On the Whitehead theorem in shape theory II

by

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Abstract. Recently Š. Ungar and the author have proved a relative "Hurewicz theorem" in shape theory [10]. In the present paper this theorem is applied to obtain a homological and a cohomological version of the "Whitehead theorem" in shape theory ([7], [13]). One also studies the category of pro-groups and characterizes monomorphisms and epimorphisms in this category. In particular it is proved that bimorphisms and isomorphisms coincide in the category of pro-groups.

1. Introduction. Recently M. Moszyńska [13] has proved a "Whitehead theorem" in shape theory. It applies to shape maps $\boldsymbol{f}\colon (X, x_0)\to(Y, y_0)$ of finite-dimensional metric continua and gives sufficient conditions for $\boldsymbol{f}$ to be a shape equivalence. Precisely, if $n_0 = \max\{1+\dim X, \dim Y\}$ and $(\boldsymbol{f})_{k\#}\colon \pi_k(\boldsymbol{X}, \boldsymbol{x})\to\pi_k(\boldsymbol{Y}, \boldsymbol{y})$ is a bimorphism of pro-groups for $1\leqslant k<n_0+1$ and is an epimorphism for $k = n_0+1$, then $\boldsymbol{f}$ is a shape equivalence.

In [7] the author has reproved and extended Moszyńska's result. It was shown that the conclusion is valid also in the following cases:

(i) X is a compact Hausdorff continuum and Y a metric continuum.

(ii) X and Y are connected topological spaces but $\boldsymbol{f}$ is generated by a continuous map $f\colon (X, x_0)\to(Y, y_0)$.

Recently Š. Ungar and the author [10] proved a relative "Hurewicz theorem" in shape theory. This result is applied in the present paper to obtain additional results concerning the Whitehead theorem. One first shows that in the case of 1-shape connected spaces one can improve by 1 the dimensional assumptions in the Whitehead theorem obtaining thus the same dimensional conditions as in the classical case (see Theorem 2).

The same theorem enables one to establish a Whitehead theorem in terms of homology and cohomology respectively (see Theorems 3 and 4). The cohomology version is especially simple because it is stated in terms of the usual Čech cohomology groups.

In the last section 8, we analyze epimorphisms and monomorphisms in the category pro$(\mathscr{G})$ of pro-groups and we prove that in this case bimorphisms are actually isomorphism.

This paper can be regarded as a continuation of [7] and we use [7] as general reference for notions and notations.

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2. Homology pro-groups

2.1. Let $\mathscr{W}^2$ denote the homotopy category of pairs of spaces having the homotopy type of a simplicial pair. Let $(\boldsymbol{X}, \boldsymbol{A}) = ((X, A)_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathscr{W}^2$ over a quasi-ordered set $(\Lambda, \leqslant)$, i.e. an object of $\mathrm{pro}(\mathscr{W}^2)$. In analogy with the case of homotopy pro-groups (see 5 in [7]) one defines *k-th homology pro-group* with coefficients in G of $(\boldsymbol{X}, \boldsymbol{A})$ as the inverse system $H_k(\boldsymbol{X}, \boldsymbol{A}; G) = (H_k((X, A)_\lambda; G), (p_{\lambda\lambda'})_{k*}, \Lambda)$. It is a pro-group, i.e. an object of $\mathrm{pro}(\mathscr{G})$, where $\mathscr{G}$ denotes the category of groups. For $G = Z$ we write merely $H_k(\boldsymbol{X}, \boldsymbol{A})$ for $H_k(\boldsymbol{X}, \boldsymbol{A}; G)$.

Every morphism $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{Y}, \boldsymbol{B}) = ((Y, B)_\mu, q_{\mu\mu'}, M)$ in $\mathrm{pro}(\mathscr{W}^2)$ induces a morphism of pro-groups $(\boldsymbol{f})_{k*}\colon H_k(\boldsymbol{X}, \boldsymbol{A}; G) \to H_k(\boldsymbol{Y}, \boldsymbol{B}; G)$. If $\boldsymbol{f}$ is given by $f\colon M \to \Lambda$ and $f_\mu\colon (X, A)_{f(\mu)} \to (Y, B)_\mu$, then $(\boldsymbol{f})_{k*}$ is given by f and

$$(f_\mu)_{k*}\colon H_k((X, A)_{f(\mu)}; G) \to H_k((Y, B)_\mu; G).$$

Clearly, $(\boldsymbol{gf})_{k*} = (\boldsymbol{g})_{k*}(\boldsymbol{f})_{k*}$, $(\boldsymbol{1})_{k*} = \boldsymbol{1}$, so that isomorphic systems have isomorphic homology pro-groups.

Similarly, one defines $H_k(\boldsymbol{X}; G)$ for inverse systems $\boldsymbol{X}$ in $\mathscr{W}$, where $\mathscr{W}$ denotes the homotopy category of spaces having the homotopy type of a simplicial complex.

