in n+J_N\} = [-N,N]$ for all n. Thus any partial form $T_{n+J_N}^M(\xi;\boldsymbol{m})$ involves only $\{m_n\}_{-N}^N$. The difficulties introduced by fact (i) are of a more serious character. But although (3.3) is an essential step in the classical theory of canonical measures, we may avoid seeking for its analogue for $\mathfrak{R}_M$ by introducing the consideration of the classes $\mathfrak{R}_{MN}$.

Let $\boldsymbol{R}_*^{2N} = \{(x_{-N}, \ldots, x_N) \in \boldsymbol{R}^{2N+1}: x_0 = 1\}$. For each pair of integers $0 < L \leqslant N$, let $\varphi_N: \mathscr{M} \to \boldsymbol{R}_*^{2N}$ be the continuous map defined by $\varphi_N(\mu) = (\mu(e_{-N}), \ldots, \mu(e_N))$, and let

$$\mathscr{K}_{MN}^0 = \{M|Q|^2 - |\breve{Q}|^2, Q \in \mathscr{E}_{J_N}\}, \qquad \mathscr{K}_{MN} \text{ the cone spanned by } \mathscr{K}_{MN}^0,$$

$$\mathfrak{R}_{MN} = \{\mu \in \mathscr{M}: \mu(H) \geqslant 0, \forall H \in \mathscr{K}_{MN}^0\},$$

$$\mathfrak{R}_{MN}^+ = \{\mu \in \mathscr{M}: \mu(H) > 0, \ 0 \neq H \in \mathscr{K}_{MN}^0\},$$

$$\mathfrak{R}_{MN}(L) = \{\mu \in \mathfrak{R}_{MN}: \mu(H) = 0 \text{ for some } 0 \neq H \in \mathscr{K}_{ML}^0 \text{ but } \mu(H) > 0 \text{ if } 0 \neq H \in \mathscr{K}_{M(L-1)}^0\},$$

$$\mathscr{V}_{MN} = \varphi_N(\mathfrak{R}_{MN}), \qquad \mathscr{V}_{MN}^+ = \varphi_N(\mathfrak{R}_{MN}^+), \qquad \mathscr{V}_{MN}(L) = \varphi_N(\mathfrak{R}_{MN}(L)).$$

The $\boldsymbol{m} \in \mathscr{V}_{MN}$ and the $\mu \in \mathfrak{R}_{MN}$ are called *elements* and *measures of order* N for $\mathfrak{R}_M$.

Note that $\mathfrak{R}_{M1} \supset \mathfrak{R}_{M2} \supset \ldots$ and $\mathfrak{R}_M = \bigcap \mathfrak{R}_{MN}$, so that *the study of* $\mathfrak{R}_M$ *is reduced to that of* $\mathscr{V}_{MN}$, $N = 1, 2, \ldots$

By conveniently modifying the proofs of Section 2 and of Proposition 3.1, we obtained for $\mathfrak{R}_{MN}$ the following properties similar to those proved for $\mathfrak{R}_M$.

PROPOSITION 3.2. (a) $\mu \in \mathfrak{R}_{MN}^+$ *(respectively,* $\mathfrak{R}_{MN}$*) iff* $\boldsymbol{m} = \varphi_N(\mu) \in \mathscr{V}_{MN}^+$ *(respectively,* $\mathscr{V}_{MN}$*) and iff* $T_{J_N}^M(\xi; \boldsymbol{m})$ *is positive (non negative).*

(b) *Given* $\boldsymbol{m} = \{m_n\}_{-L}^L$, $L < N$, *there exists* $\mu \in \mathfrak{R}_{MN}$ *with* $\varphi_L(\mu) = \boldsymbol{m}$ *iff* $l(P) \geqslant 0$ *for every* $P \in \mathscr{E}_L \cap \mathscr{K}_{MN}$, *being the functional defined by* $\boldsymbol{m}$ *in* $\mathscr{E}_{J_L}$.

PROPOSITION 3.3. *If* $\mu \in \mathfrak{R}_{MN}(L)$, $0 < L \leqslant N$, *and* $\mu(M|Q|^2 - |\breve{Q}|^2) = 0$ *where* $Q = \sum_{k \in J_L} a_k e_k$, *then*

(a) *relations* (2.8) *and* (2.8a) *of Proposition* 2.4 *hold for* $n \in J_N$ *(*k *varying in* J_L*);*

(b) *there exists an absolutely continuous measure* $\mu_a = w(t)\,dt$ *with* $\mu_a(e_n) = \mu(e_n)$ *for* $n \in J_N$;

(c) *if* $\mu^0 = \sum_{-\infty}^{-1} m_n e_n = \sum_{-\infty}^{\infty} m_n^0 e_n$, *where* $m_n = \mu(e_{-n})$, *then* $\mu^0(e_n) = \mu_r(e_n)$ *for* $|n| \leqslant [(N+L+1)/2]$, *where* μ_r *has a rational density* P^0/Q, *with* $P^0 = \sum b_n e_n$, $b_n = 0$ *if* $n \notin J_L$ *and* $b_n = \sum_{k \in J_L} a_k m_{-k+n}^0$ *if* $n \in J_L$;

(d) *if* $L = N-1$, *then the* $\{m_n\}_{-N}^N$ *are the moments of a rational function* P^0/Q *and are explicitely determined by* $\{m_n\}_{-N+1}^{N-1}$, *which are said to be lifted to* $\{m_n\}_{-N}^N$.

To any finite sequence $\{m_n\}_{-N}^N$, $m_{-n} = \overline{m_n}$, we can associate the $N+1$ determinants $\Delta_0, \Delta_1, \ldots, \Delta_N$ given by expressions (2.5). If these Δ_n are all non negative for $0 \leqslant n \leqslant N$, then we denote by Γ the set of all complex numbers z such that also the sequence $\{m_{-N-1}, m_{-N}, \ldots, m_N, m_{N+1}\}$ with $m_{N+1} = z$ has non negative determinants

$$\Delta_0 \geqslant 0, \quad \ldots, \quad \Delta_N \geqslant 0, \quad \Delta_{N+1}(z) \geqslant 0. \tag{3.6}$$

The determinant of order N obtained from Δ_{N-1} by adding the j line and the k column of determinant Δ_{N+1}, for some $j, k \leqslant N+1$, will be denoted by $\Delta_{N-1}\binom{j}{k}$. We write $\Delta_{N-1}\binom{N+1}{N+1} = \Delta'_N = \Delta_{J'_N}$.

