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Regular and coregular mappings of differential spaces

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Abstract. The concepts of regularity, coregularity, weak regularity, weak coregularity, in the category of differential spaces, are introduced and some of their properties are examined. It is proved that the mapping (4), where $\mathscr{C}$ is the differential structure induced from $\mathscr{D}$ by the function f is open and that every weak coregular mapping is open. Some characterizations of differential structure induced by f (coinduced by f) are obtained and it is shown that if $\mathscr{C}$ is the differential structure induced from $\mathscr{D}$ by f, then (4) is differentially regular and differentially coregular. It is stated that regularity of mappings (4) and (26) and the coincidence of topologies induced by $\mathscr{C}$ and $\mathscr{C}'$ on the set M imply $\mathscr{C} = \mathscr{C}'$. Similarly, the requirement that mapping (4) is coregular assures that so is $\mathscr{D}$. A certain theorem useful for an examination of coregularity of the natural mapping of an equivalence relation is proved.

1. Introductory remarks. The present paper is devoted to an examination of some concepts of regularity and coregularity of mappings. These concepts are equivalent (cf. Serre [4]) in the category of manifolds. They are not equivalent in the larger category of all differential spaces. The main aim of the paper is to give some properties and to point out distinctions occurring between those related notions. First we recall the concept of a differential space and some of its properties.

By a *differential space* (cf. R. Ciampa [1], S. MacLane [2], R. Sikorski [5] and [6]) we mean the pair $(M, \mathscr{C})$, where M is an arbitrary set and $\mathscr{C}$ is a set of real functions defined on M closed with respect to localization and superposition with all functions of class C^∞ on the Cartesian spaces. The set C is called the *differential structure* of this space. More exactly, let us denote by $\tau_{\mathscr{C}}$ the weakest topology such that all functions of $\mathscr{C}$ are continuous. For any subset A of M we denote by $\mathscr{C}_A$ the set of all real functions a defined on A such that for any $p \in A$ there exists a set $B \in \tau_{\mathscr{C}}$ with $p \in B$ and a function $\beta \in \mathscr{C}$ satisfying the equality $a|(A \cap B) = \beta|(A \cap B)$. Closedness with respect to localization may be expressed in the form $\mathscr{C}_M = \mathscr{C}$. Denote by sc$\mathscr{C}$ the smallest of sets $\mathscr{C}'$ such that $\mathscr{C} \subset \mathscr{C}'$ and with the following property: if $a_1, \ldots, a_s$ belong to $\mathscr{C}'$, where s is any positive integer and φ is of class $C^\infty(\boldsymbol{R}^s)$, then the function

$$M \ni p \mapsto \varphi\big(a_1(p), \ldots, a_s(p)\big) \tag{1}$$

belongs to $\mathscr{C}'$. Closedness with respect to superposition with all functions of class C^∞ on Cartesian spaces means $\mathrm{sc}\mathscr{C} = \mathscr{C}$.

Let us consider an algebraic closure, i.e. an operation a which to every subset of some set assigns a subset of this set in such a way that:

(i) $\mathscr{C} \subset a(\mathscr{C})$;

(ii) if $\mathscr{C}' \subset \mathscr{C}$, then $a(\mathscr{C}') \subset a(\mathscr{C})$;

(iii) $a\big(a(\mathscr{C})\big) = a(\mathscr{C})$.

We prove the following lemma about two algebraic closures.

1.1. *If a and b are two closures defined on the set of all subsets of the set $\mathscr{E}$ satisfying for any subset $\mathscr{C}$ of $\mathscr{E}$ the inclusion*

$$a\big(b(C)\big) \subset b\big(a(C)\big), \tag{2}$$

then the superposition

$$\mathscr{C} \mapsto b\big(a(\mathscr{C})\big) \tag{3}$$

of these operations is an algebraic closure and for any $\mathscr{C} \subset \mathscr{E}$ the set $b\big(a(\mathscr{C})\big)$ is the smallest of sets $\mathscr{C}'$ containing $\mathscr{C}$ and closed with respect to the operations a and b.

Proof. It is easy to verify that the operation (3) is an algebraic closure. Consider any subset $\mathscr{C}$ of $\mathscr{E}$. Then, by (2), we have

$$a\Big(b\big(a(\mathscr{C})\big)\Big) \subset b\Big(a\big(a(\mathscr{C})\big)\Big) = b\big(a(\mathscr{C})\big) \quad \text{and} \quad b\Big(b\big(a(\mathscr{C})\big)\Big) = b\big(a(\mathscr{C})\big).$$

The set $b\big(a(\mathscr{C})\big)$ is closed with respect to a and b. If $\mathscr{C} \subset \mathscr{C}' \subset \mathscr{E}$ and $\mathscr{C}'$ is closed with respect to a and b, then

$$b\big(a(\mathscr{C})\big) \subset b\big(a(\mathscr{C}')\big) = b(\mathscr{C}') = \mathscr{C}',$$

which ends the proof.

As a corollary we have (cf. MacLane [2]):

1.2. *For any set $\mathscr{C}$ of real functions defined on M the set $(\mathrm{sc}\mathscr{C})_M$ is the smallest of the sets $\mathscr{C}'$ of real functions defined on M such that $\mathscr{C} \subset \mathscr{C}'$ and $(M, \mathscr{C}')$ is a differential space.*

Proof. It is easy to verify that the operations

$$\mathscr{C} \mapsto \mathscr{C}_M \quad \text{and} \quad \mathscr{C} \mapsto \mathrm{sc}\mathscr{C}$$

are algebraic closures such that

$$\mathrm{sc}(\mathscr{C}_M) \subset (\mathrm{sc}\mathscr{C})_M.$$

According to Lemma 1.1 the proof is finished.

The differential space $\big(M, (\mathrm{sc}\mathscr{C})_M\big)$ is called the *space generated by the set $\mathscr{C}$* of real functions. MacLane remarked that the differential space generated by $\mathscr{C}$ may be obtained as above by means of two closures. For the direct proof of the existence of this space see Sikorski [5].

Let $(M_1, \mathscr{C}_1)$ and $(M_2, \mathscr{C}_2)$ be differential spaces. The differential space $(M_1, \mathscr{C}_1)\times(M_2, \mathscr{C}_2)$ generated by the set

$$\{\alpha\circ\mathrm{pr}_1;\ \alpha\in\mathscr{C}_1\}\cup\{\alpha\circ\mathrm{pr}_2;\ \alpha\in\mathscr{C}_2\}$$

is called the *Cartesian product* of $(M_1, \mathscr{C}_1)$ and $(M_2, \mathscr{C}_2)$, where $\mathrm{pr}_i(u_1, u_2) = u_i$ for $(u_1, u_2)\in M_1\times M_2$, $i = 1, 2$.

We say that a function f defined on the set M and having values in the set N maps *smoothly* the differential space $(M, \mathscr{C})$ into $(N, \mathscr{D})$, in symbols:

$$f\colon (M, \mathscr{C})\to(N, \mathscr{D}), \tag{4}$$

iff for every $\beta\in\mathscr{D}$ the function $\beta\circ f\in\mathscr{C}$. The mapping (4) is said to be a *diffeomorphism* if f maps M one-to-one and onto, and

$$f^{-1}\colon (N, \mathscr{D})\to(M, \mathscr{C}).$$

This fact will also be written in the form $f\colon (M, \mathscr{C})\leftrightarrow(N, \mathscr{D})$.

