3. To show that WUD (mod N) implies Dirichlet-WUD (mod N) one has only to observe that if

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

are two Dirichlet series with non-negative and bounded coefficients with their abscissas of convergence equal to a and b respectively ($a \leqslant b$) and moreover for x tending to infinity we have

$$\sum_{n\leqslant x} a_n = \big(1+o(1)\big) \sum_{n \leqslant x} b_n,$$

then $a = b$ and $\lim_{s\to a+0} f(s)/g(s) = 1$. (See e.g. [2], § 8, Satz 8.)

We may finally state a corollary to the theorem proved in Section 2:

COROLLARY. *If f is a multiplicative function which is integer-valued and WUD* (mod N), *then it satisfies the condition* (*). ■

It would be interesting to determine, whether the Dirichlet-WUD (mod N) is in fact weaker than WUD (mod N) for multiplicative functions.

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Elementary methods in the theory of L-functions, III
The Deuring-phenomenon

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1. Deuring [2] proved in 1933 that if the class number $h(-D)$ of the imaginary quadratic field belonging to the fundamental discriminant $-D<0$ is equal to 1 for an infinite sequence of $D_n\to\infty$, then the Riemann hypothesis for $\zeta(s)$ is true. Mordell [5] proved in 1934 that if $h(-D)\not\to\infty$ for $D\to\infty$, then the Riemann hypothesis is true. These striking results showed a curious connection between the possibly existing real zeros of special real L-functions (which exist by the theorem of Hecke (see [4]), if $h(-D) < C_0\sqrt{D}/\log D$) and the non-trivial zeros of the ζ-function.

In [6] we proved that if

$$(1.1) \qquad h(-D) \leqslant \frac{\log D}{2\log\log D} \quad \text{and} \quad \chi(n) = \left(\frac{-D}{n}\right)$$

then for the greatest real zero $1-\delta$ of $L(s,\chi) = L(s)$

$$\delta = \frac{L(1)}{\prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}}\big(1+o(1)\big) = \frac{6h(-D)}{\prod_{p|D}\left(1+\frac{1}{p}\right)\pi\sqrt{D}}\big(1+o(1)\big).$$

In this paper we shall demonstrate that, assuming a little stronger upper bound for $h(-D)$ than (1.1), we can determine up to a factor $1+o(1)$ the values of the corresponding L-function in a great domain of the critical strip. Our result will also show that except for the real zero $1-\delta$ mentioned above, neither $L(s,\chi)$ nor $\zeta(s)$ has a zero in this domain. As a consequence we have also a weakened form of Mordell's theorem [5], namely that if $h(-D)\not\to\infty$ for $D\to\infty$, then $\zeta(s)$ has no zero in the half-plane $\sigma > \frac{3}{4}$.

Siegel [8] has shown that our assumption (1.1) cannot be valid for infinitely many D's, because by his theorem for an arbitrary $\varepsilon > 0$

$$(1.2) \qquad h(-D) > D^{1/2-\varepsilon} \quad \text{for} \quad D > D_0(\varepsilon).$$

But as the constant $D_0(\varepsilon)$ is ineffective, this theorem does not give any information for specific values of D's. In our theorems all the constants implied in the o and O symbols will be effective.

In our theorems $h(-D)$ will denote the class number of the imaginary quadratic field belonging to the fundamental discriminant $-D<0$; $L(s) = L(s,\chi)$ the L-function belonging to the real primitive character $\chi(n) = (-D/n)$. We shall assume that $D > D_1$, where D_1 is an absolute effective constant. Then we state

THEOREM 1. *If the inequality*

$$h(-D) \leqslant \frac{\log D}{2\log\log D} \tag{1.3}$$

holds and H denotes the domain

$$H = \left\{s;\ |1-s| \leqslant \frac{1}{\log^4 D}\right\}, \tag{1.4}$$

then $L(s)$ has a single simple real zero $1-\delta$ in H, for which one has (¹)

$$\delta = \frac{L(1)}{\prod\limits_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}}\left(1+O\left(\frac{1}{\log D}\right)\right) = \frac{6h(-D)\left(1+O\left(\frac{1}{\log D}\right)\right)}{\prod\limits_{p|D}\left(1+\frac{1}{p}\right)\pi\sqrt{D}}, \tag{1.5}$$

and for $s \in H$ the relation

$$\begin{aligned} L(s)-L(1) &= L(s) - \frac{\pi h(-D)}{\sqrt{D}} \\ &= (s-1)\prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}\left(1+O\left(\frac{1}{\log D}\right)\right) \end{aligned} \tag{1.6}$$

holds.