Observe that z (or $\bar{z}$) appears in $\Delta_{N+1}(z)$ (of order $N+2$) in the $N+2$ line and the $N+1$ column, while $\bar{z}$ (or z) is located in the $N+1$ line and the $N+2$ column. So, Δ'_N does not contain neither z nor $\bar{z}$, and depends only on the given $\{m_n\}_{-N}^N$.

In what follows $\Delta|_{(j,k)=0}$ means that in Δ the element in the j line and the k column is replaced by zero.

LEMMA 3.3. *Let* $\mu \in \mathscr{M}$ *and* $Q_N(t)$ *be the trigonometric polynomial obtained by replacing the last column of the associated determinant* Δ_N *of* μ *by the column* $\{e_j\}_{j \in J_N} = \begin{pmatrix} e_0 \\ e_{-1} \\ e_1 \\ \vdots \end{pmatrix}$ *and let* $u_0, u_{-1}, \ldots$, *be the minors of the last column in* Δ_N.

Then, for every $p = 0, 1, 2, \ldots$, *the scalar product of* $\{u_j\}_{j \in J_N}$ *by the even columns of* Δ_{N+p} *are given by the numbers* $M\mu(Q_N e_n)$, $n \geqslant 1$, $n \in J_{N+p}$, *while the scalar products of* $\{u_j\}_{j \in J_N}$ *by the odd columns of* Δ_{N+p} *are given by the numbers* $\mu((MQ_N - \breve{Q}_N)e_n)$, $n \leqslant 0$, $n \in J_{N+p}$.

(It must be understood that for $p > 0$, we form the scalar product of $\{u_j\}_{j \in J_N}$ with the columns of Δ_{N+p} truncated at the first $N+1$ elements.)

Proof. Follows easily from the expressions (2.5).

THEOREM 3.4. (a) *Let* $\{m_n\}_{-N}^N$, $m_{-n} = \overline{m_n}$, *have positive associated determinants:* $\Delta_0 > 0, \ldots, \Delta_N > 0$. *Then* $\Gamma = \Gamma_{N+1}$ *is a circle of center* c *and radius* r, *where*

$$c = -\frac{1}{M}\frac{1}{\Delta_{N-1}}\Delta_{N-1}\binom{N+1}{N}\bigg|_{(N+1,N)=0}, \tag{3.7}$$

$$r^2 = \frac{1}{M^2}\frac{\Delta_N \Delta'_N}{\Delta_{N-1}^2} > 0. \tag{3.8}$$

Moreover, z *belongs to the boundary of* Γ *iff* $\Delta_{N+1}(z) = 0$.

(b) $\mathscr{V}_{MN}^+ = \{\{m_n\}_{-N}^N: \Delta_n > 0 \text{ for } n \leqslant N\}$. $\mathfrak{R}_{MN}(L) = \{\mu \in \mathfrak{R}_{MN}: \Delta_n > 0$ *if* $n < L$, $\Delta_L = 0\}$. $\operatorname{Int}\mathscr{V}_{MN} = \mathscr{V}_{MN}^+$ *is dense in* $\mathscr{V}_{MN}$. $\mathfrak{R}_{MN}^+ \subset \operatorname{Int}\mathfrak{R}_{MN}$. *Boundary* $\mathscr{V}_{MN} = \bigcup_{L=1}^N \mathscr{V}_{MN}(L)$.

(c) $\{m \in \mathscr{V}_{MN}: \Delta_{N-2} = 0\} \subset$ closure $\{m \in \mathscr{V}_{MN}: \Delta_n > 0,\ n \leqslant N-2,\ \Delta_{N-1}\cdot\Delta'_{N-1} = 0\}$.

(d) closure $\mathscr{V}_{MN}(N) =$ closure $\{m = \{m_n\}_{-N}^{N}: \Delta_n > 0,\ n \leqslant N-1,\ \Delta'_{N-1} > 0,\ \Delta_N = 0\}$.

Proof. (a) For simplicity we develop the case $N = 2$, but the argument applies unchanged for any $N \geqslant 0$. We have then

$$\Delta_{N+1}(z) = \Delta_3(z) = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 & Mm_{-2} \\ Mm_1 & M & Mm_2 & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & M-1 & M\bar z \\ Mm_2 & Mm_1 & Mz & M \end{vmatrix},$$

$$z \in \Gamma \quad \text{iff} \quad \Delta_{N+1}(z) \geqslant 0.$$

By the known Sylvester identity for determinants,

$$\Delta_n \Delta_{n-2} = \begin{vmatrix} \Delta_{n-2}\binom{n-1}{n-1} & \Delta_{n-2}\binom{n-1}{n} \\ \Delta_{n-2}\binom{n}{n-1} & \Delta_{n-2}\binom{n}{n} \end{vmatrix}. \tag{3.9}$$

We apply (3.9) to $\Delta_{N+1}(z)$ and Δ_{N-1}:

(3.10)
$$\Delta_{N+1}(z)\Delta_{N-1} = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ (M-1)m_{-1} & Mm_{-2} & M-1 \end{vmatrix} \cdot \begin{vmatrix} M-1 & Mm_{-1} & Mm_{-2} \\ Mm_1 & M & Mm_{-1} \\ Mm_2 & Mm_1 & M \end{vmatrix} -$$
$$- \begin{vmatrix} M-1 & Mm_{-1} & Mm_{-2} \\ Mm_1 & M & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & M\bar z \end{vmatrix} \cdot \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ Mm_2 & Mm_1 & Mz \end{vmatrix}.$$

As $\Delta_{N-1} > 0$, $\Delta_{N+1}(z)$ will be $\geqslant 0$ iff the expression (3.10) is $\geqslant 0$, which amounts to

$$-M^2\Delta_1^2 z\bar z - M\Delta_1 \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ Mm_2 & Mm_1 & 0 \end{vmatrix} z -$$
$$- M\Delta_1 \begin{vmatrix} M-1 & Mm_{-1} & Mm_{-2} \\ Mm_1 & M & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & 0 \end{vmatrix} \bar z + \Delta_1\binom{2}{2}\cdot\Delta_1\binom{3}{3} -$$
$$- \begin{vmatrix} M-1 & Mm_{-1} & Mm_{-2} \\ Mm_1 & M & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & 0 \end{vmatrix} \cdot \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ Mm_2 & Mm_1 & 0 \end{vmatrix} \geqslant 0.$$

Hence $z \in \Gamma$ iff z satisfies $-z\bar z + c\bar z + \bar c z + r^2 - c\bar c \geqslant 0$, where c and r are given by (3.7) and (3.8). To prove that Γ is a circle it only remains to check that $M^2 r^2 = (\Delta_N \Delta'_N)/\Delta^2_{N-1}$ is positive. This is in effect so, because by hypothesis all Δ_n, $|n| \leqslant N$, are positive and, by Proposition 3.2 (b) this implies that also Δ'_N is positive.

