4.2. Now to derive Theorem 4, we assume that

$$|\Delta_n(\chi_1, \chi)| \leqslant \frac{\sqrt{2\log\log q}}{2\pi\Phi(q)}$$

for some positive increasing function $\Phi(q)$ which tends to ∞ as $q \to \infty$, and for positive proposition of nonprincipal characters χ_1 to modulus q. Then if χ_1 and χ_2 are such characters, then

$$|\Delta_n(\chi_1, \chi) - \Delta_n(\chi_2, \chi)| \leqslant |\Delta_n(\chi_1, \chi)| + |\Delta_n(\chi_2, \chi)| \leqslant \frac{\sqrt{2\log\log q}}{\pi\Phi(q)}.$$

Since the number of such pairs (χ_1, χ_2) is at least Aq^2 with some positive absolute constant, this contradicts with the fact in 4.1. Hence for a given χ to modulus q $|\Delta_n(\chi_1, \chi)| > \frac{\sqrt{2\log - \log q}}{2\pi\Phi(q)}$ for almost all χ_1 to modulus q. This proves Theorem 4. Theorem 3 is a special case of Theorem 4.

4.3. Similarly from Lemma 1 if $(t_2 - t_1)\log q \to \infty$ as $q \to \infty$, for any positive increasing function $\Phi(q)$ which tends to ∞ as $q \to \infty$,

$$\big|\big(N(t_2, \chi_1) - N(t_1, \chi_1)\big) - \big(N(t_2, \chi_2) - N(t_1, \chi_2)\big)\big| > \frac{\sqrt{4\log\big((t_2 - t_1)\log q\big)}}{2\pi\Phi(q)}$$

for almost all pairs (χ_1, χ_2) of characters to modulus q. From this we can see Theorems 2 and 2′ as before.

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Received on 12. 4. 1974 (564)

ACTA ARITHMETICA
XXIX (1976)

On the zeros of Dirichlet L-functions (VII)

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§ 1. Introduction. As an application of the methods which we have used in the author's previous works, [1] and [4] we add some results to Knapowski–Turán's problem which will be explained later. We will quote the above articles by (I) or (V).

Let q be a given fixed positive integer. Assume that $(b, q) = (d, q) = 1$ and $b \not\equiv d (\operatorname{mod} q)$. Let χ be a character to modulus q. We write

$$g(\chi) = \frac{1}{\varphi(q)}\big(\bar\chi(b) - \bar\chi(d)\big) \quad \text{and} \quad \mu(\varrho) = \mu_{b,d}(\varrho) = \sum_{\chi} g(\chi) m_\chi(\varrho),$$

where χ runs over all characters to modulus q and $m_\chi(\varrho)$ is the multiplicity of ϱ as a zero of Dirichlet L-function $L(s, \chi)$. Knapowski and Turán proposed the following problem in their study of prime numbers: to study

$$f(T) = \sum_{\substack{0 < \operatorname{Im}\varrho < T \\ \mu(\varrho) \neq 0}} 1$$

(cf. [6]).

To this problem Kátai (unpublished) and Grosswald ([5]) proved independently the existence of infinitely many ϱ's with $\mu(\varrho) \neq 0$. Later Turán obtained the following results (cf. [10]).

1) For $T > \psi(q)$ we have the inequality

$$f(T) > C_1 \exp\big((\log T)^{1/5}\big).$$

2) Under the assumption of the generalized Riemann hypothesis we have

$$f(T) > C_2 T^{1/2} \quad \text{for} \quad T > \psi(q),$$

where C_ν are numerical constants and $\psi(q)$ an explicit function of q.

Recently Motohashi ([7]) obtained the following results.

1) For $T > \psi(q)$ we have

$$f(T) > T^{1/10}(\log T)^{-3}.$$

Here the estimation is independent of b and d.

* Supported in part by National Science Foundation grant GP – 36418X1.

2) For any sufficiently large T there exists at least one q, with

$$\tfrac{1}{2}T^{1/2}(\log T)^{-51} \leqslant q \leqslant T^{1/2}(\log T)^{-51},$$

s.t.

$$f(T) > T^{3/28}(\log T)^{-45}$$

holds for any pair (b, d).

Now we can show

THEOREM. *For $T > \psi(q)$ we have*

$$f(T) > AT\log T,$$

where $\psi(q)$ is some explicit function of q and positive constant A may depend on q.

