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Invariant measures for semigroups

by

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Abstract. The invariant measure problem is investigated for a strongly continuous semigroup $\Gamma = \{T_t;\ t > 0\}$ of positive linear operators on the L_1-space of a finite measure space which satisfies $\sup_{b>0} \left\| \frac{1}{b} \int_0^b T_t\,dt \right\|_1 < \infty$.

1. Introduction. Let $(X, \mathcal{M}, m)$ be a probability space and $L_p(X) = L_p(X, \mathcal{M}, m)$, $1 \leqslant p \leqslant \infty$, the usual Banach spaces. Let $\Gamma = \{T_t;\ t > 0\}$ be a strongly continuous semigroup of positvie linear operators on $L_1(X)$. Throughout this paper we shall assume that Γ satisfies

$$(*) \qquad \sup_{b>0} \left\| \frac{1}{b} \int_0^b T_t\,dt \right\|_1 < \infty.$$

With this condition on Γ, more general than $\sup_{t>0} \|T_t\|_1 < \infty$ (used by Fong and Sucheston [3] and Lin [4]), we obtain a decomposition similar to that of Fong and Sucheston ([3], Propositions 2.1 and 2.2.), and necessary and sufficient conditions for the existence of a finite equivalent invariant measure.

All sets and functions introduced below are assumed measurable. All relations are assumed to hold modulo sets of m-measure zero. For a set $A \subset X$, 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. A is said to be *closed* (under Γ) if $T_t(L_1(A)) \subset L_1(A)$ for any $t > 0$. If μ is a σ-finite measure on $(X, \mathcal{M})$ equivalent to m, then $L_1(X, \mathcal{M}, \mu)$ and $L_1(X, \mathcal{M}, m)$ are isometric by the Radon–Nikodym theorem and this gives a representation $\{U_t;\ t > 0\}$ on $L_1(X, \mathcal{M}, \mu)$ of the original $\Gamma = \{T_t;\ t > 0\}$, which preserves also pointwise convergence.

2. The decomposition for $\Gamma = \{T_t;\ t > 0\}$. It is known (cf. [2], p. 686) that for any $f \in L_1(X)$ there exists a function $T_t f(x)$, measurable with respect to the product of Lebesgue measure and m, such that for almost all t, $T_t f(x)$ belongs, as a function of x, to the equivalence class of $T_t f$. Moreover, there exists a set $N(f) \subset X$ with $m(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$ then $T_t f(x)$ is integrable over every finite interval (a, b) and the integral $\int_a^b T_t f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f dt$. We first note that a similar result holds for $\{T_t^*;\ t > 0\}$. To see this, let $f \in L_\infty(X)$ and $0 \leqslant a < b < \infty$, and let $\int_a^b T_t^* f dt$ be the function in $L_\infty(X)$ defined by

$$\left\langle u, \int_a^b T_t^* f\,dt \right\rangle = \left\langle \int_a^b T_t u\,dt, f \right\rangle$$

for all $u \in L_1(X)$. Then we have the following

Lemma 1. *For any $f \in L_\infty(X)$ there exists a function $T_t^* f(x)$, measurable with respect to the product of Lebesgue measure and m, and a set $N(f) \subset X$ with $m(N(f)) = 0$ such that if $x \notin N(f)$ then $T_t^* f(x)$ is integrable over every finite interval (a, b) and the integral $\int_a^b T_t^* f(x)\,dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t^* f dt$.*

Proof. The discussion in Lin [4], Theorem 1.1, can be modified to yield a proof of the lemma.

Theorem 1. *Γ decomposes X into two sets Y and Z such that*

(i) *Z is closed under Γ,*

(ii) *if $f \in L_1(Z)$ then* $\lim\limits_{b \uparrow \infty} \left\| \frac{1}{b} \int_0^b T_t f dt \right\|_1 = 0,$

(iii) *there exists a non-negative function s in $L_\infty(Y)$ with $s > 0$ on Y and $T_t^* s = s$ for all $t > 0$.*

Proof. If we let

$$u(x) = \limsup_{b \uparrow \infty} \frac{1}{b} \int_0^b T_t^* 1(x)\,dt,$$

then an easy calculation shows that $u \in L_\infty(X)$ and $T_t^* u \geqslant u$ for any $t > 0$. Thus we can define

$$s(x) = \lim_{b \uparrow \infty} \frac{1}{b} \int_0^b T_t^* u(x)\,dt.$$

It follows that $0 \leqslant s \in L_\infty(X)$ and $T_t^* s = s$ for any $t > 0$. Put $Y = \{x \in X;\ s(x) > 0\}$ and $Z = X - Y$. If $0 \leqslant f \in L_1(Z)$, then by Fatou's lemma,

$$\lim_{b \uparrow \infty} \left\| \frac{1}{b} \int_0^b T_t f\,dt \right\|_1 = \lim_{b \uparrow \infty} \left\langle f, \frac{1}{b} \int_0^b T_t^* 1(x)\,dt \right\rangle \leqslant \int f u\,dm \leqslant \int f s\,dm = 0.$$

Thus (ii) follows. (i) is clear. The proof is complete.