From the definition of the Cartesian product of differential spaces it immediately follows that

1.3. *The mapping*

$$f\colon (M, \mathscr{C})\to(M_1, \mathscr{C}_1)\times(M_2, \mathscr{C}_2)$$

is smooth if and only if

$$\mathrm{pr}_i\circ f\colon (M, \mathscr{C})\to(M_i, \mathscr{C}_i)$$

is smooth for $i = 1, 2$. In particular, for $i = 1, 2$,

$$\mathrm{pr}_i\colon (M_1, \mathscr{C}_1)\times(M_2, \mathscr{C}_2)\to(M_i, \mathscr{C}_i).$$

Now, we suppose that the sets M_1 and M_2 are disjoint. We shall define the *disjointed union* $(M_1, \mathscr{C}_1)\oplus(M_2, \mathscr{C}_2)$ of the differential spaces $(M_1, \mathscr{C}_1)$ and $(M_2, \mathscr{C}_2)$ as follows. Let $\mathscr{C}$ be the set of all real functions α defined on $M_1\cup M_2$ such that $\alpha|M_1$ and $\alpha|M_2$ belong to $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively. From the definition of the set $\mathscr{C}$ it immediately follows that $\mathrm{sc}\mathscr{C} = \mathscr{C}$. Let $\alpha\in\mathscr{C}_M$ and $p\in M_1$. Then there exists a neighbourhood A of p open in $\tau_{\mathscr{C}}$ such that $\alpha|A = \alpha'|A$ for some function $\alpha'\in\mathscr{C}$. Thus, there exist functions $\alpha_1, \ldots, \alpha_s$ of $\mathscr{C}$ and real numbers $a_1, b_1, \ldots, a_s, b_s$ such that $p\in A_0\subset A$, where

$$A_0 = \bigcap_{i=1}^{s}\alpha_i^{-1}[(a_i;\ b_i)]\subset A.$$

Then we have $\alpha|A_1 = \alpha'|A_1 = (\alpha'|M_1)|A_1$, where $\alpha_i|M_1\in\mathscr{C}_1$ and

$$A_1 = \bigcap_{i=1}^{s}(\alpha_i|M_1)^{-1}[(a_i;\ b_i)] = A_0\cap M_1\subset A\cap M_1.$$

This yields $\alpha|M_1 \in (\mathscr{C}_1)_{M_1} = \mathscr{C}_1$. Similarly, $\alpha|M_2 \in \mathscr{C}_2$. Then $\mathscr{C}_M \subset \mathscr{C}$. Therefore, $(M, \mathscr{C})$ is a differential space called the *disjointed union* of $(M_1, \mathscr{C}_1)$ and $(M_2, \mathscr{C}_2)$. The topological space of the disjointed union of these differential spaces is the disjointed union of the topological spaces $(M_1, \tau_{\mathscr{C}_1})$ and $(M_2, \tau_{\mathscr{C}_2})$. Analogously the disjointed union of an arbitrary set of differential spaces may be defined.

Let us remark that the operation of producting leads from differentiable manifolds to a differentiable manifold.

The verification of smoothness of a mapping is faciliated by the following lemma.

1.4. *If $(M, \mathscr{C})$ is a differential space, $f\colon M \to N$, $\mathscr{D}'$ is a set of real functions defined on N, then from the condition*

(i) *if $\beta \in \mathscr{D}'$, then $\beta \circ f \in \mathscr{C}$,*

it follows that:

(ii) *if $\beta \in \mathrm{sc}\mathscr{D}'$, then $\beta \circ f \in \mathscr{C}$;*

(iii) *if $\beta \in \mathscr{D}'_N$, then $\beta \circ f \in \mathscr{C}$.*

Then, the mapping (4), *where $(N, \mathscr{D})$ is a differential space generated by $\mathscr{D}'$, is smooth if and only if condition* (i) *is satisfied.*

For the proof see [6].

2. The differential structure induced and coinduced by a mapping. Let $\mathscr{D}$ be any set of real functions defined on a set N. For an arbitrary function

$$f\colon M \to N \tag{5}$$

we may define the function f^* on the set of all real functions defined on N by the formula

$$f^*(\beta) = \beta \circ f. \tag{6}$$

So, the set $f^*[\mathscr{D}]$ being the image of the set $\mathscr{D}$ under f^* is a set of real functions defined on M.

2.1. *If $(N, \mathscr{D})$ is a differential space, then for every mapping* (5)

$$\big(M, (f^*[\mathscr{D}])_M\big) \tag{7}$$

is a differential space such that

$$f\colon \big(M, (f^*[\mathscr{D}])_M\big) \to (N, \mathscr{D}). \tag{8}$$

The set $(f^*[\mathscr{D}])_M$ is the smallest of sets $\mathscr{C}$ such that $(M, \mathscr{C})$ is a differential space and (4) *is smooth.*

The topology $\tau_{f^*[\mathscr{D}]}$ of the differential space (7) *coincides with the topology induced from the topological space $(N, \tau_{\mathscr{D}})$ by* (5).

The mapping (8) *is open, i.e. for every set A open in $\tau_{f^*[\mathscr{D}]}$ the set $f[A]$ is open in $\tau_{\mathscr{D}}$.*

Proof. From the inclusion $f^*[\mathscr{D}] \subset (f^*[\mathscr{D}])_M$ and the identity $f^*[\mathscr{D}] = \mathrm{sc}f^*[\mathscr{D}]$ it follows that the function f maps smoothly the differential space (7) into $(N, \mathscr{D})$. Now, let us suppose that (4) holds. Consider any real function $\alpha \in (f^*[\mathscr{D}])_M$. Let us take $p \in M$. Then there is a function $\alpha' \in f^*[\mathscr{D}]$ and a set $A \in \tau_{f^*[\mathscr{D}]}$ such that $\alpha|A = \alpha'|A$, $p \in A$. Therefore, we have $\alpha' = \beta \circ f$, where $\beta \in \mathscr{D}$, and $A = f^{-1}[B]$, where $B \in \tau_{\mathscr{D}}$. From (4) it follows that $\alpha' \in \mathscr{C}$ and $A \in \tau_{\mathscr{C}}$. Hence it follows that $\alpha \in \mathscr{C}_M = \mathscr{C}$. So $f^*[\mathscr{D}] \subset \mathscr{C}$. The proof of the third assertion of the lemma is obtained by an easy verification.

For the proof of the openness of the mapping (8) take any set A open in $\tau_{f^*[\mathscr{D}]}$. Let $q \in f[A]$. Thus $f(p) = q$ for some $p \in A$. Then there exist functions $\alpha_1, \ldots, \alpha_s \in f^*[\mathscr{D}]$ and reals $a_1, b_1, \ldots, a_s, b_s$ such that

$$p \in \bigcap_{i=1}^{s} \alpha_i^{-1}[(a_i;\ b_i)] \subset A. \tag{9}$$

The definition of f^* yields the existence of functions $\beta_1, \ldots, \beta_s \in \mathscr{D}$ such that $\alpha_i = \beta_i \circ f$, $i = 1, \ldots, s$. Hence, by (9), we get

$$p \in f^{-1}\Big[\bigcap_{i=1}^{s} \beta_i^{-1}[(a_i;\ b_i)]\Big] \subset A.$$

So,

$$q = f(p) \in \bigcap_{i=1}^{s} \beta_i^{-1}[(a_i;\ b_i)] \subset f[A].$$

Thus, the set $f[A]$ is open in $\tau_{\mathscr{D}}$.