In fact, we can take $\psi(q) = \exp(\exp(C_1 q))$ and $A = \exp(-C_2 q)$ with some positive absolute constants C_1 and C_2.

§ 2. Proof of Theorem. Let $N(t, \chi)$ be the number of zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ in $0 \leqslant \gamma \leqslant t$, $0 < \beta < 1$, possible zeros on $\operatorname{Im} s = 0$ or t counted one-half only. Let

$$S(t, \chi) = \frac{1}{\pi}\arg L(\tfrac{1}{2} + it, \chi)$$

as usual. If χ is a primitive character to modulus q, we know

$$\text{(1)} \quad N(t, \chi) = \frac{t}{2\pi}\log t - \frac{1 + \log\frac{2\pi}{q}}{2\pi}t + S(t, \chi) - S(0, \chi) - \frac{\chi(-1)}{8} + O\left(\frac{1}{1+t}\right) \quad \text{for} \quad t > 0.$$

If χ is not a primitive character, we write the primitive character attached to χ by χ^* and its modulus by q^*. Then we have

$$\text{(2)} \quad N(t, \chi) = N(t, \chi^*) = \frac{t}{2\pi}\log t - \frac{1 + \log\frac{2\pi}{q^*}}{2\pi}t + S(t, \chi^*) - S(0, \chi^*) - \frac{\chi^*(-1)}{8} + O\left(\frac{1}{1+t}\right) \quad \text{for} \quad t > 0.$$

Now let h be a positive number to be chosen later and let

$E = \{t \in (T, T+H)$; in $(t, t+h)$ there exists at least one ϱ with $\mu(\varrho) \neq 0\}$.

Then we see that for $t \notin E$,

$$\Big|\sum_{\chi} g(\chi)\big(N(t+h, \chi) - N(t, \chi)\big)\Big| \leqslant 1.$$

Generally,

$$\sum_{\chi} g(\chi)\big(N(t+h, \chi) - N(t, \chi)\big) = \sum_{\chi} g(\chi)\big(N(t+h, \chi^*) - N(t, \chi^*)\big)$$
$$= \sum_{\chi} g(\chi)\left(\frac{h}{2\pi}\log\frac{q^*\xi}{2\pi} + S(t+h, \chi^*) - S(t, \chi^*)\right) + O\left(\frac{1}{1+t}\right),$$

where ξ is some number in $t \leqslant \xi \leqslant t+h$. By the definition of $g(\chi)$ we see

$$\sum_{\chi} g(\chi) = \begin{cases} 1 & \text{if} \quad b \equiv 1(q), \\ -1 & \text{if} \quad d \equiv 1(q), \\ 0 & \text{otherwise.} \end{cases}$$

We may assume $b \not\equiv 1(q)$ and $d \not\equiv 1(q)$, since otherwise $f(T) > AT\log T$ trivially. Namely we may assume $\sum g(\chi) = 0$. Hence

$$\sum_{\chi} g(\chi)\big(N(t+h, \chi) - N(t, \chi)\big) = \frac{h}{2\pi}\sum_{\chi} g(\chi)\log q^* + \sum_{\chi} g(\chi)\big(S(t+h, \chi^*) - S(t, \chi^*)\big) + O\left(\frac{1}{1+t}\right).$$

For simplicity we write

$$\|f(t)\| = \Big(\int_T^{T+H} |f(t)|^{2k}\,dt\Big)^{1/2k}.$$

Then by the lemma in § 3 if

$$h\log T = C, \quad T > \exp\left(\frac{C_1 C}{\sqrt{\log C}}q^{1/2}\log\log q\right) \quad \text{and} \quad C > \exp(C_2 q)$$

with some positive absolute constants C_i,

$$\Big\|\sum_{\chi} g(\chi)\big(S(t+h, \chi^*) - S(t, \chi^*)\big)\Big\| \cong H^{1/2k}\left(\frac{\log(h\log T)}{\varphi(q)}\right)^{1/2}$$

for each k, where $f \cong g$ means $f \ll g$ and $g \ll f$. Using trivial estimate

$$\Big\|\frac{h}{2\pi}\sum_{\chi} g(\chi)\log q^*\Big\| \ll H^{1/2k}h\log q,$$

we get

$$\Big\|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big\| \cong H^{1/2k}\left(\frac{\log(h\log T)}{\varphi(q)}\right)^{1/2}+O(H^{1/2}h\log q)$$

for each k. Hence if

$$h\log T=C,\quad T>\exp\left(\frac{C}{\sqrt{\log C}}\,q^{1/2}\log q\right)\quad\text{and}\quad C>\exp(C_2 q),$$

$$\Big\|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big\| \cong H^{1/2k}\left(\frac{\log(h\log T)}{\varphi(q)}\right)^{1/2}.$$