THEOREM 2. *We define a domain $H(\varepsilon, D)$, depending on ε, for which $0<\varepsilon<1/8$, and on D, where $D > D_1(\varepsilon)$ ($D_1(\varepsilon)$ is an effective constant depending on ε). Let*

$$\begin{aligned} H(\varepsilon, D) = \{s;\ s = 1-\tau+it,\ &|1-s| \geqslant 1/\log^4 D,\ 0 \leqslant \tau \leqslant \tfrac{1}{4}-\varepsilon, \\ &|s| \leqslant D^{\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)\frac{1}{\varrho}-\frac{3}{4}} \text{ where } \varrho = \max(\tau, D^{-\varepsilon/4})\}. \end{aligned} \tag{1.7}$$

If the inequality

$$h(-D) \leqslant (\log D)^{3/4} \tag{1.8}$$

holds, then neither $L(s)$ nor $\zeta(s)$ has a zero in $H(\varepsilon, D)$, and for $s \in H(\varepsilon, D)$ one has

$$L(s) = \frac{\zeta(2s)}{\zeta(s)}\prod_{p|D}\left(1+\frac{1}{p^s}\right)\left[1+O\left(\exp\left(-\tfrac{1}{8}\log^{1/4} D\right)\right)\right]. \tag{1.9}$$

Theorem 1 will easily follow from the results of the paper II of this series [6] (the main result, the formula (1.5) is even contained in it), because the most critical point, the calculation of $L'(1)$ was made already in [6]. We mention here Theorem 1 in this formulation only to complete the result of Theorem 2. Our method for proving Theorem 2, i.e. (1.9), will be the same as applied in [6] to show (1.5).

2. For proving Theorem 1, we need besides Lemma 1, proved in [6], only some easy lemmas.

LEMMA 1. *If (1.3) is valid, then the relation*

$$L'(1) = \prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}\left(1+O\left(\frac{1}{\log D}\right)\right) > 1 \tag{2.1}$$

holds.

This is Theorem 2 in [6].

LEMMA 2. *If (1.3) holds, then for $s \in H$ one has*

$$L'(s) = L'(1)\left(1+O\left(\frac{1}{\log D}\right)\right) \neq 0. \tag{2.2}$$

Proof. Using the known inequality

$$L''(s) = O(\log^3 D) \quad \text{for} \quad s \in H \tag{2.3}$$

(which one can easily prove by partial summation) we have

$$L'(1)-L'(s) = \int_s^1 L''(z)\,dz = O(|1-s|\log^3 D) = O\left(\frac{1}{\log D}\right) = O\left(\frac{L'(1)}{\log D}\right),$$

since by Lemma 1 $L'(1) > 1$.

LEMMA 3. *If (1.3) holds, then for $s_1, s_2 \in H$, $s_1 \neq s_2$,*

$$L(s_1) \neq L(s_2).$$

Proof. Using Lemma 1 and Lemma 2 we have

$$\begin{aligned} L(s_2)-L(s_1) &= \int_{s_1}^{s_2} L'(z)\,dz = \int_{s_1}^{s_2} L'(1)\left(1+O\left(\frac{1}{\log D}\right)\right)dz \\ &= (s_2-s_1)L'(1)\left(1+O\left(\frac{1}{\log D}\right)\right) \neq 0. \end{aligned} \tag{2.4}$$

(¹) p always denotes throughout this paper a prime.