The differential space (7) will be called the *space induced from* $(N, \mathscr{D})$ *by the mapping* (5).

2.2. *If* $(M, \mathscr{C})$ *is a differential space, then for any mapping* (5)

$$(N, f^{*-1}[\mathscr{C}]) \tag{10}$$

is a differential space and the set $f^{*-1}[\mathscr{C}]$ *is the greatest of differential structures* $\mathscr{D}$ *such that the mapping* (4) *is smooth.*

Proof. From the definition of f^* we immediately obtain the equality $\mathrm{sc}f^{*-1}[\mathscr{C}] = f^{*-1}[\mathscr{C}] = (f^{*-1}[\mathscr{C}])_N$. Hence, by Lemma 1.2, it follows that (10) is a differential space. For any $\beta \in f^{*-1}[\mathscr{C}]$ we have $\beta \circ f \in \mathscr{C}$.

Now, let $\mathscr{D}$ be a differential structure such that the mapping (4) is smooth. Consider any $\eta \in \mathscr{D}$. Hence it follows that $f^*(\eta) = \eta \circ f \in \mathscr{C}$. Thus $\eta \in f^{*-1}[\mathscr{C}]$. In other words, $\mathscr{D} \subset f^{*-1}[\mathscr{C}]$. The proof is finished.

The differential space (10) will be called the *space coinduced from* $(M, \mathscr{C})$ *by the mapping* (5). The topology $\tau_{f^{*-1}[\mathscr{C}]}$ of the differential space coinduced from $(M, \mathscr{C})$ by (5) does not always coincide with the topology coinduced by (5) from the topological space $(M, \tau_{\mathscr{C}})$. Below there is given an example which illustrates such a situation.

EXAMPLE 1. Let us set $M = \boldsymbol{R}\times\{0\}\cup\boldsymbol{R}\times\{1\}$ and define the relation R as follows:

$$pRq \textit{ iff } p, q \in M \quad \textit{and} \quad p^1 = q^1 \neq 0, \quad \textit{where } p = (p^1, p^2),\ q = (q^1, q^2).$$

It is easy to verify that R is reflexive, symmetric and transitive. Let f be the natural mapping of R, i.e. the mapping (5), where N is the set of all cosets of R and to each p of M the mapping f assigns the coset $f(p)\in N$ such that $p\in f(p)$. The topology coinduced by mapping (5) is not a Hausdorff topology. It is easy to state that for every point of N there exists a neighbourhood homeomorphic to R with usual topology. Therefore, N with the topology coinduced by (5) is a T_1-space, i.e. for any different points x and y of N there exists a neighbourhood U of x open in $(M, \tau_{\mathscr{C}})/R$ ($= N$ with the topology coinduced by (5)) such that $y\notin U$. Hence it follows that such a topology must not be a topology of the form $\tau_{\mathscr{D}}$, where $\mathscr{D}$ is any set of real functions defined on N. This yields, in particular, that the topology $\tau_{f^{\bullet-1}[\mathscr{C}]}$ of the differential space coinduced from $(M, \mathscr{C})$ by (5) is different from the topology coinduced by (5) from the topological space $(M, \tau_{\mathscr{C}})$.

By a direct verification we see that any neighbourhood of the point $a_0 = \{(0, 0)\}$ in the topology of (10) includes the point $a_1 = \{(0, 1)\}$ of N. That also shows the difference between the topology coinduced from $(M, \tau_{\mathscr{C}})$ by (5) and the topology of differential space (10) coinduced by the same mapping.

Now, we shall prove properties which characterize the differential structure of the differential space induced or coinduced by the mapping (5).

2.3. *If* $(N, \mathscr{D})$ *is a differential space, then for every mapping* (5) *there exists exactly one differential structure* $\mathscr{C}$ *on* M *such that the mapping* (4) *is smooth and for every differential space* $(M', \mathscr{C}')$ *and for every mapping*

$$g\colon M'\to M \tag{11}$$

there is a smooth mapping

$$g\colon (M', \mathscr{C}')\to(M, \mathscr{C}) \tag{12}$$

if and only if the mapping

$$f\circ g\colon (M', \mathscr{C}')\to(N, \mathscr{D}) \tag{13}$$

is smooth. The differential space $(M, \mathscr{C})$ *coincides with the space* (7).

Similarly, if $(M, \mathscr{C})$ *is a differential space, then for every mapping* (5) *there exists exactly one differential structure* $\mathscr{D}$ *on* N *such that the mapping* (4) *is smooth and for every differential space* $(N', \mathscr{D}')$ *and for every mapping*

$$h\colon N\to N' \tag{14}$$

there is a smooth mapping

$$h\colon (N, \mathscr{D}) \to (N', \mathscr{D}')$$

if and only if the mapping

$$h \circ f\colon (M, \mathscr{C}) \to (N', \mathscr{D}') \tag{15}$$

is smooth. The differential space $(N, \mathscr{D})$ *coincides with the space* (10).

Proof. Let us suppose that $(N, \mathscr{D})$ is a differential space and $\mathscr{C}$ has the property mentioned, in the first part of the theorem and let a differential structure $\mathscr{C}_1$ has the same property, i.e. the mapping

$$f\colon (M, \mathscr{C}_1) \to (N, \mathscr{D})$$

is smooth and for every differential space $(M', \mathscr{C}')$ and for mapping (11) the mapping

$$g\colon (M', \mathscr{C}') \to (M, \mathscr{C}_1)$$

is smooth if and only if the mapping (13) is smooth. Hence it follows that

$$\mathrm{id}\colon (M, \mathscr{C}) \to (M, \mathscr{C}_1)$$

is smooth. Then $\mathscr{C}_1 \subset \mathscr{C}$. Similarly, $\mathscr{C} \subset \mathscr{C}_1$. Thus $\mathscr{C} = \mathscr{C}_1$. To end the first part of the theorem it sufficies to prove that the differential space (7) has the property in question. By 2.1, the mapping (8) is smooth. Let (11) be any mapping for which the mapping (13) is smooth. In other words, $\beta \circ f \circ g \in \mathscr{C}'$ when $\beta \in \mathscr{D}$. Let $\gamma \in f^*[\mathscr{D}]$. Then $\gamma = \beta \circ f$, where $\beta \in \mathscr{D}$. Hence $\gamma \circ g \in \mathscr{C}'$. By Lemma 1.4 we get the smooth mapping

$$g\colon (M', \mathscr{C}') \to \big(M, (f^*[\mathscr{D}])_M\big).$$

The proof of the first part of the theorem is completed. The proof of the second one is analogous.

As an immediate consequence of 2.3 we get

2.4. *For every mapping* (5) *the following conditions are equivalent*:

(a) *The mapping* (4) *is smooth*;

(b) $\mathscr{D}$ *is a differential structure contained in* $f^{*-1}[\mathscr{C}]$;

(c) $\mathscr{C}$ *is a differential structure containing* $(f^*[\mathscr{D}])_M$.