Now we consider the integral

$$\int_T^{T+H}\Big|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big|^2 dt.$$

This is

$$=\int_E\Big|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big|^2 dt+$$

$$+\int_{E^C}\Big|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big|^2 dt$$

$$\leqslant\sqrt{|E|}\left(\int_T^{T+H}\Big|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big|^4 dt\right)^{1/2}+H$$

$$\leqslant C_3 H^{1/2}\left(\frac{\log(h\log T)}{\varphi(q)}\right)\sqrt{|E|}+H.$$

On the other hand

$$\int_T^{T+H}\Big|\sum_{\chi} g(\chi)\big(N(t+h,\chi)-N(t,\chi)\big)\Big|^2 dt\geqslant C_4 H\frac{\log(h\log T)}{\varphi(q)}.$$

Hence we get by taking C_2 sufficiently large

$$|E|>C_5 H.$$

Dividing the interval $(T,T+H)$ into subintervals of length h, we get as usual

$$f(T,T+H)>AH\log T$$

if $T>\exp\big(\exp(C_6 q)\big)$, where we write

$$f(T,T+H)=\sum_{\substack{T<\gamma<T+H\\ \mu(q)\neq 0}}1$$

and we can take $A=\exp(-C_7 q)$. $C_1, C_2, \ldots$ above are all positive absolute constants. Q.E.D.

§ 3. Lemma

3.1. In this paragraph we will prove the following

LEMMA. *Assume that $T^{1/2+a}\leqslant H\leqslant T$, where*

$$0<a<\tfrac12,\qquad h\log T=C,$$

$$T>\exp\left(C_1\frac{C}{\sqrt{\log C}}\,q^{1/2}\log\log q\right)\quad\textit{and}\quad C>\exp(C_2 q),$$

where C_1 and C_2 are suitable, positive, absolute constants. Then

$$\int_T^{T+H}\Big|\sum_{\chi} g(\chi)\big(S(t+h,\chi^*)-S(t,\chi^*)\big)\Big|^{2k} dt\cong H\left(\frac{\log(h\log T)}{\varphi(q)}\right)^k$$

for each $k=1,2,\ldots$

3.2. For simplicity we prove only for $k=1$ and 2. In (I) we saw that if $T^{1/4-b}\geqslant q$,

$$S(t,\chi^*)-\frac{1}{\pi}\operatorname{Im}\sum_{p<x^3}\frac{\chi^*(p)}{p^{1/2+it}}=O\big(B_{x^3}(t,\chi^*)\big),$$

where for $x=T^{a/60k}$

(3) $$\|B_{x^3}(t,\chi^*)\|\ll_k H^{1/2k}.$$

Also for $x=T^{a/60k}$ and h in $0<h<H-(H/\sqrt{T})^{1/3}$

(4) $$\|B_{x^3}(t+h,\chi^*)\|\ll_k H^{1/2k}.$$

(Cf. [1].)

Hence if we write

$$A_{x^3}(t)=\sum_{\chi} g(\chi)\big(S(t+h,\chi^*)-S(t,\chi^*)\big)-$$

$$-\frac{1}{\pi}\sum_{\chi} g(\chi)\operatorname{Im}\sum_{p<x^3}\frac{\chi^*(p)}{p^{1/2+it}}\big(\exp(-ih\log p)-1\big)$$

$$=\sum_{\chi} g(\chi)O\big(B_{x^3}(t+h,\chi^*)\big)+\sum_{\chi} g(\chi)O\big(B_{x^3}(t,\chi^*)\big),$$