(b) The first two assertions follow from Proposition 3.2 (a) and they imply that $\mathrm{Int}\,\mathscr{V}_{MN} = \mathscr{V}^+_{MN}$ and the last assertion. Since φ_N is continuous, $\mathfrak{R}^+_{MN} \subset \mathrm{Int}\,\mathfrak{R}_{MN}$. Finally, if μ_0 is the Lebesgue measure, then $\mu_0(e_n) = 0$ for $n \neq 0$, and its $\Delta_n > 0$ for all n, so $\mu_0 \in \mathscr{V}^+_{MN}$, all N. For every $\mu \in \mathscr{V}_{MN}$ and every $p = 1, 2, \ldots$ we have $\mu + \frac{1}{p}\mu_0 \in \mathscr{V}^+_{MN}$ and $\mu + \frac{1}{p}\mu_0 \to \mu$. Hence $\mathscr{V}^+_{MN}$ is dense in $\mathscr{V}_{MN}$.

(c) Let $\{m_n\}_{-N}^{N} \in \mathscr{V}_{MN}$ such that $\Delta_1 \geqslant 0, \ldots, \Delta_{N-3} \geqslant 0$, $\Delta_{N-2} = 0$. By (b) there is a $\{m_n^+\}_{-N}^{N} \in \mathscr{V}^+_{MN}$ arbitrarily close to $\{m_n\}$ with $\Delta_1^+ > 0, \ldots, \Delta^+_{N-2} > 0$, $\Delta^+_{N-1} > 0$; $\Delta^+_N > 0$. Hence, by (a) $\Delta^{+\prime}_{N-1} > 0$ and $m^+_{N-1} \in G = \mathrm{Int}\,\Gamma^+_{N-1} \cap \mathrm{Int}\,\Gamma^{+\prime}_{N-1}$ ($\Gamma'_{N-1} = \{m_{N-1}: \Delta'_{N-1} \geqslant 0\}$). If we fix m_n^+ for $n \leqslant N-2$ and replace m^+_{N-1} by any $m^0_{N-1} \in G$, all the determinants of the new sequence would be > 0, and also $\Delta^{0\prime}_{N-1} > 0$, so we can take m^0_N interior to Γ^0_N and obtain a sequence from $\mathscr{V}_{MN}$. By passing to the limit, the same is true if we take $m^0_{N-1} \in \mathrm{bdary}\,G \subset \mathrm{bdary}\,\Gamma^+_{N-1} \cup \mathrm{bdary}\,\Gamma^{+\prime}_{N-1}$. But $m^+_{N-1} \in \Gamma^+_{N-1}$ is close to the fixed point m_{N-1}, and so is m^0_{N-1}, since the radius of Γ^+_{N-1} is small because Δ^+_{N-2} is close to $\Delta_{N-2} = 0$. As $\Delta^0_{N-1} = 0$ or $\Delta^{0\prime}_{N-1} = 0$, the assertion is proved.

(d) Assume $\Delta_n > 0$ for $n \leqslant N-1$, $\Delta'_{N-1} > 0$, $\Delta_N = 0$ and let us show that $\boldsymbol{m} \in \mathscr{V}_{MN}$. By (a), there is a $m^0_N \in \mathrm{Int}\,\Gamma_N$ arbitrarily close to m_N, and $\{m_n\}_{-N+1}^{N-1} \cup \{m^0_N\} \in \mathscr{V}^+_{MN}$. Hence $\boldsymbol{m}$ is in the closure of $\mathscr{V}^+_{MN}$.

Now let $\boldsymbol{m} \in \mathscr{V}_{MN}(N)$, so that $\Delta_n > 0$ for $n < N$, $\Delta_N = 0$. It is enough to consider the case $\Delta'_{N-1} = 0$. Let $\boldsymbol{m}^+ \in \mathscr{V}^+_{MN}$ be arbitrarily close to $\boldsymbol{m}$, with $\Delta^+_n > 0$ for $n \leqslant N$, $\Delta^{+\prime}_{N-1} > 0$. As above, $\{m_n\}_{-N+1}^{N-1} \cup \{m^0_N\} \in \mathscr{V}_{MN}$ for every $m^0_N \in \Gamma^+_N$, and we can take $m^0_N \in$ boundary Γ^+_N so that $\Delta^0_N = 0$. And m^0_N is close to m_N because Γ^+_N is small since Δ^+_{N-1} is close to $\Delta_{N-1} = 0$. Q.E.D.

Corollary 3.5. *If $\boldsymbol{m} \in \mathscr{V}_{MN}(N)$ and $\Delta'_{N-1} = 0$ then $m_N = c_N$ is uniquely determined. The same conclusion holds for $\boldsymbol{m} \in \mathscr{V}_{MN}$ with $\Delta_{N-2} = 0$.*

In the cases of Corollary 3.5 or Proposition 3.3 (d), $\boldsymbol{m}$ is called the *lifted* of $\{m_n\}_{-N+1}^{N-1}$.

4. Determination of the simple elements and reduction theory. Let $\mathscr{V}_M(N) = \{\boldsymbol{m} \in \mathscr{V}_{MN}(N): \Delta'_{N-1} > 0\}$, $\mathfrak{R}_M(N) = \varphi_N^{-1}(\mathscr{V}_M(N))$. The $\boldsymbol{m} \in \mathscr{V}_M(N)$ and $\mu \in \mathfrak{R}_M(N)$ will be called the *canonical elements and measures of order* N (for $\mathfrak{R}_M$). The $\boldsymbol{m} \in \mathscr{V}_M(N) \cap \mathrm{extr}\,\mathscr{V}_{MN}$ and $\mu \in \varphi_N^{-1}(\mathscr{V}_N(N) \cap \mathrm{extr}\,\mathscr{V}_{MN})$ will be called the *simple elements and measures of order* N.