2.2. Every object $(\boldsymbol{X}, \boldsymbol{A})$ of $\mathrm{pro}(\mathscr{W}^2)$ determines objects $\boldsymbol{A}$ and $\boldsymbol{X}$ of $\mathrm{pro}(\mathscr{W})$ and morphisms $\boldsymbol{i}\colon \boldsymbol{A} \to \boldsymbol{X}$, $\boldsymbol{j}\colon \boldsymbol{X} \to (\boldsymbol{X}, \boldsymbol{A})$ given by the inclusions $i_\lambda\colon A_\lambda \to X_\lambda$, $j_\lambda\colon X = (X_\lambda, \emptyset) \to (X_\lambda, A_\lambda)$, $\lambda \in \Lambda$. We then obtain induced morphisms of pro-groups

$$(\boldsymbol{i})_k\colon H_k(\boldsymbol{A}; G) \to H_k(\boldsymbol{X}; G), \quad (\boldsymbol{j})_k\colon H_k(\boldsymbol{X}; G) \to H_k(\boldsymbol{X}, \boldsymbol{A}; G).$$

We also define morphisms

$$\partial_k\colon H_k(\boldsymbol{X}, \boldsymbol{A}; G) \to H_{k-1}(\boldsymbol{A}; G)$$

given by the boundary homomorphism $\partial_\lambda\colon H_k(X_\lambda, A_\lambda; G) \to H_{k-1}(A_\lambda; G)$. We thus have the *homology sequence of pro-groups* of $(\boldsymbol{X}, \boldsymbol{A})$:

$$\ldots \to H_k(\boldsymbol{A}; G) \to H_k(\boldsymbol{X}; G) \to H_k(\boldsymbol{X}, \boldsymbol{A}; G) \to H_{k-1}(\boldsymbol{A}; G) \to \ldots$$
$$\ldots H_1(\boldsymbol{X}, \boldsymbol{A}; G) \to H_0(\boldsymbol{A}; G) \to H_0(\boldsymbol{X}; G) \to H_0(\boldsymbol{X}, \boldsymbol{A}; G) \to 0.$$

Since the corresponding sequence for each λ is exact in the category of groups $\mathscr{G}$, we conclude that this sequence is exact in $\mathrm{pro}(\mathscr{G})$ (see 5.2 of [7]).

2.3. By 5.3 of [7] the following holds:

Let $(\boldsymbol{X}, \boldsymbol{A})$ be an object of $\mathrm{pro}(\mathscr{W}^2)$. If for a given $k \geqslant 1$, $\boldsymbol{i}_k\colon H_k(\boldsymbol{A}; G) \to H_k(\boldsymbol{X}; G)$ is an epimorphism in $\mathrm{pro}(\mathscr{G})$ and $\boldsymbol{i}_{k-1}\colon H_{k-1}(\boldsymbol{A}; G) \to H_{k-1}(\boldsymbol{X}; G)$ is a monomorphism in $\mathrm{pro}(\mathscr{G})$, then $H_k(\boldsymbol{X}, \boldsymbol{A}; G) = 0$.

2.4. For a pair of topological spaces (X, A) one can define homology pro-groups up to isomorphic objects in $\mathrm{pro}(\mathscr{G})$ as $H_k(\boldsymbol{X}, \boldsymbol{A}; G)$, where $(\boldsymbol{X}, \boldsymbol{A})$ is any inverse system in $\mathscr{W}^2$ associated with (X, A) in the sense of Morita ([12]; see also 3 in [7]). Notice that $H_k(\boldsymbol{X}, \boldsymbol{A}; G) = 0$ if $k > \dim X$, because (X, A) admits a system $(\boldsymbol{X}, \boldsymbol{A})$ associated with (X, A) and such that $\dim X_\lambda \leqslant \dim X$ for each $\lambda \in \Lambda$ (see 3.4 of [7]).

The inverse limit $\varprojlim H_k(\boldsymbol{X}, \boldsymbol{A}; G)$ is the Čech homology group $H_k(X, A; G)$ based on all open normal coverings of (X, A) [12].

3. The Hurewicz isomorphism theorem

3.1. Let $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x})$ be an object of $\mathrm{pro}(\mathscr{W}_0^2)$ and let $\pi_k(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = (\pi_k(X, A, x)_\lambda, (p_{\lambda\lambda'})_{k\#}, \Lambda)$, $k \geqslant 1$, be the *k-th homotopy pro-group* (see 5 in [7]). The Hurewicz homomorphism

$$\Phi_k\colon \pi_k(X, A, x)_\lambda \to H_k(X, A)_\lambda, \quad \lambda \in \Lambda,$$

induces a morphism of pro-groups

$$\boldsymbol{\Phi}_k\colon \pi_k(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) \to H_k(\boldsymbol{X}, \boldsymbol{A})$$

called the *Hurewicz morphism.*

For a pointed pair of topological spaces (X, A, x_0) one has also a Hurewicz morphism of the homotopy pro-groups into corresponding homology pro-groups.

DEFINITION. A pair (X, A, x_0) is said to be *n-shape connected* (or approximatively n-connected) provided both X and A are connected and $\pi_k(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = 0$ for $1 \leqslant k \leqslant n$. Similarly one defines n-shape connectedness of (X, x_0).

Now we can state the "Hurewicz theorem" proved in [10] (Theorem 3).

THEOREM A. *Let (X, A, x_0) be an $(n-1)$-shape connected pair of topological spaces, $n \geqslant 2$ and let (A, x_0) be 1-shape connected. Then*

(i) $H_k(\boldsymbol{X}, \boldsymbol{A}) = 0$ *for* $1 \leqslant k \leqslant n-1$,

(ii) $\boldsymbol{\Phi}_n\colon \pi_n(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) \to H_n(\boldsymbol{X}, \boldsymbol{A})$

is an isomorphism of pro-groups.

