p-convex iteration groups

by

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Abstract. Let $f\colon (a,b)\to \boldsymbol{R}$ be a strictly increasing function (of class $C^p((a,b))$), k-convex for all $k\in\{1,\dots,p\}$, and such that $a<f(x)<x$ in (a,b) and $s:=\lim\limits_{x\to a+} f'(x)\in(0,1]$. A necessary and sufficient condition is given in order that f possess an iteration group $\{f^u\}_{u\in\boldsymbol{R}}$ such that for every positive u the function f^u is k-convex for all $k\in\{1,\dots,p\}$.

§ 1. Let p denote a positive integer, in the sequel regarded as fixed. A function $f\colon (a,b)\to\boldsymbol{R}$, $-\infty\leqslant a<b\leqslant+\infty$, is said to be *$p$-convex* iff the inequality

$$\Delta_h^{p+1}f(x)=\sum_{j=0}^{p+1}(-1)^{p+1-j}\binom{p+1}{j}f(x+jh)\geqslant 0 \tag{1}$$

holds for all pairs $(x,h)\in(a,b)\times(0,\infty)$ such that $x+(p+1)h\in(a,b)$. For $p=1$ the functional inequality (1) may equivalently be written in the form

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2},\qquad x,y\in(a,b). \tag{2}$$

Functions for which (2) is fulfilled (i.e., 1-convex functions) will simply be referred to as convex functions. It is well known (see, for example, [2] and [6]) that even very weak regularity assumptions on a p-convex function f, for instance Lebesgue measurability, imply that

$$f\in C^{p-1}((a,b)). \tag{3}$$

In the sequel we shall deal with monotonic p-convex functions only and so (3) will automatically be valid. In that case (cf. [6]) relation (1) implies that $f^{(p-1)}$ is convex.

We shall adopt the following hypothesis:

(H_f) $f\colon(a,b)\to\boldsymbol{R}$ is k-convex for all $k\in\{1,\dots,p\}$, f is strictly increasing in (a,b) and $a<f(x)<x$ holds for all $x\in(a,b)$.

Definition. An iteration group $\{f^u\}_{u\in\boldsymbol{R}}$ of a function f is said to be *p-convex* iff for every positive u and for every $k\in\{1,\dots,p\}$ f^u is k-convex.

The notion of an iteration group may be found e.g. in [1], [3], [5]. The problem of investigation of convex iteration groups was first raised by M. Kuczma [3]. In [5], [9], [10] and [11] convex and so called absolutely monotonic iteration groups were studied. The latter may now be defined as iteration groups which are p-convex for all positive integers p.

The following question arises in a natural way: find necessary and sufficient conditions for the existence of a p-convex iteration group of a function f fulfilling (H_f). The aim of this paper is to give a solution of the question just presented. This yields simultaneously a new proof of a theorem concerning absolutely monotonic iteration groups (see [11]). In Section 4 we also give an example of a convex iteration group which is not 2-convex.

§ 2. In this section we shall give a necessary condition for the existence of a p-convex iteration group of a function f for which (H_f) is satisfied.

Note that on account of (3) and of the convexity of $f^{(p-1)}$ the one-sided derivatives $f_l^{(p)}$ and $f_r^{(p)}$ exist on (a, b) and fulfil the inequalities

$$0<f_l^{(p)}\leqslant f_r^{(p)}.$$

Define Z_f as the set of all points x in (a, b) at which $f^{(p)}(x)$ does not exist, i.e.

$$Z_f = \{x\in(a,b)\colon 0<f_l^{(p)}(x)<f_r^{(p)}(x)\}.$$

It is not difficult to check the following

LEMMA 1. *If functions φ and ψ of the type $(a,b)\to \boldsymbol{R}$ are both from the class $C^{p-1}((a,b))$, $p\geqslant 2$, the derivatives $\varphi_l^{(p)}$, $\varphi_r^{(p)}$, $\psi_l^{(p)}$, $\psi_r^{(p)}$ and the superposition $\varphi\circ\psi$ do exist on (a,b), and if ψ is an increasing function on (a,b), then $(\varphi\circ\psi)_l^{(p)}$ and $(\varphi\circ\psi)_r^{(p)}$ exist on (a,b) and*

$$(\varphi\circ\psi)_l^{(p)} = (\varphi_l^{(p)}\circ\psi)(\psi')^p+(\varphi'\circ\psi)\cdot\psi_l^{(p)}+\sum_{q,q_1,\ldots,q_n\leqslant p-1} c_{q,q_1,\ldots,q_n}(\varphi^{(q)}\circ\psi)\cdot\psi^{(q_1)}\cdot\ldots\cdot\psi^{(q_n)},$$

$$(\varphi\circ\psi)_r^{(p)} = (\varphi_r^{(p)}\circ\psi)(\psi')^p+(\varphi'\circ\psi)\cdot\psi_r^{(p)}+\sum_{q,q_1,\ldots,q_n\leqslant p-1} c_{q,q_1,\ldots,q_n}(\varphi^{(q)}\circ\psi)\cdot\psi^{(q_1)}\cdot\ldots\cdot\psi^{(q_n)}.$$

