Euler characteristics of 2-manifolds and light open maps *

by

John D. Baildon (Dunmore, Penn.)

Abstract. Whyburn has developed a formula for light open maps between 2-manifolds relating the degree of the map, the cardinalities of the singular set and its inverse and the Euler characteristics of the spaces involved. (Analytic Topology: X; 7.3) A modification of this is found for the situation where the domain is a union of oriented 2-manifolds without boundary intersecting in arcs and the range is an oriented 2-manifold without boundary.

1. Introduction. S. Stoilow [3, 4] began the study of light open maps on manifolds by analyzing them for the case where the domain and range were both regions on a 2-sphere or plane. Whyburn continued this with an extensive study of light open maps on 2-manifolds [6, X], showing, among other things, that they are finite to one and that the image is necessarily a 2-manifold. He also showed that a light open map between 2-manifolds without boundary is locally $w = z^k$ for some k and that one between compact 2-manifolds is simplicial. The result with which we shall be interested establishes a relationship between the degree (order) of the map, the Euler characteristics of the spaces involved, and the cardinalities of the singular set and its inverse. E. E. Floyd [2] has proven a similar appearing formula for periodic homeomorphisms of prime period on certain spaces.

2. Whyburn's result and the main theorem. Whyburn's result when restricted to the case of a map between compact manifolds without boundary says the following [6, X, 7.3]:

THEOREM 1 (Whyburn). *If A and B are compact 2-dimensional manifolds without boundary and $f(A) = B$ is a light open map of degree k (i.e., is k to one), then $k\chi(B) - \chi(A) = kr - n$, where r and n are the numbers of points in Y and $f^{-1}(Y)$, respectively, when Y is the set of all y in B such that $f^{-1}(y)$ contains a point of A at which f is not locally topological (i.e., Y is the image of the singular set of f) and $\chi(Z)$ is the Euler characteristic of Z.*

* This paper is a portion of the author's Ph. D. thesis written under the direction of Professor Louis F. McAuley at the State University of New York at Binghamton.

Our main effort will be to get a formula similar to Whyburn's (although more involved) when f is a light open map onto an oriented 2-manifold from a space consisting of a finite number of oriented 2-manifolds intersecting each other only in a finite number of disjoint one dimensional arcs. Throughout the paper we will take arc to mean one dimensional arc. We will show that f will be k to one for some k. For notation, let Y denote the set of all points in B whose inverses contain a point where $f/_{A_i}$ is not locally topological for some i. Let r be the number of points in Y and n the number in $f^{-1}(Y)$. Furthermore, let Q denote $f^{-1}(Y)\cdot\big(\bigcup_{i\neq j}(A_i\cdot A_j)\big)$, where $V\cdot W$ is the intersection of V and W. The symbol $\# V$ will represent the cardinality of V. We give the statement of our main theorem now, but defer the proof until the end of the paper.

THEOREM 2. *Let f be a light open map of $\bigcup_{i=1}^{n} A_i$ onto B, where the A_i and B are oriented 2-manifolds without boundary and $\bigcup (A_i\cdot A_j)$ is a finite union of disjoint arcs. Then $k\chi(B)-\sum_{i=1}^{n}\chi(A_i) = kr-n-\sum_{x\in Q} d(x)$, where k is the order of f (f is k to one) and $d(x) = -1+\#\{A_i\colon A_i \text{ contains } x\}$ (i.e., the number of duplications we get by counting x in each of the manifolds containing it).*

3. Simplicial maps. We begin work toward our main result by considering a restricted case of it:

THEOREM 3. *Suppose f is a k to one open simplicial map of $A_1\cup A_2$ onto B, where A_1, A_2, and B are 2-manifolds without boundary, and $A_1\cdot A_2$ is the union of a finite number of disjoint arcs. Then $k\chi(B)-\chi(A_1)-\chi(A_2) = kr-n-\#(f^{-1}(Y)\cdot A_1\cdot A_2)$.*

We will prove this by showing f restricted to each of the A_i maps it openly onto B and then applying Whyburn's result, Theorem 1.