4. Shape deformation retracts

4.1. The main step in the proof of the Whitehead theorem in [7] was the following result (see Theorem 4 of [7]).

THEOREM B. *Let (X, A, x_0) be a pair of pointed topological spaces and let $\dim X = n < \infty$. If (X, A, x_0) is $(n+1)$-shape connected, then the inclusion $i\colon (A, x_0) \to (X, x_0)$ induces a shape equivalence.*

Using the Hurewicz theorem we shall now strengthen Theorem B in the 1-shape connected case to the following

THEOREM 1. *Let (X, A, x_0) be a pair of pointed topological spaces and let $\dim X = n < \infty$. If (X, A, x_0) is n-shape connected and (A, x_0) is 1-shape connected, the*

the inclusion $i\colon (A, x_0)\to(X, x_0)$ *induces a shape equivalence, i.e.* (A, x_0) *is a shape deformation retract of* (X, x_0).

Proof. If $n = 0$, then $\pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = 0$ because one can assume that $\dim X_\lambda = 0$ for all $\lambda\in\Lambda$ and therefore $\pi_k(X, A, x)_\lambda = 0$, for $k>0$. If $n\geqslant 1$, Theorem A implies that $\pi_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x})\approx H_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x})$. One can assume that $\dim X_\lambda\leqslant n$ and therefore $H_{n+1}(X, A, x)_\lambda = 0$ which implies $H_{n+1}(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = 0$. We have proved thus that (X, A, x_0) is actually $(n+1)$-shape connected so that Theorem B applies.

COROLLARY 1. *Let* (X, x_0) *be a topological space,* $\dim X = n<\infty$. *If* (X, x_0) *is* n*-shape connected, then* $\operatorname{Sh}(X, x_0) = 0$.

5. The Whitehead theorem

5.1. We recall from Sections 7.1 and 7.3 of [7] some facts concerning the "mapping cylinder":

THEOREM C. *Let* $f\colon (X, x_0)\to(Y, y_0)$ *be a shape map of connected topological spaces. We assume in addition that either*

(i) X *is compact Hausdorff and* Y *compact metric; or*

(ii) f *is induced by a continuous map* f.

Then there is a pair of connected spaces (Z, X, x_0) *and an embedding* $j\colon (Y, y_0)\to(Z, x_0)$ *which admits a shape inverse* $g\colon (Z, x_0)\to(Y, y_0)$. *Furthermore,* $f = gi$, *where* $i\colon (X, x_0)\to(Z, x_0)$ *is the inclusion map and*

$$\dim Z\leqslant n_0 = \max(1+\dim X, \dim Y).$$

Now we can derive from Theorems C and 1 an improved Whitehead theorem for 1-shape connected spaces following the proof of Theorems 6 and 7 given in [7]. We obtain

THEOREM 2. *Let* $f\colon (X, x_0)\to(Y, y_0)$ *be a shape map of connected* 1*-shape connected finite-dimensional topological spaces. We assume in addition that either*

(i) X *is compact Hausdorff and* Y *is compact metric; or*

(ii) f *is induced by a continuous map* f.

If $(f)_{k\#}\colon \pi_k(\boldsymbol{X}, \boldsymbol{x})\to\pi_k(\boldsymbol{Y}, \boldsymbol{y})$ *is an isomorphism of pro-groups for* $1\leqslant k<n_0 = \max\{1+\dim X, \dim Y\}$ *and an epimorphism for* $k = n_0$. *Then* f *is a shape equivalence.*

Proof. The assumptions on $(f)_{k\#}$ carry over to $(i)_{k\#}\colon \pi_k(\boldsymbol{X}, \boldsymbol{x})\to\pi_k(\boldsymbol{Z}, \boldsymbol{x})$ because of Theorem C. From the exactness of the sequence of homotopy pro-groups (5.4 in [7]), it follows by 5.3 in [7] that $\pi_k(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}) = 0$ for $1\leqslant k\leqslant n_0$, i.e. that (Z, X, x_0) is n_0-shape connected. Since $\dim Z\leqslant n_0$, it follows by Theorem 1 that $i\colon (X, x_0)\to(Z, x_0)$ is a shape equivalence. The same is true of f because $f = gi$.

6. The Whitehead theorem in terms of homology

6.1. For 1-shape connected spaces one has also a homological version of the Whitehead theorem.

THEOREM 3. *Let* $f\colon (X, x_0)\to(Y, y_0)$ *be a shape map of* 1*-shape connected finite-dimensional topological spaces. We assume in addition that either*

(i) X *is compact Hausdorff and* Y *compact metric; or*

(ii) f *is induced by a continuous map* f.

If $(f)_{k*}\colon H_k(X)\to H_k(Y)$ *is an isomorphism of pro-groups for* $2\leqslant k<n_0 = \max\{1+\dim X, \dim Y\}$ *and an epimorphism for* $k = n_0$, *then* f *is a shape equivalence.*

Proof. By Theorem A, $H_1(X) = H_1(Y) = 0$ so that $(f)_{1*}$ also is an isomorphism. The assumption on $(f)_{k*}$ carry over to $(i)_{k*}\colon H_k(X)\to H_k(\boldsymbol{Z})$ because of Theorem C. From the exactness of the sequence of homology pro-groups (see 2.2), it follows that $H_k(\boldsymbol{Z}, \boldsymbol{X}) = 0$ for $1\leqslant k\leqslant n_0$ (see 2.3).