then for x in $x_0=T^{a/20k}\leqslant x\leqslant H^{1/k}$

$$\|A_x(t)\|\ll\frac{1}{\varphi(q)}\sum_{\chi}\|B_{x_0}(t+h,\chi^*)\|+$$

$$+\frac{1}{\varphi(q)}\sum_{\chi}\|B_{x_0}(t,\chi^*)\|+\frac{1}{\varphi(q)}\sum_{\chi}\Big\|\sum_{x_0\leqslant p\leqslant x}\frac{a(p,\chi^*)}{p^{1/2+it}}\Big\|,$$

where $a(p,\chi^*) = \chi^*(p)\big(\exp(-ih\log p)-1\big)$. Hence for x in $x_0 \leqslant x \leqslant H^{1/k}$, $\|A_x(t)\| \ll H^{1/2k}$ by (3), (4) and Lemma 5 in (I). Hereafter we take $x = T^{(1+a)/5k}$.

3.3. We write $\tau(\chi^*) = \dfrac{1}{\pi}\sum_{p<x}\dfrac{a(p,\chi^*)}{p^{1/2+it}}$, where $a(p,\chi^*)$ is the same as in 3.2. Then

$$\frac{1}{\pi}\sum_{\chi} g(\chi)\,\mathrm{Im}\sum_{p<x}\frac{a(p,\chi^*)}{p^{1/2+it}} = \frac{1}{2i}\sum_{\chi} g(\chi)\big(\tau(\chi^*)-\overline{\tau(\chi^*)}\big)$$

$$= \frac{1}{2\pi i}\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}} - \frac{1}{2\pi i}\sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}},$$

where

$$b(p) = \sum_{\chi} g(\chi)\chi^*(p), \qquad b'(p) = \sum_{\chi} g(\chi)\overline{\chi^*(p)},$$

and

$$a(p) = \exp(-ih\log p)-1.$$

Now

$$\left|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}} - \sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\right|^2 = \sum_{p<x}\frac{b(p_1)\overline{b(p_2)}a(p_1)\overline{a(p_2)}}{\sqrt{p_1p_2}}\left(\frac{p_2}{p_1}\right)^{it} +$$

$$+\sum_{p<x}\frac{b'(p_1)\overline{b'(p_2)}\,\overline{a(p_1)}a(p_2)}{\sqrt{p_1p_2}}\left(\frac{p_1}{p_2}\right)^{it} -$$

$$-\sum_{p<x}\frac{b'(p_1)\overline{b(p_2)}\,\overline{a(p_1)}\,\overline{a(p_2)}}{\sqrt{p_1p_2}}(p_1p_2)^{it} -$$

$$-\sum_{p<x}\frac{b(p_1)\overline{b'(p_2)}a(p_1)a(p_2)}{\sqrt{p_1p_2}}\,\frac{1}{(p_1p_2)^{it}}.$$

Hence also

$$\left|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}} - \sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\right|^4$$

$$= \sum_{p<x}\frac{b(p_1)\overline{b(p_2)}b(p_3)\overline{b(p_4)}a(p_1)\overline{a(p_2)}a(p_3)\overline{a(p_4)}}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_2p_4}{p_1p_3}\right)^{it} +$$

$$+\sum_{p<x}\frac{b'(p_1)\overline{b'(p_2)}b'(p_3)\overline{b'(p_4)}\,\overline{a(p_1)}a(p_2)\overline{a(p_3)}a(p_4)}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_1p_3}{p_2p_4}\right)^{it} +$$

$$+\sum_{p<x}\frac{b'(p_1)\overline{b(p_2)}b'(p_3)\overline{b(p_4)}\,\overline{a(p_1)}\,\overline{a(p_2)}\,\overline{a(p_3)}\,\overline{a(p_4)}}{\sqrt{p_1p_2p_3p_4}}(p_1p_2p_3p_4)^{it} +$$

$$+\sum_{p<x}\frac{b(p_1)\overline{b'(p_2)}b(p_3)\overline{b'(p_4)}a(p_1)a(p_2)a(p_3)a(p_4)}{\sqrt{p_1p_2p_3p_4}(p_1p_2p_3p_4)^{it}} +$$

$$+\sum_{p<x}\frac{b(p_1)\overline{b(p_2)}b'(p_3)\overline{b'(p_4)}a(p_1)\overline{a(p_2)}\,\overline{a(p_3)}a(p_4)}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_2p_3}{p_1p_4}\right)^{it} +$$

