ACTA ARITHMETICA
XXX (1976)

Distribution mod 1 of additive functions on the set of divisors

by

I. KÁTAI (Budapest)

For any real number x, let $\{x\} = x - [x]$ denote the fractional part of x, and $\|x\| = \min(\{x\}, 1-\{x\})$ the distance of x to the nearest integer. Let d denote a general divisor of the integer n, and the symbol $\sum_d$ denote a summation extended over the all positive divisors of n. Let $\tau(n)$ be the number of divisors of n.

Let $f(n)$ be an additive function. Our aim is to investigate the distribution of $\{f(d)\}$.

Let

$$h_n(x) = \frac{1}{\tau(n)} \sum_{\{f(d)\}<x} 1. \tag{1}$$

We shall give a necessary and sufficient condition to that on a sequence of asymptotic density 1,

$$h_n(x) \to x \tag{2}$$

uniformly for $x \in [0, 1]$. We define the discrepancy

$$\Delta(n) = \sup_{0\le\alpha<\beta<1} |h_n(\beta) - h_n(\alpha) - (\beta-\alpha)|. \tag{3}$$

It is well-known that the relation (2) is equivalent to the relation

$$\Delta(n) \to 0. \tag{4}$$

THEOREM. *The relation $\Delta(n) \to 0$ holds on a sequence of asymptotic density 1 if and only if the series*

$$\sum_p \frac{\|2mf(p)\|^2}{p} = \infty \tag{5}$$

for $m = 1, 2, \ldots$

Proof. We need the following result of Erdös and Turán [1], which we state now as

LEMMA 1. *Let $x_1, \ldots, x_N$ be any real numbers,*

$$l(x) = \frac{1}{N} \sum_{\{x_j\}<x} 1,$$

$$S_m = \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i m x_j} \quad (m = 1, 2, \ldots),$$

and

$$\Delta = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |l(\beta) - l(\alpha) - (\beta - \alpha)|.$$

Then

$$\Delta \leqslant c\left(\frac{1}{T} + \sum_{m=1}^{T} \frac{|S_m|}{m}\right), \tag{6}$$

c being an absolute constant.

LEMMA 2. *Let $g(n)$ be a multiplicative function and satisfy*

$$0 \leqslant g(n) \leqslant 1.$$

Then

$$\sum_{n \leqslant x} g(n) \leqslant c_1 x \exp\left(\sum_{p<x} \frac{g(p)-1}{p}\right),$$

where c_1 denotes an absolute positive constant.

For the proof see for example [2].

Let

$$s_m(n) = \prod_{p^\alpha \| n} (1 + e^{2\pi i m f(p)} + \ldots + e^{2\pi i m f(p^\alpha)}),$$

and

$$g_m(n) = \frac{|s_m(n)|}{\tau(n)}. \tag{7}$$

We obtain immediately that $0 \leqslant g_m(n) \leqslant 1$, and

$$g_m(p) = |\cos 2\pi m f(p)|.$$

First we consider the sum

$$A_m(x) = \sum_{p<x} \frac{1 - g_m(p)}{p}.$$

Since $|\cos \pi\alpha|$ is an even function and periodical mod 1, we can put

$$g_m(p) = \big|\cos\big(\pi \|2mf(p)\|\big)\big|.$$

By elementary calculation we have

$$1 - |\cos \pi\alpha| \geqslant c_2 \alpha^2, \quad 0 \leqslant \alpha \leqslant 1/2, \tag{8}$$

c_2 being a positive constant.

Hence

$$A_m(x) \geqslant c_2 B_m(x) \quad (c_2 > 0),$$

where

$$B_m(x) = \sum_{p \leqslant x} \frac{\|2mf(p)\|^2}{p}. \tag{9}$$

From Lemma 1 we get

$$K(x) \overset{\text{def}}{=} \sum_{n \leqslant x} \Delta(n) \leqslant c\,\frac{x}{T} + c \sum_{m \leqslant T} \frac{1}{m} \sum_{n \leqslant x} g_m(n),$$

whence by Lemma 2 we get

$$K(x) \leqslant c\,\frac{x}{T} + c \cdot c_1 \sum_{m \leqslant T} \frac{1}{m} \exp\big(-A_m(x)\big). \tag{10}$$

Suppose now that the condition (5) holds. Then $A_m(x) \to \infty$ for every fixed m, and so uniformly for $m \leqslant T = T(x)$, when $T(x)$ tends to infinity sufficiently slowly. Thus we get

$$\frac{K(x)}{x} \to 0 \quad (x \to \infty),$$

whence the assertion of our theorem immediately follows. Now we prove the necessity of (5). Suppose that

$$\sum_p \frac{\|2mf(p)\|^2}{p} < \infty \tag{11}$$

for a fixed m. This is equivalent to the condition

$$\sum_p \frac{1 - |\cos 2\pi m f(p)|}{p} < \infty.$$

Thus for the multiplicative function $g_m(n)$ the following conditions hold:

$$0 \leqslant g_m(n) \leqslant 1, \qquad \sum_p \frac{1 - g_m(p)}{p} < \infty.$$

Then, by using a theorem of H. Delange,

$$\frac{1}{x} \sum_{\substack{n \leq x \\ (n,2)=1}} g_m(n) \to M(g) \neq 0.$$

Thus by a suitable positive δ we get that

$$\frac{|s_m(n)|}{\tau(n)} > \delta$$

on a set of integers having positive density. As it is well-known, in this set

$$\Delta(n) > \delta.$$

By this the proof of the necessity of condition (5) is completed.

References

[1] P. Erdös and P. Turán, *On a problem in the theory of uniform distribution I, II*, Indag. Math. 10 (1948), pp. 370–378, 406–413.

[2] R. R. Hall, *Halving an estimate obtained from Selberg's upper bound method*, Acta Arith. 25 (1974), pp. 347–351.

Received on 28. 1. 1975 (668)