LEMMA 4. *Let H and K be 2-complexes with underlying spaces X and Y which are 2-manifolds without boundary. Let f be a simplicial map of X into Y taking each simplex of H homeomorphically to one of K. Then the set S of all 1-simplices in H where f is not locally one to one (where folding occurs) has the property that S^* has no endpoints, where S^* denotes the set of all points belonging to a member of S.*

Proof. If p is an endpoint of S^*, then there is only one 1-simplex t_0 in S containing p. Since X is a 2-manifold without boundary, the 2-simplices containing p, $s_1, s_2, \ldots, s_n$, form a 2-ball. Because t_0 is in S, the two 2-simplices containing it, s_1 and s_n, will be mapped to the same 2-simplex in K. However, any other pair of adjoining 2-simplices containing p must get sent to different 2-simplices in K since t_0 is the only 1-simplex in S containing p. It will follow then that $f(s_2) = f(s_{n-1})$, $f(s_3) = f(s_{n-2})$, and so on. If n is even, we get $f(s_{n/2}) = f(s_{n/2+1})$, contradicting the remark about adjoining 2-simplices. If n is odd, we will be led to the conclusion that the faces of $s_{(n+1)/2}$ containing p will both be sent to the same 1-simplex in K, contradicting the assumption that the 2-simplices of H get mapped homeomorphically to those of K. Thus S^* has no endpoints. ∎

COROLLARY 5. *Let f be a k to one open simplicial map of X onto Y, where X is the union of a finite number of 2-manifolds without boundary, A_i, such that the union of their pairwise intersections is at most a finite number of pairwise disjoint arcs and Y is also a 2-manifold without boundary. Then $f/_{A_i}$ maps each A_i openly onto Y.*

Proof. Since Y has no boundary and f is open on X, the only 1-simplices where f might not be locally one to one would have to be contained in the intersecting arcs. Restricting our attention to $f_i = f/_{A_i}$, the set S^* of the previous lemma must then be a finite union of arcs and hence is empty. Thus if a point is in the interior of a 1- or 2-simplex in H_i, the underlying complex in A_i, then f_i will be open there. If the point, p, is a vertex where f_i is not open, then the image of the star neighborhood of p in H_i has to miss a 2-simplex in the star neighborhood of $f(p)$. But this requires the image to have a free edge and thus results in S being non-empty, contrary to what we have shown. Thus f_i is open. Whyburn has shown that the light open image of a 2-manifold is itself a 2-manifold [6, X, 4.4]. Since S is empty, $f(A_i)$ must therefore be a 2-manifold without boundary and consequently must be Y or we would have a separation of Y. ∎

We now return to out proof of Theorem 3: Let Y_i be the set of points in B whose inverses contain a point of A_i where f_i is not locally one to one. Let r_i be the number of points in Y_i, n_i be the number in $f_i^{-1}(Y_i)$, and k_i be the degree of f_i. Clearly $Y = Y_1\cup Y_2 = Y_1'\cup Y_2'\cup(Y_1\cdot Y_2)$, where $Y_1' = Y_1 - Y_2$ and $Y_2' = Y_2 - Y_1$ and the last union is a disjoint one. Let r_i' and n_i' be the cardinalities of Y_i' and $f_i^{-1}(Y_i')$. Whyburn has shown that if f_i is locally one to one at each point of $f_i^{-1}(p)$, then $\# f_i^{-1}(p) = k_i$ [6, X, 6.3]. In $f^{-1}(Y_1')$ there are $k_2 r_1'$ preimages in A_2, $n_1-\#(f^{-1}(Y_1\cdot Y_2)\cdot A_1)$ preimages in A_1, and $\#(f^{-1}(Y_1')\cdot A_1\cdot A_2)$ preimages in both. Thus

$$\# f^{-1}(Y_1') = k_2 r_1' + n_1 - \#(f^{-1}(Y_1\cdot Y_2)\cdot A_1) - \#(f^{-1}(Y_1')\cdot A_1\cdot A_2)$$