We say that $\boldsymbol{m} \in \mathscr{V}_{MN}(N) \cup \mathscr{V}_{MN}(N-1)$ is *reducible* to a lesser order if it is the lifted of $\{m_n\}_{-N+1}^{N-1}$. In this section we prove that each $\boldsymbol{m} \in \mathscr{V}_{MN}$ can be obtained from the simple elements of order $\leqslant N$ by the operations of lifting, taking convex hulls and passing to the limit. (Note that lifting is not considered in classical moment theory.) Since bdary $\mathscr{V}_{MN} = \mathscr{V}_M(N) \cup \Big(\bigcup_{L=1}^{N-1} \mathscr{V}_{MN}(L)\Big) \cup \big(\mathscr{V}_{MN}(N) - \mathscr{V}_M(N)\big)$, and $\mathscr{V}_M(N)$ is determined by extr $\overline{\mathscr{V}_M(N)} \subset$ extr $\mathscr{V}_M(N) \cup \overline{\big(\mathscr{V}_M(N) - \mathscr{V}_M(N)\big)} \subset$ extr $\mathscr{V}_M(N) \cup \Big(\bigcup_{L=1}^{N-1} \mathscr{V}_{MN}(L)\Big) \cup \big(\mathscr{V}_{MN}(N) - \mathscr{V}_M(N)\big)$, and the $\boldsymbol{m} \in \mathscr{V}_{MN}(N) - \mathscr{V}_M(N)$ are reducible (Coroll. 3.5), the proof will be concluded by showing that the $\boldsymbol{m} \in \bigcup_{L=1}^{N-1} \mathscr{V}_{MN}(L)$ are reducible and that $\operatorname{extr} \mathscr{V}_M(N) = \mathscr{V}_M(N) \cap \operatorname{extr} \mathscr{V}_{MN}$.

THEOREM 4.1. (a) $\boldsymbol{m} \in \mathscr{V}_M(N)$ *iff* $\Delta_n > 0$ *for* $n \leqslant N-1$, $\Delta'_{N-1} > 0$, $\Delta_N = 0$, *and then* $\boldsymbol{m} = \varphi_N(\mu)$ *where* μ *has a rational density* $|P(t)/Q(t)|^2$, $Q(t)$ *being the trigonometric polynominal obtained by replacing the elements of the last column of* Δ_N *by* $e_0(t), e_{-1}(t), e_1(t), \ldots,$ *and* $|P|^2 = \overline{P^0 Q} + \overline{P^0 Q} + |Q|^2$, P^0 *as in Proposition* 3.3 (c).

(b) *If* $\boldsymbol{m} \in \mathscr{V}_{MN}(L)$ *then* $\Delta_L = \Delta_{L+1} = \ldots = \Delta_N = 0$. *In particular, the* $\boldsymbol{m} \in \mathscr{V}_{MN}(L)$, $L \leqslant N-1$ *are reducible.*

(c) *If* $\mu \in \mathfrak{R}_M$ *is canonical of order* N *then* μ *is of order* N *in the sense that* $\nu \in \mathfrak{R}_M$ *and* $\nu(F) = \mu(F)$ *for* $F \in \mathscr{E}_N$ *imply* $\nu \equiv \mu$.

Proof. (a) The first assertion follows directly from Theorem 3.4(d). Now *let* $u_0, u_{-1}, u_1, \ldots$ *be the minors of the last column of* Δ_N *and*

$$Q_N(t) = u_0 e_0 + u_{-1} e_{-1} + u_1 e_1 + \ldots = \sum_{j \in J_N} u_j e_j(t). \tag{4.1}$$

Since $\Delta_N = 0$, by Lemma 3.3 we have

$$\mu(Q_N e_k) = 0 \ \text{ for } k \geqslant 1,\ k \in J_N \quad \text{and} \quad \mu\big((MQ_N - \check{Q}_N) e_k\big) = 0 \ \text{ for } k \leqslant 0,\ k \in J_N. \tag{4.2}$$

By Proposition 2.6 (b), $\mu(M|Q|^2 - |Q|^2) = 0$ and Proposition 3.3 (c) ends the proof.

(b) Let $u_0, u_{-1}, \ldots$ be the minors of the last column of Δ_L and $Q_L(t) = Q(t) = u_0 e_0(t) + u_{-1} e_{-1}(t) + \ldots$ Then, as in the proof of (a), $\mu(M|Q|^2 - |\check{Q}|^2) = 0$, and by Proposition 3.3 (a), conditions (2.7) and (2.7a) hold with the restriction $k \in J_N$. By Lemma 3.3 this implies that the first $L+1$ lines of any determinant Δ_{L+p} are linearly dependent, hence $\Delta_{L+p} = 0$ for $L+p \leqslant N$.

(c) Since the determinants of μ satisfy $\Delta_n > 0$, $n < N$, $\Delta_N = 0$, and they depend only on the $m_n = \mu(e_{-n}) = \nu(e_{-n})$, $|n| \leqslant N$, we have that these determinants are also the determinants of ν and by part (a), ν and μ are both given by the same expression (cf. Corollary 2.5). Moreover, let $\Delta_n > 0$ for $n \leqslant N$ and let us prove that there exists $\nu \in \mathfrak{R}_{MN}$ such that $\nu(e_n) = \mu(e_n)$ for $n \leqslant N$ but $\nu \neq \mu$. In fact, by Theorem 3.4 (a), Γ_N has then a non zero radius and we can take $m_N^0 \in \Gamma_N$, $m_N^0 \neq m_N$, and by Theorem 3.4 (d), there exists $\nu \in \mathfrak{R}_{MN}$ such that $\nu(e_{-n}) = m_n = \mu(e_{-n})$ for $n < N$ and $\nu(e_{-N}) = m_N^0 \neq \mu(e_{-N})$. Q.E.D.

Theorem 4.1 (a) gives explicit formulae for all the canonical μ for $\mathfrak{R}_M$ and shows that they are of a very special type, and that those of order N are characterized by the condition $m_N \in$ boundary of Γ_N, $r_N > 0$.

The elements of $\mathscr{V}_{MN}$ which are extremal and canonical of order N are called the *simple elements for* $\mathfrak{R}_M$ *of order* N. Let now $\mathscr{V}_M(N)$ be the set of all canonical elements of order N for $\mathfrak{R}_M$. If $\boldsymbol{m} \in \mathscr{V}_M(N)$ we know that if $\{u_j\}_{j \in J_N}$ are the minors of the last column of Δ_N and $Q_N = \sum_{j \in J_N} u_j e_j$, then (4.2) and Lemma 3.3 give $N+1$ linear relations between $\{u_j\}_{j \in J_N}$ and $\{m_n\}_{-N}^{N}$ of which only the first N may be independent. If in these N linear relations we consider $\{u_j\}$ as coefficients and $\{m_n\}$, $n \neq 0$, as $2N$ unknowns, if we write $u_j = v_j + iw_j$, $m_j = x_j + iy_j$, and separate real and imaginary parts, we shall obtain a system $S_N(\mu)$ of $2N$ equations for the $2N$ variables x_j, y_j and *the determinant of this system will be denoted by* Δ_N^*.