Note that $\operatorname{Sh}(Z, x_0) = \operatorname{Sh}(Y, y_0)$ implies $\pi_1(\boldsymbol{Z}, \boldsymbol{x}) = \pi_1(\boldsymbol{Y}, \boldsymbol{y}) = 0$. Since also $\pi_1(\boldsymbol{X}, \boldsymbol{x}) = 0$, it follows that $(i)_{1\#}\colon \pi_1(\boldsymbol{X}, \boldsymbol{x})\to\pi_1(\boldsymbol{Z}, \boldsymbol{x})$ and $(i)_{0\#}\colon \pi_0(\boldsymbol{X}, \boldsymbol{x})\to\pi_0(\boldsymbol{Z}, \boldsymbol{x})$ are isomorphisms. By exactness of the sequence of homotopy pro-groups one obtains $\pi_1(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}) = 0$ (see 5.2 and 5.3 of [7]). Therefore, one can apply Theorem A and conclude that (Z, X, x_0) is n_0-shape connected. Since $\dim Z\leqslant n_0$, it follows from Theorem 1 that $i\colon (X, x_0)\to(Z, x_0)$ is a shape equivalence. The same holds for f because $f = gi$ and g is a shape equivalence.

6.2. COROLLARY 2. *Let* (X, x_0) *be a* 1*-shape connected finite-dimensional space. If* $H_k(X) = 0$ *for* $2\leqslant k\leqslant\dim X$, *then* $\operatorname{Sh}(X, x_0) = 0$.

Remark 1. D. S. Kahn [4] has exhibited an ∞-dimensional metric continuum (X, x_0) such that $\pi_k(\boldsymbol{X}, \boldsymbol{x}) = H_k(\boldsymbol{X}, \boldsymbol{x}) = 0$ for all k but $\operatorname{Sh}(X, x_0)\neq 0$ as shown by D. Handel and J. Segal [3]. The author and Segal [8] have considered a 1-dimensional metric continuum (X, x_0) (the Case-Chamberlin curve) for which $H_1(X) = 0$ and $\operatorname{Sh}(X, x_0)\neq 0$. However (X, x_0) is not 1-shape connected.

7. The Whitehead theorem in terms of cohomology

7.1. For every pair of topological spaces (X, A), every Abelian group G and integer $k\geqslant 0$ one defines the *k-th cohomology group* $H^k(X, A; G)$ as the direct limit of the direct system $H^k(\boldsymbol{X}, \boldsymbol{A}; G) = \big(H^k((X, A)_\lambda; G), (p_{\lambda\lambda'})^{k*}, \Lambda\big)$, where $(\boldsymbol{X}, \boldsymbol{A}) = \big((X, A)_\lambda, p_{\lambda\lambda'}, \Lambda\big)$ is any inverse system in $\mathscr{W}^2$ associated with (X, A). If $G = Z$ we simplify the notation to $H^k(X, A)$. Note that any two systems associated with (X, A) are isomorphic and therefore determine $H^k(X, A; G)$ up to an isomorphism. This definition yields the usual Čech cohomology groups based on open normal coverings (see 3 in [11]). Shape maps f induce natural homomorphisms of cohomology groups.

It is well-known that the direct limit of an exact sequence is exact (see e.g. [2], Theorem 5.4, p. 225). Therefore, the following sequence is exact

$$\ldots\to H^{k-1}(X; G)\to H^{k-1}(A; G)\to H^k(X, A; G)\to H^k(X; G)\to\ldots$$

The cohomology group $H^k(X, A; G) = 0$ if $k>\dim X$.

7.2. In order to prove a cohomological version of the Whitehead theorem, we need the following

LEMMA 1 [1]. *Let (X, A) be a pair of compact Hausdorff spaces.*

(i) *If the groups $H^k(X, A) = H^{k+1}(X, A) = 0$, then the pro-group $H_k(\boldsymbol{X}, \boldsymbol{A}) = 0$.*

(ii) *If the pro-groups $H_{k-1}(\boldsymbol{X}, \boldsymbol{A}) = H_k(\boldsymbol{X}, \boldsymbol{A}) = 0$, then the group $H^k(X, A) = 0$.*

Proof. Let $(\boldsymbol{X}, \boldsymbol{A}) = ((X, A)_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathscr{W}^2$ associated with (X, A) and consisting of pairs of finite simplicial complexes. Let us assume that $H^k(X, A) = H^{k+1}(X, A) = 0$. Then for every $\xi \in H^k(X, A)_\lambda$ there is a $\lambda' \geqslant \lambda$ such that $(p_{\lambda\lambda'})^{k*}(\xi) = 0$. Since the groups $H^{k+1}(X, A)_\lambda$ are finitely generated, each λ admits a $\lambda' \geqslant \lambda$ such that $(p_{\lambda\lambda'})^{k+1*} = 0$. Similarly, each λ' admits a $\lambda'' \geqslant \lambda'$ such that $(p_{\lambda'\lambda''})^{k*} = 0$.

Consider now the diagram

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Ext}(H^{k+1}(X, A)_{\lambda''}, Z) & \to & H_k(X, A)_{\lambda''} & \to & \operatorname{Hom}(H^k(X, A)_{\lambda''}, Z) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Ext}(H^{k+1}(X, A)_{\lambda'}, Z) & \to & H_k(X, A)_{\lambda'} & \to & \operatorname{Hom}(H^k(X, A)_{\lambda'}, Z) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Ext}(H^{k+1}(X, A)_{\lambda}, Z) & \to & H_k(X, A)_{\lambda} & \to & \operatorname{Hom}(H^k(X, A)_{\lambda}, Z) & \to 0 .
\end{array}$$

The rows are functorial exact sequences and therefore the diagram commutes (see [14], Theorem 12, p. 248).