So from Lemma 2 and Lemma 3 we know that $L(s)$ can have at most one, simple zero in the domain H. On the other hand for the real $s = 1-\tau \in H$ we have by (2.4)

$$L(1-\tau) = L(1) - \tau L'(1)\left(1 + O\left(\frac{1}{\log D}\right)\right).$$

Here

$$L(1-\tau) > 0 \quad \text{for} \quad \tau = \tau_1 = \frac{L(1)}{2L'(1)},$$

$$L(1-\tau) < 0 \quad \text{for} \quad \tau = \tau_2 = \frac{2L(1)}{L'(1)},$$

because here $1-\tau_1$ and $1-\tau_2$ belong to H.

Thus $L(s)$ has a real zero $1-\delta$, for which the relation

$$\delta = \frac{L(1)}{L'(1)}\left(1+O\left(\frac{1}{\log D}\right)\right) = \frac{L(1)}{\prod\limits_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}}\left(1+O\left(\frac{1}{\log D}\right)\right)$$

holds, and (2.4) gives with $s_1 = 1$ the relation

$$L(s) - L(1) = (s-1)\prod_{p|D}\left(1+\frac{1}{p}\right)\frac{\pi^2}{6}\left(1+O\left(\frac{1}{\log D}\right)\right)$$

which proves Theorem 1.

3. To prove Theorem 2 we need the generalization of Lemma 2 of [6] (which we proved in [7]) for complex values of s.

LEMMA 4. *Let χ be any real or complex non-principal character modulo D, $L(s) = L(s, \chi)$.*

Then with the notations

$$g(n) = \sum_{d|n} \chi(d), \quad 1 \neq s = 1-\tau+it, \quad A_s = \max\left(1, \frac{1}{|1-s|}\right) \tag{3.1}$$

for $0 \leqslant \tau \leqslant \frac{1}{2}$ and for an arbitrary real $x \geqslant |s|^2 A_s \sqrt{D}$ the relation

$$\sum_{n\leqslant x}\frac{g(n)}{n^s} = L(s)\zeta(s) + \frac{L(1)x^{1-s}}{1-s} + O\,\frac{x^\tau |s| \sqrt[4]{D}\sqrt{A_s}\log D \log x}{\sqrt{x}} \tag{3.2}$$

holds.

Proof. Let z be a number, to be chosen later, for which $1 \leqslant z \leqslant x$. Then

$$\sum_{n\leqslant x}\frac{g(n)}{n^s} = \sum_{d\leqslant x}\frac{\chi(d)}{d^s}\sum_{m\leqslant x/d}\frac{1}{m^s} = \sum\nolimits_1 + \sum\nolimits_2. \tag{3.3}$$

Using that for $s \neq 1-\tau+it$, $0 \leqslant \tau \leqslant \frac{1}{2}$

$$\sum_{m\leqslant M}\frac{1}{m^s} = \zeta(s) + \frac{M^{1-s}}{1-s} + O\left(\frac{|s|}{M^{1-\tau}}\right)$$

(where the constant in O symbol is absolute), we get

$$\sum\nolimits_1 = \sum_{d\leqslant z}\frac{\chi(d)}{d^s}\left(\zeta(s) + \frac{x^{1-s}}{(1-s)d^{1-s}} + O\left(\frac{|s|d^{1-\tau}}{x^{1-\tau}}\right)\right) \tag{3.4}$$

$$= \left(\sum_{d\leqslant z}\frac{\chi(d)}{d^s}\right)\zeta(s) + \frac{x^{1-s}}{1-s}\sum_{d\leqslant z}\frac{\chi(d)}{d} + O\left(\frac{|s|zx^\tau}{x}\right).$$

Using the Pólya–Vinogradov inequality

$$\left|\sum_{d=a}^{b}\chi(d)\right| \ll \sqrt{D}\log D,$$

by partial summation we get the inequalities

$$\sum_{d>z}\frac{\chi(d)}{d^s} = O\left(\frac{|s|\sqrt{D}\log D z^\tau}{z}\right), \tag{3.5}$$

$$\sum_{d>z}\frac{\chi(d)}{d} = O\left(\frac{\sqrt{D}\log D}{z}\right) \tag{3.6}$$

and

$$\sum\nolimits_2 = \sum_{z<d\leqslant x}\frac{\chi(d)}{d^s}\sum_{m\leqslant x/d}\frac{1}{m^s} = O\left(\frac{|s|\sqrt{D}\log D x^\tau \log x}{z}\right). \tag{3.7}$$