EXAMPLE 1. If $N = 3$,

$$\Delta_3 = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 & Mm_{-2} \\ Mm_1 & M & Mm_2 & Mm_{-1} \\ (M-1)m_{-1} & Mm_{-2} & M-1 & Mm_{-3} \\ Mm_2 & Mm_1 & Mm_3 & M \end{vmatrix},$$

$$Q = u_0 e_0 + u_{-1} e_{-1} + u_1 e_1 + u_{-2} e_{-2},$$

$$MQ - \check{Q} = (M-1) u_0 e_0 + M u_{-1} e_{-1} + (M-1) u_1 e_1 + M u_{-2} e_{-2},$$

where

$$u_0 = \begin{vmatrix} Mm_1 & M & Mm_2 \\ (M-1)m_{-1} & Mm_{-2} & M-1 \\ Mm_2 & Mm_1 & Mm_3 \end{vmatrix}, \text{ etc.}$$

The relations (4.2) are now

$$(*)\qquad \begin{aligned} (M-1)u_0+Mu_{-1}m_1+(M-1)u_1m_{-1}+Mu_{-2}m_2 &= 0,\\ Mu_0m_{-1}+Mu_{-1}+Mu_1m_{-2}+Mu_{-2}m_1 &= 0,\\ (M-1)u_0m_1+Mu_{-1}m_2+(M-1)u_1+Mu_{-2}m_3 &= 0,\\ Mu_0m_{-2}+Mu_{-1}m_{-1}+Mu_1m_{-3}+Mu_{-2} &= 0. \end{aligned}$$

The last equation is a consequence of the other three. Letting $u_j = v_j+iw_j$, $m_j = x_j+iy_j$, $j = 1, 2, 3$, and separating real and imaginary parts in the first three equations of $(*)$ we get that the left hand side of the system $S_3(\mu)$ is

$$\begin{aligned} &\big(Mv_{-1}+(M-1)v_1\big)x_1+\big(-Mw_{-1}+(M-1)w_1\big)y_1+Mv_{-2}x_2-Mw_{-2}y_2\\ &\big(Mw_{-1}+(M-1)w_1\big)x_1+\big(Mv_{-1}-(M-1)v_1\big)y_1+Mw_{-2}x_2+Mv_{-2}y_2\\ &M(v_{-2}+v_0)x_1+M(w_0-w_{-2})y_1+Mv_1x_2+Mw_1y_2\\ &M(w_{-2}+w_0)x_1+M(v_{-2}-v_0)y_1+Mw_1x_2-Mv_1y_2\\ &(M-1)v_0x_1-(M-1)w_0y_1+Mv_{-1}x_2-Mw_{-1}y_2+Mv_{-2}x_3-Mw_{-2}y_3\\ &(M-1)w_0x_1+(M-1)v_0y_1+Mw_{-1}x_2+Mv_{-1}y_2+Mw_{-2}x_3+Mv_{-2}y_3. \end{aligned}$$

So, the determinant of $S_3(\mu)$ is

$$\Delta_3^* = \begin{vmatrix} Mv_{-1}+(M-1)v_1 & (M-1)w_1-Mw_{-1} & Mv_{-2} & -Mw_{-2} & 0 & 0\\ Mw_{-1}+(M-1)w_1 & Mv_{-1}-(M-1)v_1 & Mw_{-2} & Mv_{-2} & 0 & 0\\ M(v_{-2}+v_0) & M(w_0-w_{-2}) & Mv_1 & Mw_1 & 0 & 0\\ M(w_{-2}+w_0) & M(v_{-2}-v_0) & Mw_1 & -Mv_1 & 0 & 0\\ (M-1)v_0 & -(M-1)w_0 & Mv_{-1} & -Mw_{-1} & Mv_{-2} & -Mw_{-2}\\ (M-1)w_0 & (M-1)v_0 & Mw_{-1} & Mv_{-1} & Mw_{-2} & Mv_{-2} \end{vmatrix}$$

THEOREM 4.2. (a) *A measure μ is a simple measure of order N for $\mathfrak{R}_M$ iff $\mu\in\mathfrak{R}_M(N)$ and $\Delta_N^*\neq 0$.*

Moreover the simple elements of order N are the extremal points of the set $\mathscr{V}_M(N)$ itself.

(b) *The convex combinations of the lifted simple elements are dense (in the weak topology) in $\mathscr{V}_{MN}$.*

More precisely, given $\mu\in\mathfrak{R}_{MN}^+$ and $\varepsilon>0$, there exists a convex combination ν of canonical elements of order $\leqslant N$, such that

$$|\nu(e_n)-\mu(e_n)|<\varepsilon \quad \textit{for} \quad |n|\leqslant N.$$

Proof. (a) Let $\boldsymbol{m}\in\mathscr{V}_M(N)$ with $\Delta_N^*\neq 0$, and let us show that $\boldsymbol{m}$ is an extremal point of $\mathscr{V}_{MN}$, that is, that $\boldsymbol{m} = (\boldsymbol{m}'+\boldsymbol{m}'')/2$, $\boldsymbol{m}'$, $\boldsymbol{m}''\in\mathscr{V}_{MN}$ imply $\boldsymbol{m} = \boldsymbol{m}' = \boldsymbol{m}''$, $\boldsymbol{m}^{(i)} = \varphi_N(\mu^{(i)})$.

We know that $\mu(F) = 0$ where $F = M|Q_N|^2-|\check{Q}_N|^2$, and since $F\in\mathscr{K}_M^0$, $\mu'(F)\geqslant 0$, $\mu''(F)\geqslant 0$, we must have also $\mu'(F) = 0 = \mu''(F)$. By Proposition 2.4 the relations (4.2) will hold also for μ' and μ''. If $\{m_n'\}_{-N}^N$ are the moments of μ', $m_n' = x_n'+iy_n'$, then by (4.2) and Lemma 3.3 the x_n', y_n' satisfy the same system $S_N(\mu') = S_N(\mu)$, with the determinant $\Delta_N^*\neq 0$. This implies $x_n' = x_n$, $y_n' = y_n$, so $\boldsymbol{m}' = \boldsymbol{m}$.

Conversely, let $\boldsymbol{m}\in\mathscr{V}_M(N)$ with $\Delta_N^* = 0$ and let us show that $\boldsymbol{m}$ *is not at extremal point of the set* $\mathscr{V}_M(N)$, and a fortiori neither of $\mathscr{V}_{MN}$.