$$+\sum_{p<x}\frac{b'(p_1)\overline{b(p_2)}b(p_3)\overline{b'(p_4)}\,\overline{a(p_1)}\,\overline{a(p_2)}a(p_3)a(p_4)}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_1p_2}{p_3p_4}\right)^{it} -$$

$$-\sum_{p<x}\frac{b(p_1)\overline{b(p_2)}a(p_1)\overline{a(p_2)}b(p_3)\overline{b'(p_4)}a(p_3)a(p_4)}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_2}{p_1p_3p_4}\right)^{it} -$$

$$-\sum_{p<x}\frac{b'(p_1)\overline{b'(p_2)}\,\overline{a(p_1)}a(p_2)b'(p_3)\overline{b(p_4)}\,\overline{a(p_3)}a(p_4)}{\sqrt{p_1p_2p_3p_4}}\left(\frac{p_1p_3p_4}{p_2}\right)^{it}.$$

Hence

$$\int_T^{T+H}\left|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}} - \sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\right|^2 dt$$

$$= H\left(\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p} + \sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}\right) + O\left(\sum_{\substack{p_1\neq p_2\\ p<x}}\frac{1}{\sqrt{p_1p_2}\,|\log p_2/p_1|}\right)$$

$$= H\left(\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p} + \sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}\right) + O(T^{1/5}\log T).$$

Also

$$\int_T^{T+H}\left|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}} - \sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\right|^4 dt$$

$$= 2H\left(\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p}\right)^2 + 2H\left(\sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}\right)^2 +$$

$$+2H\left(\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p}\right)\left(\sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}\right) +$$

$$+2H\left(\sum_{p<x}\frac{b(p)b'(p)|a(p)|^2}{p}\right)\left(\sum_{p<x}\frac{\overline{b(p)b'(p)}|a(p)|^2}{p}\right) +$$

$$+O\left(\sum_{\substack{p_1p_4\neq p_2p_3\\ p<x}}\frac{1}{\sqrt{p_1p_2p_3p_4}\left|\log\dfrac{p_2p_4}{p_1p_3}\right|}\right).$$

Now we see

$$\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p}=\sum_{\substack{p<x\\ p\equiv b(q)}}\frac{|a(p)|^2}{p}+\sum_{\substack{p<x\\ p\equiv d(q)}}\frac{|a(p)|^2}{p}+O\Bigl(\sum_{\substack{p|q\\ p<x}}\frac{|a(p)|^2}{p}\Bigr)$$

and

$$\sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}=\sum_{\substack{p<x\\ p\equiv b^*(q)}}\frac{|a(p)|^2}{p}+\sum_{\substack{p<x\\ p\equiv d^*(q)}}\frac{|a(p)|^2}{p}+O\Bigl(\sum_{\substack{p|q\\ p<x}}\frac{|a(p)|^2}{p}\Bigr),$$

where $bb^*\equiv 1(q)$ and $dd^*\equiv 1(q)$. Moreover,

$$\sum_{p\equiv b(q)}\frac{|a(p)|^2}{p}=2\sum_{\substack{p<x\\ p\equiv b(q)}}\frac{1-\cos(h\log p)}{p}=\frac{2}{\varphi(q)}\log(h\log x)+O(1)$$

and

$$\sum_{\substack{p<x\\ p|q}}\frac{|a(p)|^2}{p}=O\bigl(h^2(\log\log q)^2\bigr).$$

Hence

$$\sum_{p<x}\frac{|b(p)|^2|a(p)|^2}{p}=\frac{4}{\varphi(q)}\log(h\log T)+O\bigl(h^2(\log\log q)^2\bigr)+O(1)$$

and

$$\sum_{p<x}\frac{|b'(p)|^2|a(p)|^2}{p}=\frac{4}{\varphi(q)}\log(h\log T)+O\bigl(h^2(\log\log q)^2\bigr)+O(1).$$

Hence

$$\int_T^{T+H}\Bigl|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}}-\sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\Bigr|^2dt$$
$$=\frac{8}{\varphi(q)}H\log(h\log T)+O\bigl(Hh^2(\log\log q)^2\bigr)+O(T^{1/5}\log T)+O(H).$$