The composition of the middle vertical arrows $(p_{\lambda\lambda''})^{k*}$ must be 0 because $(p_{\lambda'\lambda''})^{k*} = (p_{\lambda\lambda'})^{k+1*} = 0$. This means however, that the pro-group $H_k(\boldsymbol{X}, \boldsymbol{A}) = 0$, and (i) is established.

The functorial exact sequence ([14], Theorem 3, p. 243)

$$0 \to \operatorname{Ext}(H_{k-1}(X, A)_\lambda, Z) \to H^k(X, A)_\lambda \to \operatorname{Hom}(H_k(X, A)_\lambda, Z) \to 0$$

is used to obtain a similar proof for the assertion (ii). Note that (ii) holds even without the compactness restriction.

7.3. THEOREM 4. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a shape map of 1-shape connected finite-dimensional compact Hausdorff spaces. We assume in addition that either*

(i) *Y is metric; or*

(ii) *f is induced by a continuous map f.*

If $(f)^{k*}\colon H^k(Y) \to H^k(X)$ is an isomorphism for $2 \leqslant k < n_0 = \max\{1 + \dim X, \dim Y\}$ and a monomorphism for $k = n_0$, then f is a shape equivalence.

Proof. By Theorem A, $\pi_1(\boldsymbol{X}, \boldsymbol{x}) = \pi_1(\boldsymbol{Y}, \boldsymbol{y}) = 0$ implies, $H_1(\boldsymbol{X}) = H_1(\boldsymbol{Y}) = 0$. By (ii) of Lemma 1, it follows $H^1(X) = H^1(Y) = 0$. Consequently, $(f)^{k*}$ is an isomorphism also for $k = 1$.

By Theorem C the assumptions on $(f)^{k*}$ carry over to $(i)^{k*}\colon H^k(Z) \to H^k(X)$. From the exactness of cohomology it follows $H^k(Z, X) = 0$ for $2 \leqslant k \leqslant n_0$. Notice

[1] The author is indebted to Professor Y. Kodama for correcting an error in the first draft of the proof of Lemma 1.

Added in proof. Essentially the same result was proved by R. C. Lacher, *Cellularity criteria for maps*, Michigan Math. J. 17 (1970), pp. 385–396.

that also $H^{n_0+1}(Z, X) = 0$ because $\dim Z \leqslant n_0$. We conclude now by part (i) of Lemma 1 that $H_k(\boldsymbol{Z}, \boldsymbol{X}) = 0$ for $2 \leqslant k \leqslant n_0$ and the proof proceeds as in the case of Theorem 3.

7.4. COROLLARY 3. *Let (X, x_0) be a 1-shape connected finite-dimensional compact Hausdorff space. If $H^k(X) = 0$ for $2 \leqslant k \leqslant \dim X$, then* $\operatorname{Sh}(X, x_0) = 0$.

COROLLARY 4. *If X is a finite-dimensional Hausdorff continuum and $H^k(X) = 0$ for $1 \leqslant k \leqslant \dim X$, then the suspension ΣX of X is of trivial shape.*

Proof. ΣX is also a finite-dimensional Hausdorff continuum. Furthermore, $H^k(\Sigma X) = 0$ for $2 \leqslant k \leqslant \dim \Sigma X$ because $H^k(\Sigma X) = H^{k-1}(X)$. If

$$(\boldsymbol{X}, \boldsymbol{x}) = ((X, x)_\lambda, p_{\lambda\lambda'}, \Lambda)$$

is an inverse system of finite polyhedra X_λ with $\varprojlim(\boldsymbol{X}, \boldsymbol{x}) = (X, x_0)$, then $\Sigma(\boldsymbol{X}, \boldsymbol{x}) = ((\Sigma X_\lambda, x_\lambda), \Sigma p_{\lambda\lambda'}, \Lambda)$ is an inverse system of finite polyhedra with $\varprojlim \Sigma(\boldsymbol{X}, \boldsymbol{x}) = (\Sigma X, x_0)$ and is therefore associated with $(\Sigma X, x_0)$. Since each $(\Sigma X_\lambda, x_\lambda)$ is simply connected (see [16], Corollary 3, p. 454), we conclude that $(\Sigma X, x_0)$ is 1-shape connected. Consequently, Corollary 3 implies $\operatorname{Sh}(\Sigma X, x_0) = 0$.

Remark 2. The Case-Chamberlin curve has trivial cohomology groups and a non-trivial shape [8]. However, it was shown in [5] that its suspension is of trivial shape. This also follows from Corollary 4.

8. Bimorphisms in the category of pro-groups

8.1. In the original formulation of the theorem of Moszyńska [13] appears the assumption that $(f)_{k\#}$ is a bimorphism of pro-groups for $1 \leqslant k < n_0 + 1$. In every category isomorphisms are bimorphisms, i.e. are at the same time epimorphisms and monomorphism. In general the converse is false. The purpose of this section is to show that in the category $\operatorname{pro}(\mathscr{G})$ of pro-groups the converse does hold, i.e. bimorphisms of progroups are isomorphisms.