By the same argument as above we shall have now infinite solutions for the system $S_N(\mu)$. Of course $\{m_n = x_n+iy_n\}$ is one solution; if $\{m_n^0\}_{-\infty}^{\infty}$ is another, then $\{m_n' = m_n+\lambda(m_n-m_n^0)\}$ is also a solution. We choose λ so that the m_n' are arbitrarily close to the m_n. Since $\Delta_n>0$ for $0\leqslant n<N$, we shall also have $\Delta_n^{(\prime)}>0$ if $n<N$, where $\Delta_n^{(\prime)}$ are the determinants of $\{m_n'\}$. And since $\{m_n'\}$ satisfy S_N this means that the scalar product of $\{u_j\}$ by the columns of $\Delta_N^{(\prime)}$ are equal to zero and we have then $\Delta_N^{(\prime)} = 0$. By Theorem 3.4 (d), there exists a $\mu'\in\mathfrak{R}_{MN}$ such that $\mu'(e_{-n}) = m_n'$ for $|n|\leqslant N$; moreover $\mu'\in\mathfrak{R}_M(N)$.

Letting $m_n'' = m_n-\lambda(m_n-m_n^0)$, we shall have similarly that $m_n'' = \mu''(e_{-n})$ with similar μ'', and since $(m_n'+m_n'')/2 = m_n$, we have that

$$\mu(e_n) = \big(\mu'(e_n)+\mu''(e_n)\big)/2 \quad \text{for} \quad |n|\leqslant N.$$

The measure $\nu = (\mu'+\mu'')/2\in\mathfrak{R}_{MN}$ coincides with μ in $\mathscr{E}_N$, in other words $\varphi_N(\mu) = \varphi_N\big((\mu'+\mu'')/2\big)$. Since $\boldsymbol{m}'$, $\boldsymbol{m}''\in\mathscr{V}_M(N)$, $\boldsymbol{m}$ is not extremal of $\mathscr{V}_M(N)$.

Finally, if $\boldsymbol{m}\in\mathscr{V}_M(N)$ is an extremal point of the set $\mathscr{V}_M(N)$ then Δ_N^* must be positive, because we have shown that $\Delta_N^* = 0$ implies $\boldsymbol{m} = (\boldsymbol{m}'+\boldsymbol{m}'')/2$, $\boldsymbol{m}'$, $\boldsymbol{m}''\in\mathscr{V}_M(N)$. But $\Delta_N^*>0$ implies that $\boldsymbol{m}$ is simple.

(b) By the Krein–Milman theorem and since the extremal points of the convex hull of $\mathscr{V}_M(N)$ are in closure $\mathscr{V}_M(N)$, the convex combinations of the lifted simple elements of order N are dense in $\mathscr{V}_M(N)$.

So it is sufficient to prove that given $\mu\in\mathfrak{R}_{MN}$, there exists a convex combination ν of canonical measures of order $\leqslant N$ such that $\mu(e_n) = \nu(e_n)$ for $|n|\leqslant N$. If $\Delta_n(\mu) = 0$ for some $n\leqslant N$ then μ itself is canonical of order $\leqslant N$, or a lifted element and we are done.

So we may suppose that $\Delta_n(\mu)>0$ for all $n\leqslant N$.

The sequence $\{m_n\}_{-N+1}^{N-1}$ has all its determinants positive, and by Theorem 3.4 (a) and Corollary 3.5, there is a circle Γ such that for each $z\in\Gamma$ there exists a $\mu_z\in\mathfrak{R}_{MN}$ with $\mu_z(e_{-n}) = m_n$ for $|n|\leqslant N-1$ and $\mu_z(e_N) = z$, and $\mu_z\in\mathfrak{R}_M(N)$ if z is on the boundary of Γ. Since $\Delta_N(\mu)>0$, m_N is interior to Γ, and therefore m_N = convex combination of points

z of the boundary of Γ. Therefore there exists a convex combination ν of elements from $\mathfrak{R}_M(N)$ such that $\nu(e_n) = \mu(e_n)$ for $|n| \leqslant N$. Q.E.D.

While the canonical elements of order N of $\mathscr{M}$ are discrete measures concentrated in N points, those for $\mathfrak{R}_M$ are absolutely continuous of "rational" type. On the other hand, the only simple measures of $\mathscr{M}$ are the canonical of order 1 (which are the Dirac measures) and none of the canonical measures of order N, $N > 1$, is simple. The situation is totally different for $\mathfrak{R}_M$ as shown by the following

COROLLARY 4.3. (a) *If $N = 1, 2$, every element of $\mathfrak{R}_M(N)$ is simple. If $N = 3$, none of the elements of $\mathfrak{R}_M(N)$ is simple.*

(b) *For all $M \geqslant M_0$ there are in $\mathfrak{R}_M(N)$ simple measures of order N with arbitrary high N.*

Remark. As for $N = 0$, $\Delta_0 = M-1 > 0$ and there are no canonical measures of order 0.

Proof. (a)

1° If $N = 1$, then for every $\mu \in \mathfrak{R}_M(1)$,

$$\Delta_1 = M(M-1) - M^2|m_1|^2 = 0,$$

$m_1 = \sqrt{\dfrac{M-1}{M}}\, e^{i\theta}$, so that $\mu = (\mu' + \mu'')/2$ (in $\mathscr{E}_1$) implies

$$\sqrt{\frac{M-1}{M}}\, e^{i\theta} = \frac{1}{2}\left(\sqrt{\frac{M-1}{M}}\, e^{i\theta'} + \sqrt{\frac{M-1}{M}}\, e^{i\theta''}\right)$$

which gives $\theta = \theta' = \theta''$, $\mu = \mu' = \mu''$.