And

$$\int_T^{T+H}\Bigl|\sum_{p<x}\frac{b(p)a(p)}{p^{1/2+it}}-\sum_{p<x}\frac{b'(p)\overline{a(p)}}{p^{1/2-it}}\Bigr|^4dt$$
$$=c(b,d)H\Bigl(\frac{\log(h\log T)}{\varphi(q)}\Bigr)^2+O\bigl(Hh^4(\log\log q)^4\bigr)+O(H)+$$
$$+O\Bigl(H\frac{\log(h\log T)}{\varphi(q)}h^2(\log\log q)^2\Bigr)$$

where

$$c(b,d)=\begin{cases}128 & \text{if (i) } b^*=b \text{ and } d^*=d\\ & \text{or (ii) } b^*=d,\\ 104 & \text{if (i) } b^*=b \text{ and } d^*\neq d\\ & \text{or (ii) } d^*=d \text{ and } b^*\neq b,\\ 96 & \text{otherwise}\end{cases}$$

since

$$b(p)b'(p)=\begin{cases}1 & \text{if (i) } p\equiv b \text{ and } p\equiv b^*(\mathrm{mod}\,q)\\ & \text{or (ii) } p\equiv d \text{ and } p\equiv d^*(\mathrm{mod}\,q),\\ -1 & \text{if (i) } p\equiv b \text{ and } p\equiv d^*(\mathrm{mod}\,q)\\ & \text{or (ii) } p\equiv d \text{ and } p\equiv b^*(\mathrm{mod}\,q),\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\int_T^{T+H}\Bigl|\sum_\chi g(\chi)\bigl(S(t+h,\chi^*)-S(t,\chi^*)\bigr)\Bigr|^2dt$$
$$=\frac{2}{\pi^2}H\frac{\log(h\log T)}{\varphi(q)}+O\Bigl(H\Bigl(\frac{\log(h\log T)}{\varphi(q)}\Bigr)^{1/2}\Bigr)+O\bigl(Hh^2(\log\log q)^2\bigr)+O(H).$$

And

$$\int_T^{T+H}\Bigl|\sum_\chi g(\chi)\bigl(S(t+h,\chi^*)-S(t,\chi^*)\bigr)\Bigr|^4dt$$
$$=\frac{c(b,d)}{(2\pi)^4}H\Bigl(\frac{\log(h\log T)}{\varphi(q)}\Bigr)^2+$$
$$+O\Bigl(H\Bigl(\frac{\log(h\log T)}{\varphi(q)}\Bigr)^{3/2}\Bigr)+O\bigl(Hh^4(\log\log q)^4\bigr)+$$
$$+O(H)+O\Bigl(H\frac{\log(h\log T)}{\varphi(q)}h^2(\log\log q)^2\Bigr).$$

Hence we get our lemma.

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Received on 12. 4. 1974 (565)

ACTA ARITHMETICA
XXIX (1976)

The half dimensional sieve

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§ 1. Introduction. The purpose of this paper is to describe the half dimensional sieve in its general form (with conditions (Ω_1) and (Ω_2) of Halberstam and Richert type [3]) and to show that this sieve is powerful enough to establish the asymptotic formulae. In the paper a variant of Brun's method is used (without any weights or other combinatorial devices). Estimations of the 'sifting function' which are given in Theorem 1 cannot be essentially improved. Theorem 2 shows that in a special case the main term of the upper estimation coincides with the main term of the asymptotic formula for the sifting function. The fundamental result of the paper is included in Theorem 4. From this theorem we obtain the asymptotic formula for the number of quasi-primes of the form u^2+v^2+c lying in a short interval of consecutive terms of an arithmetic progression with the difference large in comparision to the length of the interval. It is the contents of Corollary 1. From Corollary 1 we obtain in particular the following Landau's theorem

$$B(x) := \sum_{\substack{n\leqslant x\\ n=a^2+b^2}} 1 \sim B\frac{x}{\sqrt{\log x}}$$

where

$$B = \frac{1}{\sqrt{2}}\prod_{p\equiv -1 (\mathrm{mod}\, 4)}\left(1-\frac{1}{p^2}\right)^{-1/2}.$$

Our proof is elementary and does not make use of the Prime Number Theorem (in contrast to the Levin and Fainleib's iterative method [11], p. 379). In 1953 K. Prachar [7] proved

$$(1.1)\qquad B(x, k, l) := \sum_{\substack{n\leqslant x\\ n=a^2+b^2\\ n\equiv l (\mathrm{mod}\, k)}} 1 \sim B_k\frac{x}{\sqrt{\log x}}$$