So we have from the formulae (3.3)–(3.7)

$$\sum_{n\leqslant x}\frac{g(n)}{n^s} = L(s)\zeta(s) + \frac{L(1)x^{1-s}}{1-s} + \tag{3.8}$$

$$+ O\left(\frac{|s||\zeta(s)|\sqrt{D}\log D z^\tau}{z}\right) + O\left(\frac{\sqrt{D}\log D x^\tau}{z|1-s|}\right) +$$

$$+ O\left(\frac{|s|\sqrt{D}\log D x^\tau \log x}{z}\right) + O\left(\frac{|s|zx^\tau}{x}\right).$$

We shall use that for $1 \neq s = 1-\tau+it$, $0 \leqslant \tau \leqslant \frac{1}{2}$, the estimation

$$|\zeta(s)| \ll |s|^\tau \log(|s|+1) A_s \tag{3.9}$$

holds, where

$$A_s = \max\left(1, \frac{1}{1-s}\right).$$

So if we choose

$$z = \sqrt{x}\sqrt[4]{D}\sqrt{A_s} \quad (\leqslant x)$$

then as $|s|z \leqslant x$, $\log(|s|+1) < \log x$, the first three error terms are

$$O\left(\frac{|s|\sqrt{D}\log D x^\tau A_s \log x}{z}\right)$$

and so all the error terms have the required order

$$O\left(\frac{|s|\log D \log x \cdot x^\tau \sqrt[4]{D}\sqrt{A_s}}{\sqrt{x}}\right). \quad \blacksquare$$

Further on set for $s = 1-\tau+it \in H(\varepsilon, D)$

$$y = y(s) = D^{\left(\frac{1}{2}-\varepsilon\right)\frac{1}{\varrho}} \quad (\varrho = \max(\tau, D^{-\varepsilon/4})). \tag{3.10}$$

Then with the notations of Lemma 4, we have

LEMMA 5. *If* (1.8) *holds, then for* $s \in H(\varepsilon, D)$ *one has*

$$\sum_{n\leqslant y} \frac{g(n)}{n^s} = L(s)\zeta(s) + O(D^{-\varepsilon/2}). \tag{3.11}$$

Proof. For $s \in H(\varepsilon, D)$ the inequalities

$$A_s = \max\left(1, \frac{1}{|1-s|}\right) \leqslant \log^4 D,$$

$$|s|^2\sqrt{A_s}\sqrt[4]{D} \leqslant \log^2 D \cdot D^{\left(\frac{1}{2}-\varepsilon\right)\frac{1}{\varrho}-\frac{3}{2}+\frac{1}{4}} < y$$

hold, so we can apply Lemma 4 with $x = y$.

Here

$$y^\tau \leqslant y^\varrho = D^{1/2-\varepsilon}, \tag{3.12}$$

further

$$\log y \leqslant \frac{1}{\varrho}\log D \leqslant D^{\varepsilon/4}\log D.$$

Hence

$$\sqrt{A_s}\log D \log y \cdot y^\tau \sqrt[4]{D} \leqslant \log^4 D \cdot D^{\frac{\varepsilon}{4}+\frac{1}{2}-\varepsilon+\frac{1}{4}} \leqslant D^{\frac{3}{4}-\frac{\varepsilon}{2}}.$$

Thus the error term in Lemma 4 is

$$O\left(\frac{\sqrt{A_s}\log D \log y \cdot y^\tau \sqrt[4]{D}|s|}{\sqrt{y}}\right) = O\left(\frac{D^{\frac{3}{4}-\frac{\varepsilon}{2}+\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)\frac{1}{\varrho}-\frac{3}{4}}}{D^{\left(\frac{1}{4}-\frac{\varepsilon}{2}\right)\frac{1}{\varrho}}}\right) = O(D^{-\varepsilon/2}).$$