8.2. We shall first characterize epimorphisms in $\operatorname{pro}(\mathscr{G})$.

LEMMA 2. *Let H be a group and $M \subset H$ a subgroup. Then there exist a group P and two homomorphisms $\Psi, \Psi'\colon H \to P$ such that*

$$(1) \qquad \Psi|M = \Psi'|M ,$$

$$(2) \qquad (\forall h \in H \setminus M)\, \Psi(h) \neq \Psi'(h) .$$

Proof. If M is a normal subgroup, we put $P = H/M$. We take for $\Psi\colon H \to P$ the quotient map and for $\Psi'\colon H \to P$ the constant map $1 \in P$. Clearly, $\Psi|M = \Psi'|M = 1$, and if $h \in H \setminus M$, then $\Psi(h) \neq 1 = \Psi'(h)$.

Now assume that M is not a normal subgroup. Then the index of M must be $\geqslant 3$. We take for P the group of all bijections $H \to H$. For every $h \in H$ let $\Psi(h)$ be the bijection $\Psi_h\colon H \to H$ given by

$$(3) \qquad \Psi_h(x) = hx , \quad x \in H .$$

Clearly, $\Psi_{h_1 h_2} = \Psi_{h_1}\Psi_{h_2}$ so that Ψ is a homomorphism $H \to P$. We define $\Psi'\colon H \to P$ by

$$\Psi'(h) = \Psi'_h = \sigma^{-1}\Psi_h\sigma\,, \qquad h \in H\,, \tag{4}$$

where $\sigma\colon H \to H$ is a bijection still to be specified.

Since $\Psi'_{h_1h_2} = \sigma^{-1}\Psi_{h_1}\Psi_{h_2}\sigma = \Psi'_{h_1}\Psi'_{h_2}$, $\Psi'\colon H \to P$ is a homomorphism. Note that

$$\Psi'_h(x) = \sigma^{-1}\big(h\sigma(x)\big)\,, \qquad x \in H\,. \tag{5}$$

We shall now define the bijection σ. In every right coset class Mh of H, different from M, choose a representative h_α. Since the set $S = \{h_\alpha, \alpha \in A\}$ of these representatives has at least 2 elements, there is a bijection $\sigma\colon S \to S$ with no fixed points. We extend σ to a bijection $H \to H$ by putting

$$\sigma(m) = m\,, \qquad m \in M\,, \tag{6}$$

$$\sigma(mh_\alpha) = m\sigma(h_\alpha)\,, \qquad m \in M\,. \tag{7}$$

Note that $\sigma(x) = x$, $x \in H$, implies $x \in M$ because $\sigma(mh_\alpha) = m\sigma(h_\alpha) \in M\sigma(h_\alpha)$ and $(Mh_\alpha) \cap \big(M\sigma(h_\alpha)\big) = \emptyset$ because $\sigma(h_\alpha) \neq h_\alpha$.

We prove now that

$$\Psi'_h = \Psi_h\,, \qquad h \in M\,. \tag{8}$$

Denote h by $h = m' \in M$. If $x = mh_\alpha \in Mh_\alpha$, then

$$\Psi'_{m'}(x) = \Psi'_{m'}(mh_\alpha) = \sigma^{-1}(m'mh_{\alpha'}) = m'mh_\alpha\,,$$

where $h_{\alpha'} = \sigma(h_\alpha)$. On the other hand, $\Psi_{m'}(x) = m'mh_\alpha = \Psi'_{m'}(x)$. If $x = m \in M$, then $\Psi'_{m'}(x) = \sigma^{-1}(m'm) = m'm = \Psi_{m'}(x)$. We prove now that

$$\Psi'_h \neq \Psi_h\,, \qquad h \in Mh_\alpha \subset H \setminus M\,. \tag{9}$$

Assume contrary to (9) that there is an $h = mh_\alpha$, $m \in M$ such that $\Psi'_{mh_\alpha} = \Psi_{mh_\alpha}$. Since $\Psi'_{mh_\alpha}\big(\sigma^{-1}(h_\alpha^{-1})\big) = \sigma^{-1}(mh_\alpha h_\alpha^{-1}) = \sigma^{-1}(m) = m$ and

$$\Psi_{mh_\alpha}\big(\sigma^{-1}(h_\alpha^{-1})\big) = mh_\alpha\sigma^{-1}(h_\alpha^{-1})\,,$$

we would have $mh_\alpha\sigma^{-1}(h_\alpha^{-1}) = m$ and therefore $\sigma^{-1}(h_\alpha^{-1}) = h_\alpha^{-1}$, i.e. $\sigma(h_\alpha^{-1}) = h_\alpha^{-1}$. It follows $h_\alpha^{-1} \in M$ and thus $h_\alpha \in M$, which is a contradiction.

8.3. THEOREM 5. *Let $\boldsymbol{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{H} = (H_\mu, q_{\mu\mu'}, M)$ be pro-groups and let $\boldsymbol{f}\colon \boldsymbol{G} \to \boldsymbol{H}$ be a morphism of pro-groups given by $f\colon M \to \Lambda$ and by homomorphisms $f_\mu\colon G_{f(\mu)} \to H_\mu$. The morphism $\boldsymbol{f}$ is an epimorphism in* pro$(\mathscr{G})$ *if and only if it has the following property*

(e) *for each $\mu \in M$ and each $\lambda \geqslant f(\mu)$ there is a $\mu' \geqslant \mu$ such that*

$$q_{\mu\mu'}(H_{\mu'}) \subset f_\mu p_{f(\mu)\lambda}(G_\lambda)\,.$$

Proof. We first assume that $\boldsymbol{f}$ is an epimorphism. Let $\mu \in M$, $\lambda \geqslant f(\mu)$. By Lemma 2 there is a group P and there are homomorphisms $\Psi, \Psi'\colon H_\mu \to P$ such that

$$\Psi | f_\mu p_{f(\mu)\lambda}(G_\lambda) = \Psi' | f_\mu p_{f(\mu)\lambda}(G_\lambda)\,, \tag{10}$$

$$\big(\forall h_\mu \in H_\mu \setminus f_\mu p_{f(\mu)\lambda}(G_\lambda)\big)\Psi(h_\mu) \neq \Psi'(h_\mu)\,. \tag{11}$$