Similarly, for every one-to-one and onto mapping (5) *the following conditions are equivalent*:

(a′) *The mapping* (4) *is a diffeomorphism*;

(b′) $\mathscr{D}$ *coincides with* $f^{*-1}[\mathscr{C}]$;

(c′) $\mathscr{C}$ *coincides with* $f^*[\mathscr{D}]$.

If M is an arbitrary set, then the set of all functions c_M, where $c_M(p) = c$ for $p \in M$, c is any real number, forms a differential structure on M, the smallest one. The set of all real functions defined on M is also a differential structure on M, the greatest one.

2.5. *For any mapping* (5) *on* M *the set* $(f^*[\mathscr{C}])_{N-f[M]}$ *is the greatest of all differential structures on* $N-f[M]$. *The set* $f[M]$ *as well as any subset of* $N-f[M]$ *is open in the topology* $\tau_{f^{*-1}[\mathscr{C}]}$.

Proof. Set $\mathscr{D} = f^{*-1}[\mathscr{C}]$. Let γ be an arbitrary real function defined on $N-f[M]$. Then the function defined by the formula

$$\gamma_0(q) = \begin{cases} \gamma(q), & \text{where } q \in N-f[M], \\ 0, & \text{where } q \in f[M], \end{cases}$$

fulfils the equality $(\gamma_0 \circ f)(q) = 0$ for $q \in M$. Hence, $\gamma_0 \circ f \in \mathscr{C}$. In other words, $\gamma_0 \in f^{*-1}[\mathscr{C}]$. This yields

$$\gamma = \gamma_0 | (N-f[M]) \in (f^{*-1}[\mathscr{C}])_{N-f[M]}.$$

To prove the second assertion take $q \in N-f[M]$ and the characteristic function of the set $\{q\}$. We have $\chi \in f^{*-1}[\mathscr{C}]$ and

$$\{q\} = \chi^{-1}[(0;\ 2)] \in \tau_{f^{*-1}[\mathscr{C}]}.$$

3. Regularity and coregularity, weak regularity and weak coregularity. Every differentiable Hausdorff manifold of class C^∞ induces on the set M of all its points the differential structure. If the induced differential structures of such manifolds coincide, then these differentiable manifolds coincide. Then, under the above assumptions we need not make of distinction between a manifold and its differential space. We introduce the concepts of regularity and weak regularity. These concepts are there compared with the notion of differential regularity (called by R. Sikorski [6] simply: *regularity*). Further, we shall be concerned with the dual concepts of coregularity, weak coregularity and differential coregularity.

A smooth mapping (4) is said to be *regular at a point* p *of* M iff there are neighbourhoods U and V of points p and $f(p)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, a differential space $(M_0, \mathscr{C}_0)$, a point a and a diffeomorphism

(16) $$\varphi\colon (U, \mathscr{C}_U) \times (M_0, \mathscr{C}_0) \to (V, \mathscr{D}_V)$$

such that

(17) $$f|U = \varphi \circ i,$$

where i is the function defined by the formula

(18) $$i(u) = (u, a) \quad \text{for } u \in U.$$

A mapping which is regular at every point $p \in M$ is said to be *regular*. A smooth mapping (4) is said to be *weak regular at a point* p *of* M iff there exist a neighbourhood U of p and a neighbourhood V of $f(p)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, and there exists a smooth mapping

$$\varrho\colon (V, \mathscr{D}_V)\to(U, \mathscr{C}_U) \tag{19}$$

such that

$$f[U] \subset V \quad \text{and} \quad \varrho\circ f|U = \mathrm{id}_U. \tag{20}$$

A mapping which is weak regular at every point p of M is called *weak regular*.

We say that a smooth mapping (4) is *coregular at a point p of M* iff there exist neighbourhoods U of p and V of $f(p)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, a differential space $(N_0, \mathscr{D}_0)$ and a diffeomorphism

$$\psi\colon (U, \mathscr{C}_U)\to(V, \mathscr{D}_V)\times(N_0, \mathscr{D}_0) \tag{21}$$

such that

$$f|U = \mathrm{pr}_1\circ\psi. \tag{22}$$

A mapping coregular at every point of M is said to be *coregular*. A mapping (4) is said to be *weak coregular at p* iff there exist neighbourhoods U and V of points p and $f(p)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, and a mapping

$$\sigma\colon (V, \mathscr{D}_V)\to(U, \mathscr{C}_U) \tag{23}$$

such that

$$\sigma[V] \subset U, \quad f\circ\sigma = \mathrm{id}_V \quad \text{and} \quad \sigma\big(f(p)\big) = p. \tag{24}$$

A mapping weak coregular at every point of M will be called *weak coregular*.

If in the definition of regularity we set $\varrho = \mathrm{pr}_1\circ\varphi^{-1}$, then, by (16), (17) and (18), we get (19) and (20). Similarly, setting in the definition of coregularity

$$\sigma(v) = \psi^{-1}\big(v, \mathrm{pr}_2\psi(p)\big) \quad \text{for } v\in V,$$

by (21) and (22), we get (23) and (24). So, we have:

3.1. *If a mapping* (4) *is regular* (*coregular*) *at a point p, then* (4) *is weak regular* (*weak coregular*) *at this point. Every regular* (*coregular*) *mapping is weak regular* (*weak coregular*). *Every diffeomorphism is regular. The superposition of two regular* (*weak regular, coregular, weak coregular*) *mappings is also regular* (*weak regular, coregular, weak coregular*).

The inversions of the statements of the first part of above theorem are not true in general. This fact is illustrated by the following example.

EXAMPLE 2. Let $(M, \mathscr{C})$ be the natural differential space of the set $\boldsymbol{R}$ (here $\boldsymbol{R} = M$) of all reals. We set $N = \boldsymbol{R}\times\boldsymbol{R}$. Let $\mathscr{D}_0$ be the union of the set of all real C^∞-functions on $\boldsymbol{R}\times\boldsymbol{R}$ and the set of all functions $\chi_{r,t}$

defined by the formula

$$\chi_{r,t}(q) = \begin{cases} 1, & \text{where } q = (r, t), \\ 0, & \text{where } (r, t) \neq q \in N, \end{cases}$$

where r is an arbitrary rational and t is any positive rational number. Let $(N, \mathscr{D})$ be the differential space generated by the set $\mathscr{D}_0$ of real functions considered above. We set

$$f(x) = (x, 0) \quad \text{for } x \in M$$

and

$$g(y^1, y^2) = y^1 \quad \text{for } (y^1, y^2) \in N.$$

By Lemma 1.4 we get the smooth mapping (4) and the smooth mapping

$$g\colon (N, D) \to (M, C). \tag{25}$$

From the definitions of functions f and g it immediately follows that $g \circ f = \mathrm{id}_M$. Thus, we conclude that the mapping (4) is weak regular and (25) is weak coregular at every point of the set $\boldsymbol{R} \times \{0\}$. If we would suppose that the mapping (4) is coregular at some point p of M; then there should exist a neighbourhood U of p open in $\tau_{\mathscr{C}}$, a neighbourhood v of $(p, 0)$ open in $\tau_{\mathscr{D}}$, a differential space $(M_0, \mathscr{C}_0)$ and a diffeomorphism (16) such that (17) is satisfied, where i is defined by (18) and a is a certain point of M_0. The diffeomorphism (16) induces a homeomorphism of the topological space

$$(U, \tau_{\mathscr{C}} | U) \times (M_0, \tau_{\mathscr{C}_0}) \tag{26}$$

onto $(V, \tau_{\mathscr{D}} | V)$. But this is impossible, because in the topological space $(V, \tau_{\mathscr{D}} | V)$ there exist open one-point sets, whereas any non-empty open set in (26) must contain an open segment of the form $(a; b) \times \{c\}$, where a, b, c are some reals. The same argument leads to the statement that the mapping (25) is coregular at no point of $\boldsymbol{R} \times \{0\}$.