and

$$\# f^{-1}(Y_2') = k_1 r_2' + n_2 - \#(f^{-1}(Y_1\cdot Y_2)\cdot A_2) - \#(f^{-1}(Y_2')\cdot A_1\cdot A_2)$$

by a similar argument. Moreover,

$$\# f^{-1}(Y_1\cdot Y_2) = \#(f^{-1}(Y_1\cdot Y_2)\cdot A_1) + \\ + \#(f^{-1}(Y_1\cdot Y_2)\cdot A_2) - \#(f^{-1}(Y_1\cdot Y_2)\cdot A_1\cdot A_2).$$

Therefore,

$$n = \#f^{-1}(Y) = k_2r_1'+n_1-\#(f^{-1}(Y_1')\cdot A_1\cdot A_2)+k_1r_2'+n_2-$$
$$-\#(f^{-1}(Y_2')\cdot A_1\cdot A_2)-\#(f^{-1}(Y_1\cdot Y_2)\cdot A_1\cdot A_2)$$
$$= k_2r_1'+k_1r_2'+n_1+n_2-\#(f^{-1}(Y)\cdot A_1\cdot A_2)\,.$$

Now $r_i' = r_i-\#(Y_1\cdot Y_2)$, $r = r_1+r_2-\#(Y_1\cdot Y_2)$, and $k = k_1+k_2$. Thus

$$n = k_2r_1+k_1r_2-k(\#(Y_1\cdot Y_2))+n_1+n_2-\#(f^{-1}(Y)\cdot A_1\cdot A_2)$$
$$= kr_1-k_1r_1+kr_2-k_2r_2-k(\#(Y_1\cdot Y_2))+n_1+n_2-\#(f^{-1}(Y)\cdot A_1\cdot A_2)$$
$$= kr-k_1r_1-k_2r_2+n_1+n_2-\#(f^{-1}(Y)\cdot A_1\cdot A_2).$$

Rearranging terms, we have

$$k_1r_1-n_1+k_2r_2-n_2 = kr-n-\#(f^{-1}(Y)\cdot A_1\cdot A_2)\,.$$

Since f_i is an open map onto B, Theorem 1 is applicable and we get

$$k\chi(B)-\chi(A_1)-\chi(A_2) = kr-n-\#(f^{-1}(Y)\cdot A_1\cdot A_2)\,. \blacksquare$$

We generalize Theorem 3 by extending it to a finite union of 2-manifolds:

THEOREM 6. *Let f be a k to one open simplicial map of $X = \bigcup_{i=1}^{m} A_i$ onto B, where B and the A_i are 2-manifolds without boundary and $\bigcup (A_i\cdot A_j)$ is a finite number of disjoint arcs. Then $k\chi(B)-\sum_{i=1}^{m}\chi(A_i) = kr-n-\sum_{x\in Q} d(x)$ where $d(x) = -1+\#\{A_i\colon x\in A_i\}$ and $Q = f^{-1}(Y)\cdot(\bigcup(A_i\cdot A_j))$.*

Proof. By induction. From Corollary 5, $f_i = f/_{A_i}$ will map openly onto B. If $m=2$, $\sum_{x\in Q} d(x)$ is simply $\#Q = \#(f^{-1}(Y)\cdot A_1\cdot A_2)$ and so the conclusion follows from Theorem 3. Now suppose the theorem is true up to $m-1$ and let Y' denote Y on $\bigcup_{i=1}^{m-1} A_i = X'$ with k', r', n', d', and Q' playing corresponding roles. Then

(1) $$n = \#f^{-1}(Y) = \#(f^{-1}(Y'-Y_m))+\#(Y_m-Y')+\#(f^{-1}(Y'\cdot Y_m))$$

because of disjoint unions. Reasoning similar to that in Theorem 3 yields

(2) $$\#(f^{-1}(Y'-Y_m)) = k_m(\#(Y'-Y_m))+n'-\#(f^{-1}(Y'\cdot Y_m)\cdot \bigcup_{i=1}^{m-1} A_i)-$$
$$-\#(f^{-1}(Y'-Y_m)\cdot(\bigcup_{i=1}^{m-1}(A_m\cdot A_i)))\,.$$