2° If $N = 2$, then for every $\mu \in \mathfrak{R}_M(2)$, $\Delta_1 > 0$,

$$\Delta_2 = \begin{vmatrix} M-1 & Mm_{-1} & (M-1)m_1 \\ Mm_1 & M & Mm_2 \\ (M-1)m_1 & Mm_{-2} & M-1 \end{vmatrix} = 0,$$

$$Q = Q_2 = u_0 e_0 + u_{-1} e_{-1} + u_1 e_1,$$

$$MQ - \breve{Q} = (M-1) u_0 e_0 + M u_{-1} e_{-1} + (M-1) u_1 e_1,$$

where

$$u_0 = \begin{vmatrix} Mm_1 & M \\ (M-1)m_{-1} & Mm_{-2} \end{vmatrix}, \quad u_{-1} = \begin{vmatrix} M-1 & Mm_{-1} \\ (M-1)\ m_{-1} & Mm_{-2} \end{vmatrix},$$

$$u_1 = \begin{vmatrix} M-1 & Mm_{-1} \\ Mm_1 & M \end{vmatrix}.$$

The equations (4.4) are now

$$\begin{aligned} Mu_0 m_{-1} + Mu_{-1} m_0 + Mu_1 m_{-2} &= 0, \\ (M-1)u_0 m_0 + Mu_{-1} m_1 + (M-1)u_1 m_{-1} &= 0, \\ (M-1)u_0 m_1 + Mu_{-1} m_2 + (M-1)u_1 m_0 &= 0. \end{aligned}$$

In this special case we consider only the middle equation and its conjugate since they contain the single unknown m_1, and we have the system

$$\begin{aligned} Mu_{-1} m_1 + (M-1)u_1 m_{-1} &= -(M-1)u_0, \\ (M-1)\overline{u_1} m_1 + M\overline{u_{-1}} m_{-1} &= -(M-1)\overline{u_0}. \end{aligned}$$

The simplicity of the case makes it unnecessary to separate real and imaginary parts as in the general case. The determinant of the system is

$$\Delta_2^{(*)} = M^2 u_{-1}\overline{u_{-1}} - (M-1)^2 u_1 \overline{u_1}$$

and it is sufficient to prove that $\Delta_2^{(*)} \neq 0$ taking into account the result of Theorem 4.2 (a). An easy computation gives

$$\Delta_2^{(*)} = -M(M-1)\big(M\Delta_2 - (M-1)\Delta_1\big) = M(M-1)^2 \Delta_1 \neq 0.$$

3° If $N = 3$, then for every $\mu \in \mathfrak{R}_M(3)$, $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 = 0$. Δ_3, Δ_3^* and the system S_3 are as in Example 1. We have to show that $\Delta_3^* = 0$. Since the elements of this determinant are themselves determinants of order 3, to avoid the cumbersome direct computation we use the following relations of the minors $\{u_j\}_{j \in J_3}$ that are easy to check:

$$\text{(4.3)} \qquad \begin{aligned} Mu_{-2}\overline{u_{-2}} &= (M-1)u_1\overline{u_1}; & Mu_{-1}\overline{u_{-1}} &= (M-1)u_0\overline{u_0}; \\ Mu_{-1}\overline{u_{-2}} &= (M-1)u_1\overline{u_0}; & \overline{u_0}u_{-2} &= u_{-1}\overline{u_1}. \end{aligned}$$

Separating real and imaginary parts, these relations become

$$\begin{aligned} M(v_{-2}^2 + w_{-2}^2) &= (M-1)(v_1^2 + w_1^2), \\ M(v_{-1}^2 + w_{-1}^2) &= (M-1)(v_0^2 + w_0^2), \end{aligned}$$

$$\text{(4.3a)} \qquad \begin{aligned} M(v_{-2}w_{-1} - v_{-1}w_{-2}) &= (M-1)(v_0 w_1 - v_1 w_0), \\ M(v_{-1}v_{-2} + w_{-1}w_{-2}) &= (M-1)(v_0 v_1 + w_0 w_1), \end{aligned}$$

$$\begin{aligned} v_0 v_{-2} + w_0 w_{-2} &= v_1 v_{-1} + w_1 w_{-1}, \\ v_0 w_{-2} - w_0 v_{-2} &= v_1 w_{-1} - w_1 v_{-1}. \end{aligned}$$

Determinant Δ_3^* reduces to an order 3 determinant $v_1^{-2}|a_{jk}|$ where

$$\begin{aligned}
a_{11} &= Mv_{-1}v_1+(M-1)v_1^2-M(w_{-2}+w_0)w_{-2},\\
a_{12} &= Mw_{-1}v_1-(M-1)w_1v_1+M(v_{-2}-v_0)w_{-2},\\
a_{13} &= Mv_{-2}v_1-Mw_1w_{-2},\\
a_{21} &= Mv_1w_{-1}+(M-1)v_1w_1+M(w_0+w_{-2})v_{-2},\\
a_{22} &= -Mv_1v_{-1}+(M-1)v_1^2+M(v_0-v_{-2})v_{-2},\\
a_{23} &= Mw_{-2}v_1+Mw_1v_{-2},\\
a_{31} &= v_1v_{-2}+v_0v_1+w_0w_1+w_1w_{-2},\\
a_{32} &= v_1w_{-2}-v_1w_0+v_0w_1-v_{-2}w_1,\\
a_{33} &= v_1^2+w_1^2.
\end{aligned}$$

Using relations (4.3a), the minors of this determinant with respect to the elements a_{13}, a_{23}, a_{33}, of the last column are all multiplies of $(M-1)(v_1^2+w_1^2)-M(v_{-2}^2+w_{-2}^2)=0$, by the first of relations (4.3a).

(b) By Corollary 2.3 all $|m_n|$ are in absolute value less than or equal to $\big((M-1)/M\big)^{1/2}$, so all minors of the determinants of order N are bounded by a fixed constant. And so from 4.1a it follows that the density $w(t)$ of any canonical measure of order N is bounded by a fixed constant that depends of M and N. If all simple measures for $\mathfrak{R}_M$ were of order $\leqslant N$ for a fixed N, for all $M\geqslant 1$, we would have $w(t)$ bounded by a fixed constant C for every simple measure $d\mu = w(t)dt$. Since the convex combinations of simple elements determine $\mathscr{V}_{ML}$, this would imply $|w(t)|\leqslant C$ for every $w(t)dt\in\mathfrak{R}_M$ by letting $L\to\infty$. This contradicts the known fact (cf. [4]) that there are unbounded functions $w(t)=|t|^{-a}$ such that $w(t)dt\in\mathfrak{R}$, and therefore there is an M_0 for which $w(t)dt\in\mathfrak{R}_M$ for all $M\geqslant M_0$. Q.E.D.

Note. The relations (4.3) among minors $\{u_j\}$ extend for higher odd values of N and may be used in the corresponding decision of whether Δ_N^* is zero or not.

Example 2: (a) The simple elements of order 1 for $\mathfrak{R}_M$ are in 1-1 correspondence with the points $\xi\in(0,2\pi)$. For each $\xi\in(0,2\pi)$ the measure

$$d\mu_\xi = \frac{P^0\bar{Q}+\bar{P}^0Q+|Q|^2}{|Q|^2}\,dt,$$

where

$$Q=Q_1=M\left(\frac{M-1}{M}\right)^{1/2}e^{i\xi}-(M-1)e_{-1}(t),\qquad P^0=(M-1)e_{-1}(t)$$

(as in (2.9a)), is simple of order 1, and every simple element of order 1 is of this type.