3.2. *Every weak coregular mapping is open.*

Proof. Let a mapping (4) be weak coregular. Take any $A \in \tau_{\mathscr{C}}$ and put $B = f[A]$. Let $q \in B$. Then there exists a mapping (23) such that (24) is satisfied, where $p \in U \in \tau_{\mathscr{C}}$, $q \in V \in \tau_{\mathscr{D}}$. Set $V' = \sigma^{-1}[U \cap A]$. Then $\sigma(q) = p \in U \cap A$ and $V' \in \tau_{\mathscr{D}}$. So $q \in V'$. For any $z \in V'$ we have $\sigma(z) \in U \cap A$. Hence, $z = f(\sigma(z)) \in B$. In other words, V is contained in B. Therefore $B \in \tau_{\mathscr{D}}$.

If there is given a differential space $(N, \mathscr{D})$ and a mapping (5) we may consider any differential structure on M such that the mapping (4) is smooth. In particular, we may study connections between the differential spaces $(M, \mathscr{C})$ and $(M, \mathscr{C}')$ if f maps $(M, \mathscr{C})$ as well as $(M, \mathscr{C}')$ regularly into $(N, \mathscr{D})$. We prove the following theorem.

3.3. *If a mapping* (4) *and*

$$f\colon (M, \mathscr{C}') \to (N, \mathscr{D}) \tag{26}$$

are weak regular, then in order that $\mathscr{C} = \mathscr{C}'$ it suffices that the equality $\tau_{\mathscr{C}} = \tau_{\mathscr{C}'}$ be satisfied.

Proof. Let us suppose that mappings (4) and (26) are weak regular and $\tau_{\mathscr{C}} = \tau_{\mathscr{C}'}$. Take any $a \in \mathscr{C}$ and any $p \in M$. There exist a neighbourhood U of p open in $\tau_{\mathscr{C}}$, a neighbourhood V of $f(p)$ open in $\tau_{\mathscr{D}}$ and a smooth mapping (19) such that (20) is satisfied. Hence we get

$$a \circ \varrho \circ f | U = a \circ \mathrm{id}_U = a | U. \tag{27}$$

From (19) it follows that the function $a \circ \varrho$ maps smoothly $(V, \mathscr{D}_V)$ into the natural differential space of the set $\boldsymbol{R}$ of all reals. In other words, $a \circ \varrho \in \mathscr{D}_V$. Then there exist sets V_1 and V_2 open in $\tau_{\mathscr{D}}$ such that $p \in V_1$, $V_2 \cup V = N$, and a function $\eta \in \mathscr{D}$ such that $\eta(y) = 1$ for $y \in V_1$ and $\eta(y) = 0$ for $y \in V_2$. Hence it follows that the function β defined by the formula

$$\beta(y) = \begin{cases} \eta(y)\, a\big(\varrho(y)\big), & \text{where } y \in V, \\ 0, & \text{where } y \in N - V, \end{cases} \tag{28}$$

belongs to $\mathscr{D}$ and $\beta | V_1 = a \circ \varrho | V_1$. According to (27) and (28) hence we obtain the equality

$$\beta \circ f | (U \cap f^{-1}[V_1]) = a | (U \cap f^{-1}[V_1]).$$

From the smoothness of the mapping (26) it follows that $\beta \circ f \in \mathscr{C}'$. The set U is open in the topology $\tau_{\mathscr{C}'}$. Therefore, the set $U \cap f^{-1}[V_1]$ is open in $\tau_{\mathscr{C}'}$. Thus $a \in \mathscr{C}'_M = \mathscr{C}'$. Then we get the inclusion $\mathscr{C} \subset \mathscr{C}'$. Similarly, $\mathscr{C}' \subset \mathscr{C}$, which completes the proof.

It seems to be interesting that the regularity of mapping (4) does not yield the equality $\mathscr{C} = \mathscr{C}'$. Moreover, the sets of all points of two differential spaces may be identic, the regular mappings (4) and (24) may be one-to-one but the differential spaces $(M, \mathscr{C})$ and $(M, \mathscr{C}')$ may differ topologically.

EXAMPLE 3. Let $N = \boldsymbol{R} \times \boldsymbol{R}$ and $\mathscr{D}$ be the set of all real functions of class C^∞ on the plane $\boldsymbol{R} \times \boldsymbol{R}$. We shall define a function f which maps the open interval $(-\infty; 4)$ into the set N. Set

$$l(x) = \begin{cases} 0, & \text{where } x = 0, \\ \exp\left(1 - \dfrac{1}{x}\right), & \text{where } 0 < x \leqslant 1. \end{cases}$$

It is easy to see that the function l has derivatives of all orders (with respect to $\langle 0; 1\rangle$), $l^{(k)}(0) = 0$, $k = 1, 2, \ldots$, and $l(1) = 1$. The function l

is increasing in $\langle 0;\ 1\rangle$. So, there exists the function m inverse to l. Set

$$r(x) = l(x)\big(1-l^{-1}(1-x)\big) \quad \text{for } x\in\langle 0;\ 1\rangle. \tag{29}$$

The graph of this function has an infinite order of tangency with the straight line $\boldsymbol{R}\times\{0\}$ at the point $(0, 0)$ and with the line $\{1\}\times\boldsymbol{R}$ at the point $(1, 1)$. Now, we put

$$m_1(s) = h^{-1}\big(h(1)s\big) \quad \text{and} \quad m_2(s) = r\big(m_1(s)\big) \quad \text{for } s\in\langle 0;\ 1\rangle, \tag{30}$$

where

$$h(x) = \int_0^x \big(1+\big(r'(u)\big)^2\big)^{1/2}du \quad \text{for } x\in\langle 0;\ 1\rangle.$$

Next we set

$$L = (-\infty;\ 0)\times\{0\}, \quad S = f[\langle 0;\ 4)] \quad \text{and} \quad M = L\cup S,$$

where

$$f(s) = \begin{cases} \big(h(1)s, 0\big) & \text{for } s<0,\\ \big(m_1(s), m_2(s)\big) & \text{for } 0\leqslant s<1,\\ \big(m_1(2-s), 2-m_2(2-s)\big) & \text{for } 1\leqslant s<2,\\ \big(-m_1(s-2), 2-m_2(s-2)\big) & \text{for } 2\leqslant s<3,\\ \big(-m_1(4-s), m_2(4-s)\big) & \text{for } 3\leqslant s<4. \end{cases} \tag{31}$$