Corollary 1. *A necessary and sufficient condition that there exists a function s in $L_\infty(X)$ with $s > 0$ and $T_t^* s = s$ for all $t > 0$ is that for any $0 \leqslant g \in L_1(X)$ with $\|g\|_1 > 0$,*

$$\limsup_{b \uparrow \infty} \left\| \frac{1}{b} \int_0^b T_t g\,dt \right\|_1 > 0.$$

Remark. It follows from Theorem 1 and [3] that

(i) if $f \in L_1(X)$ and $0 \leqslant g \in L_1(X)$, then the ratio limit

$$\lim_{b \uparrow \infty} \Big(\int_0^b T_t f(x)\,dt \Big) \Big/ \Big(\int_0^b T_t g(x)\,dt \Big)$$

exists and is finite a.e. on $Y \cap \{x \in X;\ \int_0^\infty T_t g(x)\,dt > 0\}$,

(ii) if there exists a non-negative function $g \in L_1(Z)$ such that the set $C(g) = \{x \in X;\ \int_0^\infty T_t g(x)\,dt = \infty\}$ is non-null, then the ratio theorem fails on every non-null subset of $C(g)$.

Local ergodic theorems can also be proved on Y as in [3] (see [1]).

3. Invariant measures. Let Y, Z, and s be as in Theorem 1. Let $0 < h \in L_1(X)$ be invariant under $\Gamma = \{T_t;\ t > 0\}$, i.e., $T_t h = h$ for all $t > 0$. Define, for $t > 0$ and $f \in L_\infty(X)$,

$$S_t f = T_t^* f.$$

Since $\|(S_t f) h\|_1 = \int (T_t^* f) h\,dm = \int f h\,dm = \|fh\|_1$ for $0 \leqslant f \in L_\infty(X)$, S_t may be considered to be a positive linear contraction on $L_1(X, \mathscr{M}, \mu)$, where $\mu = h dm$, for each $t > 0$. It is clear that $S_t S_{t'} = S_{t+t'}$ for t, $t' > 0$.

Lemma 2. *$\Delta = \{S_t;\ t > 0\}$ is a strongly continuous semigroup of positive linear contractions on $L_1(X, \mathscr{M}, \mu)$ such that*

(i) *The conservative part of Δ is Y, and the dissipative part of Δ is Z,*

(ii) *$\nu = s d\mu$ is a finite invariant measure supported on Y.*

Proof. The strong continuity of Δ follows as in [4], Theorem 4.1. Since $S_t s = T_t^* s = s$ for all $t > 0$, $\nu = s\,d\mu$ is a finite invariant measure and, if $X = C + D$ denotes Hopf's decomposition for Δ (cf. [4]), then $Y \subset C$. To see that $Z \subset D$, we may assume, using the Radon–Nikodym theorem, that $m = \mu$. So $T_t 1 = 1$ and $S_t^* f = T_t f$ for all $t > 0$ and $f \in L_\infty(X)$. Hence $S_t^* 1_Z = T_t 1_Z \leqslant 1_Z$ for $t > 0$. Put

$$g(x) = \lim_{b \uparrow \infty} S_n^* 1_Z(x) \qquad (n = 1, 2, \ldots).$$

It follows that

$$\lim_{b\uparrow\infty}\left\|g-\frac{1}{b}\int_0^b T_t 1_Z\,dt\right\|_1=0,$$

and Theorem 1 implies that $g=0$ a.e. It follows that Z is contained in the dissipative part D_1 of the operator S_1. But, since $D=D_1$ [4], we conclude that $Z\subset D$, and the proof is complete.