Since Y_m-Y' has k_i preimages in each of the A_i up to $i=m-1$ and $k' = k_1+\ldots+k_{m-1}$, $f^{-1}(Y_m-Y')$ has $k'(\#(Y_m-Y'))-\sum_{x\in R} d'(x)$ points in $\bigcup_{i=1}^{m-1} A_i$ where $R = f^{-1}(Y_m-Y')\cdot\bigcup_{i=1}^{m-1}(A_i\cdot A_j)$, we conclude that

(3) $$\#(f^{-1}(Y_m-Y')) = k'(\#(Y_m-Y'))-\sum_{x\in R} d'(x)+n_m-$$
$$-\#(f^{-1}(Y_m\cdot Y')\cdot A_m)-\#(f^{-1}(Y_m-Y')\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Also,

(4) $$\#(f^{-1}(Y_m\cdot Y')) = \#(f^{-1}(Y_m\cdot Y')\cdot A_m)+\#(f^{-1}(Y_m\cdot Y')\cdot\bigcup_{i=1}^{m-1} A_i)-$$
$$-\#(f^{-1}(Y_m\cdot Y')\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Combining these four results yields

$$n = k_m(\#(Y'-Y_m))+n'+k'(\#(Y_m-Y'))+n_m-\sum_{x\in R} d'(x)-$$
$$-\#(f^{-1}(Y)\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Then

$$n = k_m(r'-\#(Y_m\cdot Y'))+n'+k'(r_m-\#(Y_m\cdot Y'))+n_m-\sum_{x\in R} d'(x)-$$
$$-\#(f^{-1}(Y)\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Since $k = k_m+k'$ and $r = r_m+r'-\#(Y'\cdot Y_m)$, we get

$$n = kr-k'r'+n'-k_mr_m+n_m-\sum_{x\in R} d'(x)-\#(f^{-1}(Y)\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Using Theorem 1 on A_m and the induction assumption, this becomes

(5) $$n = kr-k'\chi(B)+\sum_{i=1}^{m-1}\chi(A_i)-\sum_{x\in Q} d'(x)-k_m\chi(B)+\chi(A_m)-$$
$$-\sum_{x\in R} d'(x)-\#(f^{-1}(Y)\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

Now $Q'\cup R = f^{-1}(Y)\cdot\bigcup_{i,j=1}^{m-1}(A_i\cdot A_j)$ and it is easy to see that

$$\sum_{x\in Q} d(x) = \sum_{x\in R} d'(x)+\sum_{x\in Q'} d'(x)+\#(f^{-1}(Y)\cdot\bigcup_{i=1}^{m-1} A_i\cdot A_m)\,.$$

This, the fact that $k = k' + k_m$, and (5) yield

$$n = kr - k\chi(B) + \sum_{i=1}^{m} \chi(A_i) - \sum_{x \in Q} d(x) . \blacksquare$$

4. Preliminary general results. We would like to get a more general result than that just obtained by dropping the k to one and simplicial conditions. A natural extension of a theorem of Whyburn [6, X, 3.2] will show that a light open map on spaces of the type we have been considering is necessarily finite to one.

THEOREM 7. *Let f be a light open map of $\bigcup_{i=1}^{m} A_i$ onto B where the A_i and B are 2-manifolds without boundary and $\bigcup_{i,j=1}^{m} (A_i \cdot A_j)$ is a finite number of disjoint arcs. Then f is finite to one.*