Here $C_{q,q_1,q_2,\ldots,q_n}$ denote non-negative constants for $q, q_1, q_2, \ldots, q_n\in\{1,\ldots,p-1\}$.

LEMMA 2. *Under the assumptions of the preceding Lemma the inclusion*

$$Z_\psi\subset Z_{\varphi\circ\psi} \tag{4}$$

holds provided φ' and ψ' are both positive and $\varphi_l^{(p)}\leqslant\varphi_r^{(p)}$. Moreover, in that case,

$$\psi((a,b))\cap Z_\varphi\subset\psi(Z_{\varphi\circ\psi}). \tag{5}$$

Proof. Take an $x\in Z_\psi$. Then $\psi_l^{(p)}(x)<\psi_r^{(p)}(x)$. Since $\varphi_l^{(p)}(\psi(x))\leqslant\varphi_r^{(p)}(\psi(x))$ and

$$\varphi'(x)>0,\quad \psi'(x)>0\quad\text{and}\quad \varphi'(\psi(x))>0, \tag{6}$$

on account of Lemma 1 we obtain the inequality

$$(\varphi\circ\psi)_l^{(p)}(x)<(\varphi\circ\psi)_r^{(p)}(x),$$

which says that $x\in Z_{\varphi\circ\psi}$.

In order to prove (5) take a $y\in\psi((a,b))\cap Z_\varphi$. Then there exists an $x\in(a,b)$ such that $y=\psi(x)$ and

$$0<\varphi_l^{(p)}(y)<\varphi_r^{(p)}(y).$$

In view of (6), with the use of Lemma 1 we obtain

$$(\varphi\circ\psi)_l^{(p)}(x)<(\varphi\circ\psi)_r^{(p)}(x),$$

which simply means that $x\in Z_{\varphi\circ\psi}$, i.e., $y=\psi(x)\in\psi(Z_{\varphi\circ\psi})$.

THEOREM 1. *Suppose that f satisfies (H_f). If f possesses a p-convex iteration group, then $f\in C^p((a,b))$.*

Proof. There exists a function φ fulfilling (H_φ) and such that

$$\varphi^2(x)=f(x)\quad\text{for}\quad x\in(a,b). \tag{7}$$

Obviously, φ has a p-convex iteration group $\{\varphi^u\}_{u\in\boldsymbol{R}}$ with $\varphi^u=f^{u/2}$. Suppose that

$$Z_f\neq\emptyset. \tag{8}$$

(8) together with (7) easily implies that $Z_\varphi\neq\emptyset$. Thus we are able to take an x_0 such that

$$x_0\in Z_\varphi. \tag{9}$$

There exists a $c\in\boldsymbol{R}$ such that the function $g\colon(-\infty,c)\to\boldsymbol{R}$ given by the formula

$$g(u)=\varphi^{-u}(x_0)$$

is continuous and strictly increasing (see [1]). Moreover, c must be positive since $g(0)=x_0$ does exist. Consequently

$$x_0=\varphi^u(g(u))\in\varphi^u((a,b))\quad\text{for all } u\in(0,c). \tag{10}$$

Now, conditions (9) and (10) imply

$$g(u)=\varphi^{-u}(x_0)\in\varphi^{-u}(Z_{\varphi}\cap\varphi^u((a,b)))\quad\text{for all } u\in(0,c). \tag{11}$$

Moreover, making use of (5) with $\psi=\varphi^u$, we get the inclusion

$$Z_{\varphi}\cap\varphi^u((a,b))\subset\varphi^u(Z_{\varphi^{u+1}}) \tag{12}$$

valid for all $u\in(0,c)$. Now the relation

$$g((0,c))\subset Z_{\varphi^{u+1}}\quad\text{for all } u\in(0,c) \tag{13}$$

results immediately from (11) and (12). Without loss of generality one may assume that $c<1$ and hence that $1-u$ is positive for $u\in(0,c)$. Applying (4) for φ^{u+1} and φ^{1-u} instead of φ and ψ, respectively, we infer by (7) that

$$(14)\qquad Z_{\varphi^{u+1}}\subset Z_{\varphi^{(u+1)+(1-u)}}=Z_{\varphi^2}=Z_f \quad \text{for all } u\in(0,c).$$