Let $\mathscr{C}$ be the differential structure coinduced by the mapping

$$f\colon (-\infty;\ 4)\to M$$

from the natural structure of the open interval $(-\infty;\ 4)$. From 2.4 it follows that the differential space $(M, \mathscr{C})$ just defined is diffeomorphic with the natural differential space of the set of all reals. Then $(M, \mathscr{C})$ is a differentiable manifold. It is easy to see, using (29), (30) and (31), that f is a mapping of class C^∞ of an open interval $(-\infty;\ 4)$ into $\boldsymbol{R}\times\boldsymbol{R}$ and $f'(u)\neq 0$ for $u\in(-\infty;\ 4)$. Hence it follows that we have a regular mapping

$$\mathrm{id}\colon (M, \mathscr{C})\to(N, \mathscr{D}). \tag{32}$$

Set

$$(M, \mathscr{C}') = (L, \mathscr{C}_L)\oplus(S, \mathscr{C}_S).$$

The topology of $(M, \mathscr{C})$ differs from that of $(M, \mathscr{C}')$, because the space $(M, \tau_{\mathscr{C}'})$, as a disjointed union of two non-empty spaces, is not connected. Hence it follows that $(M, \tau_{\mathscr{C}'})$ is not homeomorphic with $(M, \tau_{\mathscr{C}})$, nor with the space induced on M from the plane $\boldsymbol{R}\times\boldsymbol{R}$ regarded with the usual topology. On the other hand, the function $f|\langle 0;\ 4)$ may be extended to the periodic function f_0 (of period 4) having non-vanishing derivative at every point. Hence we conclude that the differential structure $\mathscr{C}_S$ is equal to $(\mathrm{id}_S^*[\mathscr{D}])_{S'}$, where $\mathrm{id}_S(x) = x$ for $x \in S$, $\mathrm{id}_S\colon S \to N$. Further, it is easy to verify that id_S maps regularly this differential space into the natural space $(N, \mathscr{D})$ of the Cartesian plane $\boldsymbol{R} \times \boldsymbol{R}$. It is evident that id_L regularly maps the differential space $(L, \mathscr{C}_L)$ into $(N, \mathscr{D})$. Then we get the regular mapping

$$\mathrm{id}\colon (M, \mathscr{C}') \to (N, \mathscr{D}).$$

As an immediate corollary from 3.3 we get

3.4. *If a mapping* (4) *is weak regular and*

$$\tau_{\mathscr{C}} = \tau_{f^*[\mathscr{D}]},$$

then $\mathscr{C} = (f^*[\mathscr{D}])_M$.

3.5. *If the mapping* (4) *is weak coregular and* (5) *is onto, then*

$$\mathscr{D} = f^{*-1}[\mathscr{C}].$$

Proof. From the smoothness of (4) it follows, by 2.4, that $\mathscr{D}$ is contained in $f^{*-1}[\mathscr{C}]$. Let us take any $\beta \in f^{*-1}[\mathscr{C}]$ and any $q \in N$. Then there exists a point p of M such that $q = f(p)$. From the weak coregularity of (4) it follows that there is a mapping (23) such that (24) is satisfied. Hence

$$\beta | V = \beta \circ \mathrm{id}_V = (\beta \circ f) \circ \sigma \in \mathscr{D}_V.$$

So $\beta \in \mathscr{D}$.

We shall say that the differential space $(M, \mathscr{C})$, where $M \subset N$, is *regularly* (*weak regularly*) *lying in* $(N, \mathscr{D})$ iff the mapping (32) is regular (weak regular). It is easy to verify that the differential space $(\boldsymbol{R} \times \{0\}, \mathscr{C})$, where $\mathscr{C}$ is the set of all real functions of class C^∞ on $\boldsymbol{R} \times \{0\}$, is weak regularly lying in the differential space $(N, \mathscr{D})$ defined in Example 2, but it is not regularly lying in this space.

3.6. *If the mapping* (4) *is regular* (*weak regular*) *and the mapping* (5) *is one-to-one, then the differential space*

$$\big(f[M], (f^{*-1}[\mathscr{C}])_{f[M]}\big) \tag{33}$$

is regularly (*weak regularly*) *lying in* $(N, \mathscr{D})$.

Proof. From 2.4 and 2.5 it follows that the mapping

$$f\colon (M, \mathscr{C}) \to \big(f[M], (f^{*-1}[\mathscr{C}])_{f[M]}\big) \tag{34}$$

is a diffeomorphism. Hence, according to the second part of 3.1, we state that regularity (weak regularity) of (4) yields the same property of the mapping

$$\mathrm{id}\colon \big(f[M], (f^{*-1}[\mathscr{C}])_{f[M]}\big) \to (N, \mathscr{D}). \tag{35}$$

Now, we prove a generalization of 3.6.

3.7. *If a mapping* (4) *is regular* (*weak regular*) *and a mapping* (34) *is open, then the differential space* (33) *is regularly* (*weak regularly*) *lying in* $(N, \mathcal{D})$.

Proof. From the regularity of (4) it follows that for every point $q \in f[M]$ there exist $U \in \tau_{\mathcal{C}}$, $V \in \tau_{\mathcal{D}}$, a diffeomorphism (16) such that (17) is satisfied, where i is defined by formula (18) and a is some point of M_0. Setting $\varrho = \mathrm{pr}_1 \circ \varphi^{-1}$ and

$$V' = f[U] \tag{36}$$

we obtain a smooth mapping (19) fulfilling (20) and a smooth mapping

$$\varrho | V' \colon (V', \mathcal{D}_{V'}) \to (U, \mathcal{C}_U).$$

Hence it follows that for any $v \in V'$ we have $f(\varrho(v)) = f(\varrho(f(u))) = f(u) = v$, where $v = f(u)$, $u \in U$. Therefore $f|U$ is a one-to-one function and $\varrho|V' = (f|U)^{-1}$. From the smoothness of (4) it follows that $f^{*-1}[\mathcal{C}] \supset \mathcal{D}$. Then $(f^{*-1}[\mathcal{C}])_{V'} \supset \mathcal{D}_{V'}$. This implies the smoothness of the mapping

$$(f|U)^{-1} \colon (V', (f^{*-1}[\mathcal{C}])_{V'}) \to (U, \mathcal{C}_U). \tag{37}$$

Setting

$$\psi(v, m) = \varphi((f|U)^{-1}(v), m) \quad \text{for } (v, m) \in V' \times M_0$$

we obtain the diffeomorphism

$$\psi \colon (V', (f^{*-1}[\mathcal{C}])_{V'}) \times (M_0, \mathcal{C}_0) \to (V, \mathcal{D}_V).$$

For any $v \in V'$ we have

$$\begin{aligned} \psi(v, a) &= \varphi((f|U)^{-1}(v), a) = \varphi(i((f|U)^{-1}(v))) \\ &= (f|U)((f|U)^{-1}(v)) = v. \end{aligned}$$

From the openness of (34) it follows that the set V' defined by (36) is open in the topology $\tau_{f^{*-1}[\mathcal{C}]}$. This ends the proof of the regularity of the mapping (35).