Proof. Let p be a point of B and suppose that $f^{-1}(p)$ is infinite. Since B is a 2-manifold and f is light, we can find $2m+1$ discs, K_i, intersecting pairwise exactly at p, forming a $2m+1$ petalled flower $K = \bigcup K_i$ and such that each component of $f^{-1}(K_i) \cdot A_j$ is contained in a Euclidean neighborhood in A_j. Since $f^{-1}(K)$ is locally connected, it has a finite number of components each mapping onto K, and so one of them, C, contains infinitely many points of $f^{-1}(p)$. The restriction of f to C will also be a light open map onto K. For each i, there is a component R_i of $f^{-1}/_C(K_i - p)$ having infinitely many points of $f^{-1}(p)$ in its boundary. The R_i can be chosen so that the intersection I of their boundaries contains infinitely many points of $f^{-1}(p)$. But in each of the 2-manifolds, we cannot have three regions meeting in more than two points. Consequently, there are a maximum of $2m$ such regions in X, contradicting our construction of $2m+1$. Thus $f^{-1}(p)$ must be finite for each p. ■

We also wish to show that even if f is not necessarily simplicial, $f/_{A_i}$ is still open:

THEOREM 8. *Let f be a light open of $\bigcup_{i=1}^{m} A_i$ onto B where A_i and B are oriented 2-manifolds without boundary and $\bigcup (A_i \cdot A_j)$ is a finite number of disjoint arcs. Then $f/_{A_i}$ is open.*

We will use a theorem of Titus and Young [5] to show $f/_{A_i}$ is quasi-open (a continuous function f from A to B is quasi-open provided that for any image point q and any open set U containing a compact component of $f^{-1}(q)$, q is in the interior of $f(U)$ relative to B). This will prove our theorem since light quasi-open maps are open.

Let M and N be oriented 2-manifolds. A map f from M to N is sense preserving at the point p of M provided that, if K denotes the component of $f^{-1}(f(p))$ that contains p, then K is compact and there is an open set W containing K such that, if U is any open subset of W that contains K but has no point of $f^{-1}(f(p))$ on its boundary, then the degree of $f/_U$ at $f(p)$ is positive [5].

THEOREM 9 (Titus and Young [5]). *Let f be a map from M to N where M and N are fixed oriented manifolds satisfying:*

(i) *there is a closed set C_f such that f is sense preserving at each point of $M - C_f$,*

(ii) *f is constant on each component of the interior of C_f, and*

(iii) *$f(C_f)$ is closed and nowhere dense in N.*

Then f is quasi open.

Proof of Theorem 8. We will show the conditions of Theorem 9 are satisfied. The set C_f will be the union of the $A_i \cdot A_j$'s. Condition (ii) of Theorem 9 is satisfied vacuously. Church and Hemmingsen [1] have shown that an open map defined on a locally compact space and having point inverses consisting of isolated points has the property that it preserves the dimension of closed subsets. Since Theorem 7 shows our point inverses are finite, $f(C_f)$ must be one dimensional and hence is nowhere dense in B, thus satisfying (iii). It remains only to be shown that f is sense preserving at each point of $A_i - C_f$ or sense reversing at each. The fact that at each point of $A_i - C_f$, f will be locally topologically equivalent to $w = z^k$ on $|z| \leqslant 1$ for some k [6, X, 5.1] will force the set of points where f is sense preserving and the set where f is sense reversing to be open sets. Since $A_i - C_f$ is connected, one of the sets is all of $A_i - C_f$, so f will satisfy the conditions of Theorem 9 (recognizing that a comparable result holds for sense reversing in (i)). ■

LEMMA 10. *If f is a finite to one open map of a space X onto a 2-manifold without boundary B and A is a compact 2-manifold without boundary contained in X, having its boundary relative to X containing no simple closed curve, then $f(A) = B$.*

Proof. Suppose C is a boundary curve of the manifold $f(A)$. Because of lifting properties of light open maps and the fact that f is finite to one, $f^{-1}(C)$ must contain a simple closed curve. But $f^{-1}(C)$ is contained in $\mathrm{Bd}_X(A)$, leading to a contradiction of our hypothesis. Thus $f(A)$ has no boundary curve and so must be all of B. ■

5. Proof of main result. From Theorem 8 and Lemma 10 we know $f/_{A_i}$ is open and onto B. Whyburn has shown that we can get subdivisions of A_i and B so that $f/_{A_i}$ is a simplicial map [6, X, 7.2]. Taking a common subdivision of B and the induced subdivisions of the A_i's, we get that f is a simplicial map from $\bigcup_{i=1}^{m} A_i$ onto B. Our result follows from Theorem 6. ■

References

[1] P. T. Church and E. Hemmingsen, *Light open maps on n-manifolds*, Duke Math. J. 27 (1960), pp. 527–536.