We can consider Ψ, Ψ' as morphisms $\boldsymbol{\Psi}, \boldsymbol{\Psi}'$ of the pro-group $\boldsymbol{H}$ into the pro-group $\boldsymbol{P}$ consisting only of the group P. Since by (11) $\Psi f_\mu p_{f(\mu)\lambda} = \Psi' f_\mu p_{f(\mu)\lambda}$, we have $\boldsymbol{\Psi f} = \boldsymbol{\Psi' f}$ and $\boldsymbol{f}$ being an epimorphism, $\boldsymbol{\Psi} = \boldsymbol{\Psi}'$ follows. This means however that there is a $\mu' \geqslant \mu$ such that

$$\Psi q_{\mu\mu'} = \Psi' q_{\mu\mu'}\,. \tag{12}$$

It follows now from (12) and (11) that for each $h_{\mu'} \in H_{\mu'}$, the element $q_{\mu\mu'}(h_{\mu'})$ must belong to $f_\mu p_{f(\mu)\lambda}(G_\lambda)$, i.e. that (e) holds.

Conversely, let us assume that (e) holds and that $\boldsymbol{g}, \boldsymbol{g}'\colon \boldsymbol{H} \to \boldsymbol{K} = (K_\nu, r_{\nu\nu'}, N)$ are morphisms, given by homomorphisms g_ν and g'_ν respectively, such that $\boldsymbol{gf} = \boldsymbol{g'f}$. There is a $\mu \geqslant g(\nu)$, $g'(\nu)$ and a $\lambda \in \Lambda$ such that $g_\nu f_{g(\nu)} p_{fg(\nu)\lambda} = g'_\nu f_{g'(\nu)} p_{fg'(\nu)\lambda}$, $\lambda \geqslant f(\mu)$, $f_{g(\nu)} p_{fg(\nu)\lambda} = q_{g(\nu)\mu} f_\mu p_{f(\mu)\lambda}$ and $f_{g'(\nu)} p_{fg'(\nu)\lambda} = q_{g'(\nu)\mu} f_\mu p_{f(\mu)\lambda}$. Then

$$g_\nu q_{g(\nu)\mu} f_\mu p_{f(\mu)\lambda} = g'_\nu q_{g'(\nu)\mu} f_\mu p_{f(\mu)\lambda}\,.$$

We choose now $\mu' \geqslant \mu$ by (e). Clearly, $g_\nu q_{g(\nu)\mu'} = g'_\nu q_{g'(\nu)\mu'}$, which shows that $\boldsymbol{g} = \boldsymbol{g}'$, i.e. proves that $\boldsymbol{f}$ is an epimorphism.

Remark 3. If $M = \Lambda$, f is the identity, and $q_{\mu\mu'} f_{\mu'} = f_\mu p_{\mu\mu'}$ for $\mu \leqslant \mu'$, we speak of a special map of systems $\underline{f}\colon \boldsymbol{G} \to \boldsymbol{H}$ (see 2.2 in [7]). In this case if all f_μ are epimorphisms, condition (e) is clearly satisfied and $\boldsymbol{f}$ is an epimorphism (see 4.3 in [7]).

8.4. We now characterize monomorphisms in pro$(\mathscr{G})$.

THEOREM 6. *Let $\boldsymbol{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{H} = (H_\mu, q_{\mu\mu'}, M)$ be pro-groups and let $\boldsymbol{f}\colon \boldsymbol{G} \to \boldsymbol{H}$ be a morphism of pro-groups given by $f\colon M \to \Lambda$ and $f_\mu\colon G_{f(\mu)} \to H_\mu$. The morphism $\boldsymbol{f}$ is a monomorphism in pro$(\mathscr{G})$ if and only if it has the following property*

(m) *for each $\lambda \in \Lambda$ there is a $\mu \in M$ and there is a $\lambda' \geqslant \lambda, f(\mu)$ such that*

$$p_{\lambda\lambda'}\big(f_\mu p_{f(\mu)\lambda'}\big)^{-1}(1) = 1\,.$$

Proof. Let us first assume that $\boldsymbol{f}$ is a monomorphism given by a special map of systems (see Remark 6). Then the kernel of $\boldsymbol{f}$ is the system $\boldsymbol{N} = (N_\mu, p_{\mu\mu'}|N_{\mu'}, M)$, where $N_\mu = (f_\mu)^{-1}(1) \subset G_\mu$ (see 4.2 in [7]). Since $\boldsymbol{f}$ is a monomorphism, $\boldsymbol{N} = 0$, i.e. for each $\mu \in M$ there is a $\mu' \geqslant \mu$ such that

$$p_{\mu\mu'} f_{\mu'}^{-1}(1) = 1\,. \tag{13}$$

In order to derive the necessary condition (m) in the general case one can apply 2.2 of [7]. There an explicit construction is described which assigns to every map of systems $\underline{f}$ an equivalent special map $\underline{f}'$. It suffices now to see how the condition (13) for $\underline{f}'$ can be expressed in terms of the original map $\underline{f}$. This is not difficult although somewhat tedious and we omit it.