On the other hand from (1.8) follows by Dirichlet's class number formula

$$L(1) = O\left(\frac{(\log D)^{3/4}}{\sqrt{D}}\right)$$

and thus we have

$$\left|\frac{y^{1-s}}{1-s}L(1)\right| = O\big(y^\tau A_s L(1)\big) = O\left(D^{1/2-\varepsilon}\log^2 D \frac{\log^{3/4} D}{\sqrt{D}}\right) = O(D^{-\varepsilon/2}). \quad \blacksquare$$

4. From Lemma 5 it is obvious that we must investigate the sum $\sum_{n\leqslant y} g(n)/n^s$.

This sum was investigated in [6] in the special case $s = 1$ and $y = D^2$ where we proved the relation

$$\sum_{n\leqslant D^2} \frac{g(n)}{n} \sim \prod_{p|D}\left(1+\frac{1}{p}\right)\zeta(2).$$

Now we shall prove the analogue for $s \in H(\varepsilon, D)$. With our previous notations we state

LEMMA 6. *If* (1.8) *holds, then for* $s \in H(\varepsilon, D)$ *one has*

$$\sum_{n\leqslant y} \frac{g(n)}{n^s} = \prod_{p|D}\left(1+\frac{1}{p^s}\right)\zeta(2s) + O\big(\exp(-\tfrac{1}{7}\log^{1/4} D)\big). \tag{4.1}$$

Proof. At first we note that

$$g(n) = \sum_{d|n}\chi(d) = \prod_{p_i^{\alpha_i}\| n}\big(1+\chi(p_i)+\ldots+\chi^{\alpha_i}(p_i)\big) \geqslant 0 \tag{4.2}$$

and $g(n) \leqslant d(n)$.

For $j = -1, 0, 1$ we define the following sets

$$A_j = \{u;\ p\,|\,u \to \chi(p) = j\}. \tag{4.3}$$

Further let

$$R = \{r;\ r = bm,\ b \in A_0,\ m \in A_{-1}\}. \tag{4.4}$$

Then an arbitrary natural n can be written in the form

$$n = ar = abm \quad \text{where} \quad a \in A_1,\ b \in A_0,\ m \in A_{-1},\ r \in R.$$

First we investigate the sum

$$\sum_{\substack{a\in\mathcal{A}_1\\1<a\leqslant y}}\frac{g(a)}{a^{\sigma}}$$

where $\sigma = 1-\tau \geqslant \frac{3}{4}+\varepsilon$ and y was defined in (3.10).

We have

$$\sum_{\substack{a\in\mathcal{A}_1\\1<a\leqslant y}}\frac{g(a)}{a^{\sigma}} \leqslant \prod_{\substack{p\leqslant y\\ \chi(p)=1}}\left(1+\frac{2}{p^{\sigma}}+\frac{3}{p^{2\sigma}}+\ldots\right)-1 \tag{4.5}$$

$$= \prod_{\substack{p\leqslant y\\ \chi(p)=1}}\frac{1}{\left(1-\frac{1}{p^{\sigma}}\right)^2}-1 \leqslant \exp\left(12\sum\frac{1}{p^{\sigma}}\right)-1.$$

It is easy to show (see Davenport [1], Hilfssatz 1 and Haneke [3], Hilfssatz 14) the inequality

$$\sum_{\substack{a\in\mathcal{A}_1\\1<a\leqslant\sqrt{D}/2}} 1 \leqslant h(-D). \tag{4.6}$$

Hence if $\chi(p)=1$, then

$$p^{h(-D)+1} > \sqrt{D}/2, \quad \text{i.e.,} \quad p > \tfrac{1}{2}D^{\frac{1}{2(h(-D)+1)}} \tag{4.7}$$

and the number of primes for which $\chi(p)=1$ and $p\leqslant\sqrt{D}/2$ according to (4.6) does not exceed $h(-D)$; thus we get

$$\sum_{\substack{p\leqslant\sqrt{D}/2\\ \chi(p)=1}}\frac{1}{p^{\sigma}} \leqslant 2h(-D)\exp\left(-\frac{\sigma\log D}{2\big(h(-D)+1\big)}\right) \tag{4.8}$$

$$\leqslant 2(\log D)^{3/4}\exp\left(\frac{-3\log D}{8(\log^{3/4}D+1)}\right) = O\big(\exp(-\tfrac{1}{3}\log^{1/4}D)\big).$$