Since g is a continuous and non-constant function (g is strictly increasing), we infer that $g((0,c))$ is an interval. However, this is impossible since Z_f is at most denumerable whereas (13) and (14) imply the inclusion $g((0,c))\subset Z_f$. Thus (8) is false, i.e., $f^{(p)}$ does exist on (a,b). Moreover, $f^{(p)}$, as a derivative of a convex function, must necessarily be continuous (see, for instance [7]).

Observe that if a function f fulfilling (H_f) possesses a p-convex iteration group $\{f^u\}_{u\in\boldsymbol{R}}$, then, for every positive $u\in\boldsymbol{R}$, the function f^u also possesses a p-convex iteration group. Namely, the formula $(f^u)^v:=f^{uv}$, $v\in\boldsymbol{R}$, defines such a group. Therefore, on account of Theorem 1 we obtain the following

COROLLARY. *If a function f satisfies (H_f) and possesses a p-convex iteration group $\{f^u\}_{u\in\boldsymbol{R}}$, then $f^u\in C^p((a,b))$ for every positive real u.*

Remark. A differentiable function f fulfilling the conditions (H_f) and $\lim\limits_{x\to a^+}f'(x)\neq 0$ may have at most one p-convex iteration group. In fact, a p-convex iteration group of a function f yields simultaneously a convex iteration group for f. On the other hand, under the above assumptions it is shown in [5] that a convex iteration group of f must be the principal iteration group and thus it is unique. Consequently, a p-convex iteration group of f, if it does exist, must be unique and must coincide with the principal iteration group of f.

Now, the question of the existence of the p-convex iteration group of a function f fulfilling (H_f) remains to be answered. Theorem 1 shows that this question should be stated for C^p-functions only.

§ 3. The main existence theorem will be preceded by the following

LEMMA 3 (cf. [7], Theorem 25.7). *Let $f_n\colon (a,b)\to\boldsymbol{R}$ be an increasing, convex and differentiable function for $n=1,2,\ldots$ If $\{f_n\}_{n=1}$ is pointwise convergent to a C^1-function f on (a,b), then $\{f'_n\}_{n=1}$ is pointwise convergent to f' on (a,b).*

Now, suppose that

$$(15)\qquad f \text{ satisfies } (\mathrm{H}_f),\ f\in C^p((a,b)) \text{ and } \lim_{x\to a^+}f'(x)=:s\in(0,1].$$

It is known (the detailed references may be found in [10]) that in the case where $s\in(0,1)$ the principal solution σ of the Schröder equation

$$(16)\qquad \sigma(f(x))=s\sigma(x)$$

generates the principal iteration group $\{f^u\}_{u\in\boldsymbol{R}}$ of f:

$$f^u(x)=\sigma^{-1}(s^u\sigma(x)) \quad \text{for} \quad x\in(a,b).$$

Likewise, in the case where $s=1$, the family $\{f^u\}_{u\in\boldsymbol{R}}$ given by the formula

$$f^u(x)=\alpha^{-1}(u+\alpha(x)),\qquad x\in(-\infty,b)\,(^1),$$

where α denotes the principal solution of Abel's equation

$$(17)\qquad \alpha(f(x))=\alpha(x)+1,$$

yields the principal iteration group of f.

Our assumptions on f guarantee (cf. also [10]) the differentiability of the principal solutions σ and α of (16) and (17), respectively, as well as the existence of the function h given by the formula

$$(18)\qquad h(x):=\frac{\partial}{\partial u}f^u(x)|_{u=0}=\lim_{u\to 0}\frac{f^u(x)-x}{u}=\begin{cases}(\ln s)\dfrac{\sigma(x)}{\sigma'(x)} & \text{for } s\in(0,1),\\[2mm] \dfrac{1}{\alpha'(x)} & \text{for } s=1.\end{cases}$$

THEOREM 2. *Let assumptions* (15) *be satisfied. Then the principal iteration group of f is p-convex if and only if the function h given by* (18) *is k-convex for $k\in\{1,\ldots,p\}$.*

Proof. In the case $p=1$ our assertion reduces to the main result from [10]. Thus, it suffices to consider $p\geqslant 2$ only.

The necessity is obvious.