If the mapping (4) is weak regular, then for any point $q \in f[M]$ there exists a mapping (19) satisfying (20), where $p \in U \in \tau_{\mathcal{C}}$ and $q = f(p) \in V \in \tau_{\mathcal{D}}$. Hence, similarly as above, setting $\varrho' = f \circ \varrho$ we get a smooth mapping

$$\varrho' \colon (V, \mathcal{D}_V) \to (V', (f^{*-1}[\mathcal{C}])_{V'})$$

such that $\varrho'(v) = v$ for $v \in V'$. From the openness of (34) it follows that the set V' defined by (36) is open in the topology $\tau_{f^{*-1}[\mathcal{C}]}$. Therefore the mapping (35) is weak regular.

Now, we give properties of weak regular (weak coregular) mappings that have no corresponding ones in the category of all regular (coregular) mappings.

3.8. *If a mapping* (4) *is weak regular* (*weak coregular*) *at a point* p *of* M *and*

$$f[M] \subset N' \subset N,$$

then the mapping

$$f\colon (M, \mathscr{C}) \to (N', \mathscr{D}_{N'}) \tag{38}$$

is weak regular (*weak coregular*) *at* p.

If a mapping (4) *is weak regular* (*weak coregular*) *at* p *and* $p \in M' \subset M$, *then the mapping*

$$f\colon (M', \mathscr{C}_{M'}) \to (N, \mathscr{D})$$

is weak regular (*weak coregular*) *at* p.

Proof. Let p be any point of M. Then there exists a mapping (19) satisfying (20), where $p \in U \in \tau_{\mathscr{C}}$ and $f(p) \in V \in \tau_{\mathscr{D}}$. Setting $V' = V \cap N'$ we have $f[U] \subset V'$, $\varrho' \circ f | U = \mathrm{id}_U$, where $\varrho' = \varrho | V'$ and

$$\varrho'\colon \big(V', (\mathscr{D}_{N'})_{V'}\big) \to (U, \mathscr{C}_U).$$

The mapping (38) is thus weak regular. The proofs of the other assertions are similar.

EXAMPLE 4. Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be the natural differential spaces of the sets $\boldsymbol{R}$ and $\boldsymbol{R} \times \boldsymbol{R}$, respectively, and $N = \boldsymbol{R} \times \{0\} \cup \{0\} \times \boldsymbol{R}$. If we set $f(x) = (x, 0)$ for $x \in M$, we obtain the regular mapping of the form (4) such that respective mapping of the form (38) is not regular at a point 0.

Let now, $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be the natural differential spaces of the sets $\boldsymbol{R} \times \boldsymbol{R}$ and $\boldsymbol{R}$, respectively, $M' = N'$ and $f = \mathrm{pr}_1$. Then, we obtain the coregular mapping of the form (4) such that respective mapping $f | M'\colon (M', \mathscr{C}_{M'}) \to (N, \mathscr{D})$ is not coregular at the point $(0, 0)$.

It is not difficult to see that the mapping $\mathrm{id}\colon (\langle 0; 1 \rangle, \mathscr{C}_{\langle 0;1 \rangle}) \to (\boldsymbol{R}, \mathscr{C})$, where $\mathscr{C}$ is the natural differential structure of $\boldsymbol{R}$, is not regular at the points 0 and 1, however, the mapping $\mathrm{id}\colon (\boldsymbol{R}, \mathscr{C}) \to (\boldsymbol{R}, \mathscr{C})$ is regular.

3.9. *If the mapping* (4) *is weak regular, then for any mapping* (11) *the mapping* (12) *is smooth if and only if* (12) *is continuous and* (13) *is smooth.*

If the mapping (4) *is weak coregular, and onto, then for any mapping* (14), h *maps smoothly* $(N, \mathscr{D})$ *into* $(N', \mathscr{D}')$ *if and only if the mapping* (15) *is smooth.*

Proof. Let us suppose that (4) is weak regular, (12) is continuous and (13) is smooth. Consider any $\alpha \in \mathscr{C}$ and any point p' of M'. Setting $p = g(p')$ we see that there exists a mapping (19) such that (20) is satisfied, where U and V are some neighbourhoods of p and $f(p)$ open in $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively. Hence it follows that $p \in g^{-1}[U]$, $g^{-1}[U]$ is open in $\tau_{\mathscr{C}'}$ and

$$g | g^{-1}[U] = \mathrm{id}_U \circ g | g o^{1}[U] = \varrho \circ f | U \circ g | g^{-1}[U] = \varrho \circ (f \circ g) | g^{-1}[U].$$

Then we state that $a|U \in \mathscr{C}_U$ and

$$a \circ g|g^{-1}[U] = (a|U \circ \varrho) \circ (f \circ g)|g^{-1}[U] \in \mathscr{C}'_{g^{-1}[U]}.$$

So $a \circ g \in \mathscr{C}'_{M'} = \mathscr{C}'$. Thus the mapping (12) is smooth. The proof of the second part of the theorem is an immediate consequence of 3.5 and the second part of 2.3.

We will prove the following theorem useful for an examination of coregularity of the natural mapping of an equivalence relation.

3.10. *If a mapping* (4) *is smooth and the following conditions are satisfied:*

(a) $\mathrm{pr}_1\colon \big(R_f, (\mathscr{C} \times \mathscr{C})_{R_f}\big) \to (M, \mathscr{C})$, *where R_f is the set of all pairs (x, y) of points of M such that $f(x) = f(y)$, is weak coregular (coregular)*;

(b) *for every point p of M there exists a weak coregular (coregular) mapping*

$$s\colon (U, \mathscr{C}_U) \to (W, \mathscr{C}_W),$$

where $p \in U \supset W$, U is open in $\tau_{\mathscr{C}}$, such that

$$f^{-1}[\{f(x)\}] \cap W = \{s(x)\} \quad \text{for } x \in U;$$

(c) *there exists a set A open in $\tau_{\mathscr{C}}$ such that*

$$f^{-1}\big[f[A]\big] = M$$

and (4) *is weak coregular (coregular) at every point of A*;

then (4) *is weak coregular (coregular).*

Proof. Let $q \in M$. From (c) it follows that $f(q) = f(p)$, where $p \in A$, A is open in $\tau_{\mathscr{C}}$ and $f|A$ is weak coregular. According to (c) we may assume that the set U, in condition (b), is contained in A. Let $s'\colon (U', \mathscr{C}_{U'}) \to (W', \mathscr{C}_{W'})$, where $q \in U' \supset W'$, U' is open in $\tau_{\mathscr{C}}$, be a weak coregular mapping such that $f^{-1}[\{f(y)\}] \cap W' = \{s'(y)\}$ for $y \in U'$. It is easy to verify that

$$f^{-1}\big[f[Y]\big] = \mathrm{pr}_1[(M \times Y) \cap R_f], \quad \text{where } Y \subset M.$$

Hence, by 3.2, it follows that the sets

$$U_0 = U \cap f^{-1}\big[f[U']\big] \quad \text{and} \quad U'_0 = U' \cap f^{-1}\big[f[U]\big]$$

are neighbourhoods of the points p and q, respectively, open in $\tau_{\mathscr{C}}$. We set $W_0 = W \cap U_0$ and $W'_0 = W' \cap U'_0$. It is easy to see that

$$s[U_0] = W_0, \quad s'[U'_0] = W'_0, \quad f[W_0] = f[W'_0],$$

$f|W_0$ and $f|W_0$ are one-to-one. Moreover,

$$f|U_0 = f|W_0 \circ s|U_0 \quad \text{and} \quad f|U'_0 = f'|W'_0 \circ s'|U'_0.$$