On the other hand from (4.2) follows

$$\Bigg(\sum_{\substack{\sqrt{D}/2<p\leqslant D\\ \chi(p)=1}}\frac{1}{p^{\sigma}}\Bigg)^2 \leqslant \sum_{D/4<n\leqslant D^2}\frac{g(n)}{n^{\sigma}} \leqslant \sum_{D/4<n\leqslant D^2}\frac{g(n)}{n^{3/4+\varepsilon}}. \tag{4.9}$$

Now applying Lemma 4 which for real s is Lemma 2 of [6], (proved in [7]) with $s=\frac{3}{4}+\varepsilon$, and with the values $x_1 = D/4$, $x_2 = D^2$, subtracting

the two equalities we get

$$\sum_{D/4<n\leqslant D^2}\frac{g(n)}{n^{3/4+\varepsilon}} \tag{4.10}$$

$$= \frac{1}{\frac{1}{4}-\varepsilon}\left((D^2)^{1/4-\varepsilon}-\left(\frac{D}{4}\right)^{1/4-\varepsilon}\right)L(1)+O\left(\frac{D^{1/4-\varepsilon}D^{1/4}\log^2 D}{\sqrt{D}}\right)$$

$$= O\left(D^{1/2-2\varepsilon}\frac{\log^{3/4}D}{\sqrt{D}}\right)+O\left(\frac{\log^2 D}{D^{\varepsilon}}\right) = O(D^{-\varepsilon/2}).$$

Thus (4.9) and (4.10) imply

$$\sum_{\substack{\sqrt{D}/2<p\leqslant D\\ \chi(p)=1}}\frac{1}{p^{\sigma}} = O(D^{-\varepsilon/4}). \tag{4.11}$$

Mutatis mutandis we have

$$\sum_{\substack{D<p\leqslant y\\ \chi(p)=1}}\frac{1}{p^{\sigma}} \leqslant \sum_{D<n\leqslant y}\frac{g(n)}{n^{1-\tau}} \leqslant \sum_{D<n\leqslant y}\frac{g(n)}{n^{1-\varrho}} \tag{4.12}$$

$$= \frac{1}{\varrho}(y^{\varrho}-D^{\varrho})L(1)+O\left(\frac{D^{\varrho}\sqrt[4]{D}\log^2 D\sqrt{1/\varrho}}{\sqrt{D}}\right)$$

$$= O\left(\frac{1}{\varrho}y^{\varrho}L(1)\right)+O\left(\frac{D^{1/4-\varepsilon}\sqrt[4]{D}\log^2 D\cdot D^{\varepsilon/8}}{\sqrt{D}}\right)$$

$$= O\left(D^{\varepsilon/4}\cdot D^{1/2-\varepsilon}\frac{(\log D)^{3/4}}{\sqrt{D}}\right)+O\left(\frac{\log^2 D}{D^{7\varepsilon/8}}\right) = O(D^{-\varepsilon/2}).$$

So (4.8), (4.11) and (4.12) together give

$$\sum_{\substack{p\leqslant y\\ \chi(p)=1}}\frac{1}{p^{\sigma}} = O\big(\exp(-\tfrac{1}{3}\log^{1/4}D)\big). \tag{4.13}$$

Hence taking into account (4.5) we get

$$\sum_{\substack{a\in\mathcal{A}_1\\1<a\leqslant y}}\frac{g(a)}{a^{\sigma}} \leqslant \exp\Bigg(12\sum_{\substack{p\leqslant y\\ \chi(p)=1}}\frac{1}{p^{\sigma}}\Bigg)-1 = O\Bigg(\sum_{\substack{p\leqslant y\\ \chi(p)=1}}\frac{1}{p^{\sigma}}\Bigg) \tag{4.14}$$

$$= O\big(\exp(-\tfrac{1}{3}\log^{1/4}D)\big).$$

Now we turn our attention to the sums

$$\sum_{\substack{r\leqslant y\\ r\in R}}\frac{g(r)}{r^s},\qquad \sum_{\substack{r\leqslant y\\ r\in R}}\frac{g(r)}{r^\sigma}.$$