Sufficiency. Suppose that h given by (18) is k-convex for $k\in\{1,\ldots,p\}$. According to [10] f possesses the unique convex iteration group $\{f^u\}_{u\in\boldsymbol{R}}$. We are going to show this group to be p-convex. For an indirect proof suppose that there exists a $k\in\{2,\ldots,p\}$ such that f^u is not k-convex for some positive u. We may assume that k is the smallest positive integer with this property, i.e. that f^v are l-convex for all $l\in\{1,\ldots,k-1\}$ and all positive v. By the corollary to Theorem 1 it follows that $f^v\in C^{k-1}((a,b))$ for every positive v.

Now, $(f^u)^{(k-1)}$ is not convex in (a,b). Consequently, by means of (2), one can find x and y such that $a<x<y<b$ and

$$(f^u)^{(k-1)}\Big(\frac{x+y}{2}\Big)>\Big(\frac{(f^u)^{(k-1)}(x)+(f^u)^{(k-1)}(y)}{2}\Big)$$

whence, obviously, the inequality

$$(19)\qquad (f^u)^{(k-1)}\Big(\frac{x+y}{2}\Big)>\varkappa\,\frac{(f^u)^{(k-1)}(x)+(f^u)^{(k-1)}(y)}{2}$$

results, with some real $\varkappa$ greater than 1. At first, we shall assume that $k\geqslant 3$. The particular case $k=2$ will be considered separately. Replacing f^u by $f^{u/2}\circ f^{u/2}$

(1) $s=1$ implies $a=-\infty$. Analogously, $s\in(0,1)$ implies $a>-\infty$.

in (19), on account of Lemma 1 and the l-convexity of $f^{u/2}$ for all $l\in\{1,\ldots,k-1\}$, we easily get the following alternative of conditions:

$$(f^{u/2})^{(k-1)}\left(f^{u/2}\left(\frac{x+y}{2}\right)\right)(f^{u/2})'\left(\frac{x+y}{2}\right)^{k-1} > \varkappa\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))(f^{u/2})'(x)^{k-1}+(f^{u/2})^{(k-1)}(f^{u/2}(y))(f^{u/2})'(y)^{k-1}}{2} \tag{20a}$$

or

$$(f^{u/2})'\left(f^{u/2}\left(\frac{x+y}{2}\right)\right)(f^{u/2})^{(k-1)}\left(\frac{x+y}{2}\right) > \varkappa\frac{(f^{u/2})'(f^{u/2}(x))(f^{u/2})^{(k-1)}(x)+(f^{u/2})'(f^{u/2}(y))(f^{u/2})^{(k-1)}(y)}{2}. \tag{20b}$$

Now, (20a) implies

$$(f^{u/2})^{(k-1)}\left(\frac{f^{u/2}(x)+f^{u/2}(y)}{2}\right) > \varkappa\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))+(f^{u/2})^{(k-1)}(f^{u/2}(y))}{2} \tag{21a}$$

while the inequality

$$(f^{u/2})^{(k-1)}\left(\frac{x+y}{2}\right) > \varkappa\frac{(f^{u/2})^{(k-1)}(x)+(f^{u/2})^{(k-1)}(y)}{2} \tag{21b}$$

results from (20b). Indeed we have

$$(f^{u/2})^{(k-1)}\left(f^{u/2}\left(\frac{x+y}{2}\right)\right) > \varkappa\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))+(f^{u/2})^{(k-1)}(f^{u/2}(y))}{2} \tag{22}$$

provided (20a) occurs (hence, since $(f^{u/2})^{(k-1)}$ is increasing and $f^{u/2}$ is convex, (21a) results immediately from (22)). To show this, suppose that (22) does not hold, i.e., that the inequality

$$(f^{u/2})^{(k-1)}\left(f^{u/2}\left(\frac{x+y}{2}\right)\right) \leqslant \varkappa\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))+(f^{u/2})^{(k-1)}(f^{u/2}(y))}{2} \tag{22'}$$

is satisfied. Since $k\geqslant 3$, $(f^{u/2})'$ and hence also $[(f^{u/2})']^{k-1}$ must be convex. Consequently we have

$$(f^{u/2})'\left(\frac{x+y}{2}\right)^{k-1} \leqslant \varkappa\frac{(f^{u/2})'(x)^{k-1}+(f^{u/2})'(y)^{k-1}}{2}. \tag{23}$$