We also omit the proof of sufficiency of (m) which is straightforward and similar in nature to the proof of the sufficiency of (e) in Theorem 5.

8.5. THEOREM 7. *Every bimorphism f: $\boldsymbol{G}\to\boldsymbol{H}$ of pro-groups is an isomorphism.*

Proof. There is no loss of generality in assuming that f is given by a special map of systems (see 2.2 in [7]), i.e. that $\boldsymbol{G} = (G_\mu, p_{\mu\mu'}, M)$, $\boldsymbol{H} = (H_\mu, q_{\mu\mu'}, M)$ and f is given by homomorphisms f_μ: $G_\mu\to H_\mu$ such that $q_{\mu\mu'}f_{\mu'} = f_\mu p_{\mu\mu'}$ for $\mu\leqslant\mu'$. Since f is a monomorphism, we conclude by Theorem 6 that every μ admits a $\mu^*\geqslant\mu$ such that

$$p_{\mu\mu^*}(f_{\mu^*})^{-1}(1) = 1 . \tag{14}$$

Since f is also an epimorphism, we conclude by Theorem 5 that there is a $\mu'\geqslant\mu^*$ such that

$$q_{\mu^*\mu'}(H_{\mu'})\subset f_{\mu^*}(G_{\mu^*}) . \tag{15}$$

Therefore, for each $x_{\mu'}\in H_{\mu'}$ there is a $y_{\mu^*}\in G_{\mu^*}$ such that

$$q_{\mu^*\mu'}(x_{\mu'}) = f_{\mu^*}(y_{\mu^*}) . \tag{16}$$

If y_{μ^*}, y'_{μ^*} are two such elements, then f_{μ^*} maps $(y_{\mu^*})^{-1}y'_{\mu^*}$ into 1 so that (14) implies $p_{\mu\mu^*}(y_{\mu^*}) = p_{\mu\mu^*}(y'_{\mu^*})$. In other words, $p_{\mu\mu^*}(y_{\mu^*})$ is independent of the choice of $y_{\mu^*}\in G_{\mu^*}$ satisfying (16). We obtain thus a well-defined map $h_{\lambda\lambda'}$: $H_{\lambda'}\to G_\lambda$ such that

$$f_\lambda h_{\lambda\lambda'} = q_{\lambda\lambda'} , \tag{17}$$

$$h_{\lambda\lambda'}f_{\lambda'} = p_{\lambda\lambda'} . \tag{18}$$

It is readily verified that $h_{\lambda\lambda'}$ is a homomorphism.

All this shows that we may assume that for any $\lambda\leqslant\lambda'$, $\lambda\neq\lambda'$, a homomorphism $h_{\lambda\lambda'}$: $H_{\lambda'}\to G_\lambda$ satisfying (17) and (18) is given. Indeed, one can define a new ordering $\leqslant'$ putting $\lambda\leqslant'\lambda'$ if $\lambda=\lambda'$ or if there is a homomorphism $h_{\lambda\lambda'}$ satisfying (17) and (18). Since each $\lambda\in\Lambda$ admits a λ' such that $\lambda\leqslant'\lambda'$, the index set $(\Lambda, \leqslant)$ can be replaced by $(\Lambda, \leqslant')$.

Note that for $\lambda<\lambda_1<\lambda_2$ by (18)

$$h_{\lambda\lambda_1}q_{\lambda_1\lambda_2}f_{\lambda_2} = h_{\lambda\lambda_1}f_{\lambda_1}p_{\lambda_1\lambda_2} = p_{\lambda\lambda_1}p_{\lambda_1\lambda_2} = p_{\lambda\lambda_2} . \tag{19}$$

Let h: $\Lambda\to\Lambda$ be any function such that $h(\lambda)>\lambda$ for each $\lambda\in\Lambda$ (we may assume that Λ has no maximal element). Let h_λ: $H_{h(\lambda)}\to G_\lambda$ be the homomorphism $h_{\lambda h(\lambda)}$. Then h and h_λ, $\lambda\in\Lambda$, determine a morphism $\boldsymbol{h}$: $\boldsymbol{H}\to\boldsymbol{G}$. Indeed, if $\lambda<\lambda'$, there is a $\lambda_1>h(\lambda)$, $h(\lambda')$. It follows from (19) that

$$h_\lambda q_{h(\lambda)\lambda_1}f_{\lambda_1} = p_{\lambda\lambda_1} = p_{\lambda\lambda'}p_{\lambda'\lambda_1} = p_{\lambda\lambda'}h_{\lambda'}q_{h(\lambda')\lambda_1}f_{\lambda_1} . \tag{20}$$

By (17), $f_{\lambda_1}h_{\lambda_1\lambda_2} = q_{\lambda_1\lambda_2}$ for any $\lambda_2>\lambda_1$ and therefore (20) yields

$$h_\lambda q_{h(\lambda)\lambda_2} = p_{\lambda\lambda'}h_{\lambda'}q_{h(\lambda')\lambda_2} . \tag{21}$$

By (17) and (18) for every $\lambda\in\Lambda$ we have $f_\lambda h_\lambda = q_{\lambda h(\lambda)}$ and $h_\lambda f_{h(\lambda)} = p_{\lambda h(\lambda)}$, which proves that $\boldsymbol{fh} = \boldsymbol{1}$ and $\boldsymbol{hf} = \boldsymbol{1}$. This completes the proof of Theorem 7

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