COROLLARY 2. *If $\Gamma=\{T_t;\ t>0\}$ has a finite equivalent invariant measure then the following hold:*

(i) *For any $f\in L_\infty(X)$ there exists a function $T_t^*f(x)$, measurable with respect to the product of Lebesgue measure and m, such that for almost all t, $T_t^*f(x)$ belongs, as a functions of x, to the equivalence class of T_t^*f.*

(ii) *For any $f\in L_\infty(X)$, the limits*

$$\lim_{b\uparrow\infty}\frac{1}{b}\int_0^b T_t^*f(x)\,dt \quad \textit{and} \quad \lim_{b\downarrow 0}\frac{1}{b}\int_0^b T_t^*f(x)\,dt$$

exist and are finite a.e.

(iii) *For any $f\in L_\infty(X)$ and any $0\leqslant g\in L_\infty(X)$, the limit*

$$\lim_{b\uparrow\infty}\Big(\int_0^b T_t^*f(x)\,dt\Big)\Big/\Big(\int_0^b T_t^*g(x)\,dt\Big)$$

exists and is finite a.e. *on* $\{x\in X;\ \int_0^\infty T_t^*g(x)\,dt>0\}$.

Next let us assume $X=Y$, and define another semigroup $\Gamma'=\{V_t;\ t>0\}$ of positive linear contractions on $L_1(X)=L_1(X,\mathscr{M},m)$ as in Fong and Sucheston [3]. For $t>0$ and $sf\in L_1(X)$, where $f\in L_1(X)$, let

$$V_t(sf)=s(T_tf).$$

Since $\|V_t(sf)\|_1=\int s(T_tf)\,dm=\int sf\,dm=\|sf\|_1$ for $0\leqslant f\in L_1(X)$ and $\{sf;\ f\in L_1(X)\}$ is a dense subspace of $L_1(X)$ in the norm topology, V_t may be considered to be a positive linear contraction on $L_1(X)$ for any $t>0$. It is easy to see that (i) $V_tV_{t'}=V_{t+t'}$ for $t,\ t'>0$, and (ii) the mapping $t\to V_tf$ is strongly continuous for each $f\in L_1(X)$.

LEMMA 3. *The following conditions are equivalent.*

(i) *There exists a function $f_0\in L_1(X)$ with $f_0>0$ and $T_tf_0=f_0$ for all $t>0$.*

(ii) *There exists a function $g_0\in L_1(X)$ with $g_0>0$ and $V_tg_0=g_0$ for all $t>0$.*

Proof. (i)$\Rightarrow$(ii): Obvious.

(ii)$\Rightarrow$(i): We may assume without loss of generality that $g_0=1$. An easy calculation shows that

$$(1)\qquad m(A)>0 \quad\text{implies}\quad \inf_t\int_A T_t1\,dm>0.$$

Let φ be an invariant mean on the additive semigroup $(0,\infty)$ and define a positive linear functional Ψ on $L_\infty(X)$ by the relation

$$\Psi(u)=\varphi\left(\left\langle\frac{1}{b}\int_0^b T_t1\,dt,\ u\right\rangle\right)\quad\text{for}\quad u\in L_\infty(X).$$

If $T_{t_0}^{**}$ denotes the adjoint of $T_{t_0}^*$ then we have

$$(T_{t_0}^{**}\Psi-\Psi)(u)=\Psi(T_{t_0}^*u-u)=\varphi\left(\frac{1}{b}\int_0^b\langle T_t1,\ T_{t_0}^*u-u\rangle\right)\geqslant 0$$

for each $0\leqslant u\in L_\infty(X)$. Thus $T_{t_0}^{**}\Psi\geqslant\Psi$. But, since $T_{t_0}^*s=s$, $(T_{t_0}^{**}\Psi-\Psi)(s)=0$. Since $s>0$, it follows that if λ is a countably additive measure on $(X,\mathscr{M})$ satisfying $0\leqslant\lambda\leqslant(T_{t_0}^{**}\Psi-\Psi)$, then $\lambda=0$. Hence if we denote by μ the maximal (countably additive) measure with $0\leqslant\mu\leqslant\Psi$, then $T_{t_0}^{**}\mu=T_{t_0}\mu\leqslant\mu$, and hence $T_{t_0}\mu=\mu$, since $T_{t_0}^*s=s$ and $s>0$. Now put $f_0=d\mu/dm$. It follows from (1) and [5], Theorem 4, that $f_0>0$. This completes the proof.