It follows from (4.2) and from (4.4) that for $r=bm\in R$ ($b\in A_0$, $m\in A_{-1}$) the relation

$$g(r)=g(b)g(m)=g(m)=\begin{cases}1, & \text{if } m=l^2,\\ 0, & \text{if } m\neq l^2\end{cases}\tag{4.15}$$

holds. So we have

$$\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^\sigma}\leqslant\sum_{r\in R}\frac{g(r)}{r^{3/4}}\leqslant\sum_{\substack{b\in A_0\\ \mu(b)\neq 0}}\frac{1}{b^{3/4}}\sum_{l=1}^{\infty}\frac{1}{(l^2)^{3/4}}\tag{4.16}$$

$$=\prod_{p|D}\left(1+\frac{1}{p^{3/4}}\right)\zeta(\tfrac{3}{2})=O\left(\exp\left(\sum_{p|D}\frac{1}{p^{3/4}}\right)\right).$$

If the number of distinct prime divisors of D is $\nu(D)=\nu$, further $p_1,\ldots,p_\nu$ are the first ν primes and $D_\nu=p_1p_2\ldots p_\nu$, then since $D\geqslant D_\nu$ the inequality

$$\sum_{p|D}\frac{1}{p^{3/4}}\leqslant\sum_{j=1}^{\nu}\frac{1}{p_j^{3/4}}=O\left(\frac{(\log D)^{1/4}}{\log\log D_\nu}\right)\leqslant\frac{1}{120}\log^{1/4}D\tag{4.17}$$

holds. With (4.16) this gives

$$\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^\sigma}=O\left(\exp\left(\tfrac{1}{120}\log^{1/4}D\right)\right).\tag{4.18}$$

Combining this result with (4.14) we get

$$\left|\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^s}\sum_{\substack{a\in A_1\\ 1<a\leqslant y/r}}\frac{g(a)}{a^s}\right|\leqslant\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^\sigma}\sum_{\substack{a\in A_1\\ 1<a\leqslant y}}\frac{g(a)}{a^\sigma}\tag{4.19}$$

$$=O\left(\exp\left(\tfrac{1}{120}\log^{1/4}D\right)\right)O\left(\exp\left(-\tfrac{1}{3}\log^{1/4}D\right)\right)=O\left(\exp\left(-\tfrac{1}{4}\log^{1/4}D\right)\right).$$

Hence

$$\sum_{n\leqslant y}\frac{g(n)}{n^s}=\sum_{r\in R,\,r\leqslant y}\frac{g(r)}{r^s}\sum_{a\in A_1,\,a\leqslant y/r}\frac{g(a)}{a^s}\tag{4.20}$$

$$=\sum_{r\in R,\,r\leqslant y}\frac{g(r)}{r^s}+\sum_{r\in R,\,r\leqslant y}\frac{g(r)}{r^s}\sum_{\substack{a\in A_1\\ 1<a\leqslant y/r}}\frac{g(a)}{a^s}$$

$$=\sum_{r\in R,\,r\leqslant y}\frac{g(r)}{r^s}+O\left(\exp\left(-\tfrac{1}{4}\log^{1/4}D\right)\right).$$

From (4.15) follows

$$\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^s}=\sum_{\substack{b|D\\ \mu(b)\neq 0}}\frac{g(b)}{b^s}\sum_{\substack{r\in R\\ r^2\leqslant y/b}}\frac{1}{r^{2s}}.\tag{4.21}$$