[2] E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. 72 (1952), pp. 138–147.

[3] S. Stoilow, *Sur les transformations continues et la topologie des fonctions analytiques*, Annales Scientifiques de l'Ecole Normale Superieure 63 (1928), pp. 347–382.

[4] — Annales de l'Institut Henri Poincaré 2 (1932), pp. 233–266.

[5] C. J. Titus and G. S. Young, *The extension of interiority with some applications*, Trans. Amer. Math. Soc. 103 (1962), pp. 329–340.

[6] G. T. Whyburn, *Analytic Topology*, 2nd ed. Providence: Amer. Math. Soc. (1963).

WORTHINGTON SCRANTON CAMPUS
PENNSYLVANIA STATE UNIVERSITY
Dunmore, Pennsylvania

Accepté par la Rédaction le 8. 4. 1974

Souslin-Kleene does not imply Beth

by

Finn V. Jensen (Warszawa)

Abstract. It is proved that $L_{\min}$ (first-order logic without quantifiers) fulfils the Craig interpolation theorem. An extension of $L_{\min}$ is given, which satisfies the Souslin–Kleene interpolation theorem, but not the Beth definability theorem.

Comparing with [3] we then have shown that for extensions of $L_{\min}$, the only valid implications between Craig, Beth, S-K and weak Beth, are the following:

$$\begin{tikzcd}[column sep=small] & \text{Craig} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{S-K} \arrow[dr, Rightarrow] & & \text{Beth} \arrow[dl, Rightarrow] \\ & \text{weak Beth} & \end{tikzcd}$$

For basic definitions, readers are refered to [3].

DEFINITION 1. Let $\mathfrak{B}$ be a structure of type τ, let A be a set of constant-symbols, and let $\{b_a|\ a\in A\}\subseteq|\mathfrak{B}|$. Then $\mathfrak{B}(\langle b_a|\ a\in A\rangle)$ denotes the structure $\mathfrak{B}'$ of type $\tau\cup A$, s.t. $\mathfrak{B}'\restriction\tau=\mathfrak{B}$, and $a^{\mathfrak{B}'}=b_a$ for all $a\in A$. The *open formula* $K(X/A)$ with $X=\{x_a|\ a\in A\}$ as free variables, is the following class of sets:

$$\{K(X/A)^{\mathfrak{B}}|\ \mathfrak{B}\text{ a structure of type }\tau\},$$

where

$$K(X/A)^{\mathfrak{B}}=\{B\subseteq|\mathfrak{B}||\ B=\{b_a|\ a\in A\}\text{ and }\mathfrak{B}(\langle b_a|\ a\in A\rangle)\in K\},$$

i.e. $K(X/A)^{\mathfrak{B}}$ is the set of tuples from $|\mathfrak{B}|$ satisfying K.

DEFINITION 2. A logic L has

1) *Souslin-Kleene* property if for any PC_L, P, if $\bar{P}$ is PC_L then P is EC_L.

2) *Beth property* if the following holds: Let R be an n-ary relation symbol and let K be an EC_L of type $\tau\cup\{R\}$ s.t. for each structure $\mathfrak{A}$ of type τ there exists at most one $\mathfrak{B}\in K$ s.t. $\mathfrak{A}=\mathfrak{B}\restriction\tau$, then there exists an open formula F of type τ with n free variables s.t. for each $\mathfrak{A}\in K$ we have $R^{\mathfrak{A}}=F^{\mathfrak{A}\restriction\tau}$.