THEOREM 2. *The following conditions are equivalent:*

(i) *There exists a function $h\in L_1(X)$ with $h>0$ and $T_th=h$ for all $t>0$.*

(ii) $m(A)>0$ *implies* $\liminf_{b\uparrow\infty}\frac{1}{b}\int_0^b T_t^*1_A(x)\,dt\not\equiv 0.$

(iii) $m(A)>0$ *implies* $\liminf_{b\uparrow\infty}\frac{1}{b}\int_0^b\langle 1,\ T_t^*1_A\rangle\,dt>0.$

(iv) $m(A)>0$ *implies* $\limsup_{b\uparrow\infty}\frac{1}{b}\int_0^b\langle 1,\ T_t^*1_A\rangle\,dt>0.$

(v) $m(A)>0$ *implies* $\limsup_{b\uparrow\infty}\frac{1}{b}\int_0^b T_t^*1_A(x)\,dt\not\equiv 0.$

Proof. (i)$\Rightarrow$(ii): We may assume without loss of generality that $h=1$. Then, by Theorem 1,

$$\lim_{b\uparrow\infty}\left\|\frac{1}{b}\int_0^b T_t1_Y\,dt-1\right\|_1=0.$$

Hence if $m(A)>0$ then

$$\Big\langle\int_0^b T_t1_Y\,dt,\ 1_A\Big\rangle=\int_0^b\langle 1_Y,\ T_t^*1_A\rangle\,dt>0$$

for sufficiently large b. This implies that for some t_0 and $0 \leqslant f \in L_\infty(Y)$ with $\|f\|_\infty > 0$, $T_{t_0} 1_A \geqslant f$. Corollary 2 implies that

$$\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* 1_A(x)\,dt \geqslant \lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* f(x)\,dt \not\equiv 0,$$

since $\left\langle 1, \frac{1}{b} \int_0^b T_t^* f(x)\,dt \right\rangle = \langle 1, f\rangle \neq 0$ for any $b > 0$.

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) follow from Fatou's lemma, and (iii) $\Rightarrow$ (iv) is obvious.

(v) $\Rightarrow$ (i): It may be readily seen from Lemma 3 and [4], Theorem 5.3, that there exists a function $f \in L_1(Y)$ with $f > 0$ on Y and $T_t f \geqslant f$ for any $t > 0$. Let

$$g = \text{strong-}\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t f\,dt \quad \text{and} \quad E = X - \operatorname{supp} g.$$

It is clear that $T_t g = g$ for all $t > 0$ and $E \subset Z$. Since $T_t^* 1_E \in L_\infty(E)$ for all $t > 0$, $\int_0^b T_t^* 1_E(x)\,dt = 0$ on $X - E$ for any $b > 0$. This together with the fact that $s = 0$ on Z implies that

$$\lim_{b\uparrow\infty} \frac{1}{b} \int_0^b T_t^* 1_E(x)\,dt = 0 \text{ a.e.}$$

(v) implies $m(E) = 0$, and the proof is complete.

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Local eigenvectors for group representations

by

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Abstract. We prove that every unitary representation V of a group G has a local eigenvector (i.e., a common eigenvector for all $V(g)$, g ranging over a neighborhood of the identity) if and only if G_0, the connected component of the identity, is compact and abelian. It follows as a simple corollary that for G_0 compact and abelian, cocycle representations of G also have local eigenvectors. The proof uses Mackey's little group method.

Let V be a unitary or cocycle representation of a locally compact group G on a Hilbert space $\mathscr{H}$. A non-zero vector x in $\mathscr{H}$ is a local eigenvector (respectively, local fixed point) for V if x is a common eigenvector (respectively, fixed point) for all the unitary operators $V(g)$, as g ranges over some neighborhood of the identity e in G. It is known that all unitary representations of G have local fixed points if and only if G is totally disconnected. We extend this result by proving that all unitary representations of G have local eigenvectors if and only if G_0, the connected component of the identity, is compact and abelian. The proof uses a preliminary lemma that in fact all cocycle representations of totally disconnected groups have local eigenvectors, and as a simple corollary to the main theorem we show that indeed so do all cocycle representations of group G with G_0 compact and abelian. These results have application in determining the structure space of certain C^*-algebras associated with transformation groups ([3], Theorem 4.4).

As the proof of the main theorem involves application of Mackey's little group method, we assume that all groups are second countable and all Hilbert spaces are separable. All unitary representations are continuous and all cocycle representations are Borel. A cocycle representation with cocycle a will be called simply an *a-representation*. For terminology and basic results on cocycle representations we refer the reader to [1], Chapter I, Section 4. Throughout the paper we shall use without further explicit mention the simple observations that a local eigenvector for a unitary or cocycle representation V is a common eigenvector for all $V(k)$ as k ranges over some open subgroup containing G_0, and that if a, the cocycle of V, is cohomologous on an open subgroup K to a cocycle a' of K it suffices, in order to prove the existence of local eigenvectors, to replace V by the