Hence we get

$$(f|W_0')^{-1}\circ f|U_0\circ \mathrm{pr}_1|U_0\times U_0' = s'|U_0'\circ \mathrm{pr}_2|U_0\times U_0'.$$

From (a) it follows that the mapping $\mathrm{pr}_2\colon (R_f, (\mathscr{C}\times\mathscr{C})_{R_f})\to(M, \mathscr{C})$ is weak coregular. Thus, we have a smooth mapping

$$(f|W_0')^{-1}\colon (f[W_0], \mathscr{D}_{f[W_0]})\to(W_0', \mathscr{C}_{W_0'}).$$

Therefore, $f|U_0'$ is weak coregular as a superposition of the weak coregular mapping $s'|U_0'$, the diffeomorphism

$$f|W_0'\colon (W_0', \mathscr{C}_{W_0'})\to(f[W_0], \mathscr{D}_{f[W_0]})$$

and the inclusion $i\colon (f[W_0], \mathscr{D}_{f[W_0]})\to(N, \mathscr{D})$.

The same proof may be repeated if we consider coregularity instead of weak coregularity.

4. Differential regularity and differential coregularity. A mapping (4) is said to be *differentially regular at the point* p (= regular at p in the book [6]) iff the differential

$$(df)_p\colon (M, \mathscr{C})_p\to(N, \mathscr{D})_{f(p)} \quad .$$

of (4) at the point p, defined by the formula $(df)_p(v)(\beta) = v(\beta\circ f)$ for $\beta\in D$, is a monomorphism. A mapping (4) is said to be *differentially regular* (*differentially coregular*) iff it is differentially regular at p (differentially coregular at p) for every p of M.

4.1. *If a mapping* (4) *is weak regular* (*weak coregular*) *at* p, *then it is differentially regular* (*differentially coregular*) *at* p.

Proof. If (4) is weak regular at p, then there exists the mapping (19) satisfying (20), where $p\in U\in\tau_{\mathscr{C}}$, $f(p)\in V\in\tau_{\mathscr{D}}$. Hence

$$(d\varrho)_{f(p)}\circ(df|U)_p = (d\,\mathrm{id}_U)_p\colon (U, \mathscr{C}_U)_p\to(U, \mathscr{C}_U)_p$$

is an isomorphism. Then

$$(df|U)_p\colon (U, \mathscr{C}_U)_p\to(V, \mathscr{D}_V)_{f(p)}$$

is a monomorphism. From the openness of the sets U and V in $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, it follows that there exist natural isomorphisms

$$i\colon (M, \mathscr{C})_p\to(U, \mathscr{C}_U)_p \quad \text{and} \quad j\colon (V, \mathscr{D}_V)_{f(p)}\to(N, \mathscr{D})_{f(p)}$$

such that $(df)_p = j\circ(df|U)_p\circ i$. This yields that (39) is a monomorphism. The case of weak regularity is quite analogous.

We give below an example of a differentially regular mapping at p and a differentially coregular one which is not weak regular and not weak coregular.

EXAMPLE 5. Set $M = N = \boldsymbol{R}$. Let $\mathscr{D}$ be the natural differential structure induced from $\mathscr{D}$ by the function f defined by the formula $f(p) = p^2$ for $p \in M$. Then we obtain a smooth mapping (4) which, according to 4.2, is neither differentially regular nor differentially coregular. If the mapping (4) were weak regular at the point 0, then there would exists a mapping (19) satisfying (20), where $U = (-a;\ a)$, $a > 0$. Hence it follows that the function $f|U$ is one-to-one. This is impossible. If we suppose that (4) is coregular, then it is open. Therefore, the set $f[(-1;\ 1)]$ would be open in the usual topology of the set $\boldsymbol{R}$. But this set is equal to $\langle 0;\ 1)$.

4.3. *For every differential space* $(N, \mathscr{D})$ *and every mapping* (5) *the mapping* (4), *where* $\mathscr{C}$ *is the differential structure induced from* $(N, \mathscr{D})$ *by* (5), *is differentially regular and differentially coregular.*

If $(M, \mathscr{C})$ *is any differential space such that*

$$(f^*[f^{*-1}[\mathscr{C}]])_M = \mathscr{C}, \tag{40}$$

then the mapping (4), *where* $\mathscr{D}$ *is the differential structure coinduced from* $(M, \mathscr{C})$ *by* (5), *is differentially coregular.*

Proof. Consider any differential space $(N, \mathscr{D})$. Let $(M, \mathscr{C})$ be the differential space induced from $(N, \mathscr{D})$ by the mapping (5). Let p be any point of M and let v be any vector tangent to the differential space (7) at p. Suppose that $(df)_p(v) = 0$. Then $v(\beta\circ f) = 0$ for every $\beta \in \mathscr{D}$. In other words, $v(a) = 0$ for every $a \in f^*[\mathscr{D}]$. This yields $v = 0$. So the mapping (8) is differentially regular.

Now, let us suppose that (40) holds and take any vector w tangent to the differential space $(N, \mathscr{D})$ at the point $f(p)$, where $\mathscr{D}$ is the differential structure coinduced from $(M, \mathscr{C})$ by the mapping (5). Assume $\beta\circ f|A = \beta_1\circ f|A$, where $p \in A \in \tau_{\mathscr{C}}$ and $\beta,\ \beta_1 \in f^{*-1}[\mathscr{C}]$. Hence it follows that $\beta|f[A] = \beta_1|f[A]$. According to 2.1 the set $f[A]$ is open in $\tau_{f^{*-1}[\mathscr{C}]}$. Therefore $w(\beta) = w(\beta_1)$. Take any $a \in \mathscr{C}$. From (40) it follows that there exist $\beta \in f^{*-1}[\mathscr{C}]$ and a set $A \in \tau_{\mathscr{C}}$ such that $a|A = \beta\circ f|A$. So we may define a vector v by the formula $v(a) = w(\beta)$, tangent to $(M, \mathscr{C})$ at p and such that $(df)_p(v) = w$. The linear mapping (39) is then onto. In other words, the mapping (4) is differentially coregular.

To prove the coregularity of the mapping (4) in the situation of the first part of theorem we remark that, in this case,

$$f^*[f^{*-1}[\mathscr{C}]] = f^*[f^{*-1}[f^*[\mathscr{D}]]] = f^*[\mathscr{D}] = \mathscr{C}.$$

Condition (40) is then satisfied. This ends the proof.

From the definitions of weak regularity, differential regularity, weak coregularity and differential coregularity it immediately follows that

4.4. *If a mapping* (4) *and a mapping*

$$g\colon (N, \mathscr{D}) \to (P, \mathscr{F}) \tag{41}$$

are smooth and a mapping

$$g \circ f\colon (M, \mathscr{C}) \to (N, \mathscr{D}) \tag{42}$$

is weak regular at p (differentially regular at p), then the mapping (4) *is weak regular at p (differentially regular at p).*

If (4) *and* (41) *are smooth and* (42) *is weak coregular at p (differentially coregular at p), then* (41) *is weak coregular at $f(p)$ (differentially coregular at $f(p)$).*

Example 2 shows that analogous statements are not true for the concepts of regularity and coregularity.

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