(b) The simple elements of order 2 for $\mathfrak{R}_M$ are in 1-1 correspondence with the pairs of complex numbers (ξ_1,ξ_2), where ξ_1 is an arbitrary complex number satisfying $|\xi_1|^2<(M-1)/M$ and ξ_2 is such that

$$\begin{vmatrix} M-1 & M\bar{\xi}_1 & (M-1)\xi_1\\ M\xi_1 & M & M\xi_2\\ (M-1)\bar{\xi}_1 & M\bar{\xi}_2 & M-1\end{vmatrix}=0.$$

The corresponding measure $d\mu_{(\xi_1,\xi_2)}$ is as in Theorem 4.1a with

$$\begin{aligned}Q=Q_2=M\big(M\xi_1\bar{\xi}_2-(M-1)\bar{\xi}_1\big)-M(M-1)(\bar{\xi}_2-\bar{\xi}_1^2)e_{-1}(t)+\\ +M\big((M-1)-M|\xi_1|^2\big)e_1(t).\end{aligned}$$

5. Remarks. Let us mention briefly some questions rised by the previous considerations.

(a) There is a full grown theory of the Toeplitz forms (one aspect of which is exposed in [3]) and it may be of some interest to examine which of its results extend to quasi Toeplitz forms.

Of special interest is the extension of Bochner's theorem, as generalized by Berezanski and Maurin, for quasi Toeplitz forms.

According to Bochner's theorem if $\sum m_{n-k}\xi_n\bar{\xi}_k$ is a non negative ordinary Toeplitz form then there exists a $\mu\in\mathscr{M}$ such that

$$m_{n-k}=\int_0^{2\pi}e^{i(n-k)t}d\mu(t)=\int\gamma_{n-k}(t)\,d\mu(t),$$

where the $\sum\gamma_{n-k}(t)\,\xi_n\bar{\xi}_k$ are "elementary" forms satisfying

$$L_n\big(\gamma_{n-k}(t)\big)=\lambda\gamma_{n-k},\qquad \lambda=e^{it},$$

where $L_n(\gamma_{n-k})=\gamma_{n+1-k}$ is a special difference operator such that

$$L_n(m_{n-k})=\bar{L}_k(m_{n-k}).$$

The elementary $\{\gamma_{n-k}\}$ correspond to the Dirac (simple) measures of $\mathscr{M}$.

In the case of μ acting in $\boldsymbol{R}$ (or $\boldsymbol{R}^n$) we have similar results with the operator

$$Lf=L_xf=i\frac{\partial}{\partial x}f.$$

Berezanski and Maurin extended these results for general positive definite forms and operators L. We are interested in the determination of the operators L for the case of quasi Toeplitz forms where the elementary $\{\gamma_{n-k}\}$ correspond to the simple elements for $\mathfrak{R}_M$.

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(b) In Corollary 2.3 we saw that for all $\mu \in \mathfrak{R}_M$ and all n,

$$-\sqrt{\frac{M-1}{M}} \leqslant \mu\left(\frac{e_n+e_{-n}}{2}\right) \leqslant \sqrt{\frac{M-1}{M}}.$$

It is important to know the exact value of $\sup\{\mu((e_n+e_{-n})/2),\ \mu \in \mathfrak{R}_M\}$ and of $\inf\{\mu((e_n+e_{-n})/2),\ \mu \in \mathfrak{R}_M\}$, as well of $\sup\{\mu(Q),\ \mu \in \mathfrak{R}_M\}$, for $Q \in \mathscr{E}^R$. This corresponds to the minimum problem of the classical moment theory and can be treated as a problem of Linear Programming. We are also interested in extending to $\mathfrak{R}_M$ some of the deeper results of the classical moment theory, which may give useful information about the Riesz measures. In particular, for further study of the elements of $\mathfrak{R}_M$ it is necessary to have more information concerning the determinants Λ_N^* and the zeros of the polynomials Q_N associated with the canonical measures. (While in the case of $\mathscr{M}$ all the zeros of Q_N are located on the unit circle, in the case of $\mathfrak{R}_M$, none of the zeros of Q_N are on that circle.)

(c) We know that the order $\prec$ given by the cone $\mathscr{K}_M$ has a unit. Also the following archimedean property holds: if $F \succ 0$, $G \succ 0$ and $nF \prec G$ for $n = 1, 2, \ldots$, then $F = 0$ (because $G = nF + \sum_j (M|F_j|^2 - |\breve{F}_j|^2) = nF + \sum_j (M-1)|F_j|^2 + \sum_j (|F_j|^2 - |\breve{F}_j|^2)$, hence $\int F\,dt \leqslant (1/n)\int G\,dt$, then $\int F\,dt = 0$, and since $F = \sum_k (M|G_k|^2 - |\breve{G}_k|^2)$, we get $(M-1)\int \sum |G_k|^2\,dt \leqslant 0$, $G_k(t) = 0$ p.p., and $F = 0$ p.p.).

On the other hand, $\mathscr{K}_M$ is invariant under rotations $x \to x+\theta$ and dilations $x \to nx$, $n = 1, 2, \ldots$ It might be of interest to find all the linear automorphisms of the cone $\mathscr{K}_M$.

If we identify $\boldsymbol{R}^{2n+1} = \boldsymbol{R}^N$ with the subspace $\mathscr{E}_N \subset \mathscr{E}$ then $\mathscr{E}_N \cap \mathscr{K}_M$ is identified with a cone $\mathscr{K}_M(N)$ in $\boldsymbol{R}^N$. We may apply then some of the Koecher–Vinberg results concerning the characteristic functions and Riemannian structures associated with $\mathscr{K}_M(N)$. Though the cone $\mathscr{K}_M(N)$ is not homogeneous, many of those results will apply here and might have interesting interpretations from the point of view of moment theory in $\mathfrak{R}_M$.

(d) In the case $d\mu = dt$ the Riesz inequality (1.1) was generalized by Mazaev and Gohberg–Krein for Volterra operators in Hilbert spaces. The class $\mathfrak{R}_M$ and the problems of the present paper can be considered in Mazaev's case. There it will be natural to treat pairs of measures (μ, ν) satisfying $\int |\breve{F}|^p\,d\nu \leqslant \int M|F|^p\,d\mu$.

(e) Finally, the most important question is to extend the above considerations to L^p and $\boldsymbol{R}^n$. Since we do not know any complete exposition of the n-dimensional reduced moment problem, $n > 1$, the method of remark (a) may provide the essential tool for the extension of the theory to $\boldsymbol{R}^n$.

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