As we have shown in (4.7) if $\chi(p)=1$, then

$$p>R_0=\frac{1}{2}\exp\left(\frac{\log D}{2\big(h(-D)+1\big)}\right)\geqslant\exp\left(\tfrac{1}{3}(\log D)^{1/4}\right),$$

i.e., for $n\leqslant R_0$ we have $n\in R$, and so as $\sqrt{y/D}\geqslant R_0$ we get

$$\sum_{\substack{r\in R\\ r^2\leqslant y/b}}\frac{1}{r^{2s}}=\sum_{n=1}^{\infty}\frac{1}{n^{2s}}+O\left(\sum_{l>R_0}\frac{1}{l^{2\sigma}}\right)\tag{4.22}$$

$$=\zeta(2s)+O\left(\frac{1}{\sqrt{R_0}}\right)=\zeta(2s)+O\left(\exp\left(-\tfrac{1}{16}\log^{1/4}D\right)\right).$$

In (4.17) we proved the estimation

$$\prod_{p|D}\left(1+\frac{1}{p^{3/4}}\right)=O\left(\exp\left(\tfrac{1}{120}(\log D)^{1/4}\right)\right).\tag{4.23}$$

Thus from (4.21), (4.22) and (4.23) we have

$$\sum_{\substack{r\in R\\ r\leqslant y}}\frac{g(r)}{r^s}=\sum_{\substack{b|D\\ \mu(b)\neq 0}}\frac{1}{b^s}\sum_{\substack{r\in R\\ r\leqslant y/b}}\frac{1}{r^{2s}}\tag{4.24}$$

$$=\sum_{\substack{b|D\\ \mu(b)\neq 0}}\frac{1}{b^s}\left(\zeta(2s)+O\left(\exp\left(-\tfrac{1}{6}\log^{1/4}D\right)\right)\right)$$

$$=\zeta(2s)\prod_{p|D}\left(1+\frac{1}{p^s}\right)+O\left(\exp\left(-\tfrac{1}{6}\log^{1/4}D\right)\right)\prod_{p|D}\left(1+\frac{1}{p^{3/4}}\right)$$

$$=\zeta(2s)\prod_{p|D}\left(1+\frac{1}{p^s}\right)+O\left(\exp\left(-\tfrac{1}{7}\log^{1/4}D\right)\right).$$

From (4.20) and (4.24) we have (4.1), i.e.,

$$\sum_{n\leqslant y}\frac{g(n)}{n^s}=\zeta(2s)\prod_{p|D}\left(1+\frac{1}{p^s}\right)+O\left(\exp\left(-\tfrac{1}{7}\log^{1/4}D\right)\right).\ \blacksquare\tag{4.25}$$

5. Now Theorem 2 will be the easy consequence of Lemma 5 and Lemma 6. Namely, using (4.17) for $s\in H(\varepsilon, D)$, we have

$$\left|\zeta(2s)\prod_{p|D}\left(1+\frac{1}{p^s}\right)\right| \geqslant \prod_{p|D}\left(1-\frac{1}{p^{3/4}}\right) \geqslant \exp\left(-\sum_{p|D}\frac{1}{p^{3/4}}\right) \geqslant \exp\left(-\tfrac{1}{120}\log^{1/4}D\right). \tag{5.1}$$

Further the error term of Lemma 5 is obviously

$$O(D^{-\varepsilon/2}) = O\left(\exp\left(-\tfrac{1}{7}\log^{1/4}D\right)\right). \tag{5.2}$$

So from (3.11), (4.1), (5.1) and (5.2) for $s\in H(\varepsilon, D)$ follows the relation

$$\begin{aligned} L(s)\zeta(s) &= \sum_{n\leqslant y}\frac{g(n)}{n^s}+O(D^{-\varepsilon/2}) \\ &= \zeta(2s)\prod_{p|D}\left(1+\frac{1}{p^s}\right)+O\left(\exp\left(-\tfrac{1}{7}\log^{1/4}D\right)\right) \\ &= \zeta(2s)\left(1+\frac{1}{p^s}\right)\left(1+O\left(\exp\left(-\tfrac{1}{7}\log^{1/4}D\right)\right)\right) \neq 0 \end{aligned} \tag{5.3}$$

which proves Theorem 2.

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