Multiplying (22′) and (23), we obtain

$$\begin{aligned}(f^{u/2})^{(k-1)}&\left(f^{u/2}\left(\frac{x+y}{2}\right)\right)(f^{u/2})'\left(\frac{x+y}{2}\right)^{k-1}\\ &\leqslant\varkappa\left[\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))(f^{u/2})'(x)^{k-1}+(f^{u/2})^{(k-1)}(f^{u/2}(y))(f^{u/2})'(y)^{k-1}}{4}+\right.\\ &\quad\left.+\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))(f^{u/2})'(y)^{k-1}+(f^{u/2})^{(k-1)}(f^{u/2}(y))(f^{u/2})'(x)^{k-1}}{4}\right]\\ &\leqslant\varkappa\frac{(f^{u/2})^{(k-1)}(f^{u/2}(x))(f^{u/2})'(x)^{k-1}+(f^{u/2})^{(k-1)}(f^{u/2}(y))(f^{u/2})'(y)^{k-1}}{2}.\end{aligned}$$

The latter inequality results simply from the fact that

$$[(f^{u/2})^{(k-1)}(f^{u/2}(x))-(f^{u/2})^{(k-1)}(f^{u/2}(y))][(f^{u/2})'(y)^{k-1}-(f^{u/2})'(x)^{k-1}]\leqslant 0,$$

which, of course, remains true in view of the monotonicity of $(f^{u/2})^{(k-1)}$, $f^{u/2}$ and $[(f^{u/2})']^{k-1}$. Thus, a contradiction with (20a) is obtained, which shows the validity the of implication (20a)⇒(22) (and hence also of (20a)⇒(21a)).

The implication (20b)⇒(21b) can be derived analogously.

Put $x_1 := f^{u/2}(x)$, $y_1 := f^{u/2}(y)$ whenever (21a) occurs and $x_1 := x$, $y_1 := y$ in the case where (21b) is satisfied (if (21a) and (21b) are both fulfilled, then we put $x_1 := x$, $y_1 := y$). Consequently, we have

$$(f^{u/2})^{(k-1)}\left(\frac{x_1+y_1}{2}\right) > \varkappa\frac{(f^{u/2})^{(k-1)}(x_1)+(f^{u/2})^{(k-1)}(y_1)}{2}.$$

By induction, we construct two sequences, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, both contained in (a,b) and having the properties

$$\begin{gathered}x_n<y_n,\\ (f^{u/2})^{(k-1)}\left(\frac{x_n+y_n}{2}\right) > \varkappa\frac{(f^{u/2^n})^{(k-1)}(x_n)+(f^{u/2^n})^{(k-1)}(y_n)}{2}\end{gathered} \tag{24}$$

for all positive integers n. Moreover, it is readily seen that $x_n = f^{u_n}(x)$ as well as $y_n = f^{u_n}(y)$ where $u_n := u\sum_{i=1}^{n}2^{-i}c_i$, $c_i\in\{0,1\}$, $i=1,\ldots,n$, $n=1,2,\ldots$ Clearly, both of the sequences just constructed are convergent: $x_n\to f^{u_0}(x) =: x_0$ and $y_n \to f^{u_0}(y) =: y_0$; here $u_0 := u\sum_{i=1}^{\infty}2^{-i}c_i$. Moreover,

$$a<x_0<y_0<b. \tag{25}$$

One the other hand,

$$\frac{2^n}{u}(f^{u/2^n})^{(k-1)}\to h^{(k-1)} \quad\text{as}\quad n\to\infty, \tag{26}$$

uniformly on every compact subset of (a, b). Indeed, recalling that $k \geqslant 3$, since $\frac{2^n}{u}(f^{u/2^n})^{(k-2)}$ is convex one can make use of Lemma 3 and apply the well-known theorem for convergent sequences of continuous monotonic functions. It is also well known that uniform convergence on every compact subset of (a, b) is equivalent to so called continuous convergence. Consequently, by means of (24), (25) and (26) we get

$$h^{(k-1)}\left(\frac{x_0+y_0}{2}\right) \geqslant \varkappa \frac{h^{(k-1)}(x_0)+h^{(k-1)}(y_0)}{2} > \frac{h^{(k-1)}(x_0)+h^{(k-1)}(y_0)}{2}$$

as $\varkappa > 1$. However, this is incompatible with the assumed k-convexity of h.

It remains to consider the particular case $k = 2$. The formula (19) becomes

$$(f^u)'\left(\frac{x+y}{2}\right) > \varkappa \frac{(f^u)'(x)+(f^u)'(y)}{2},$$

whence, by putting $f^u = f^{u/2} \circ f^{u/2}$, the alternative

$$(f^{u/2})'\left(\frac{f^{u/2}(x)+f^{u/2}(y)}{2}\right) > \sqrt{\varkappa} \frac{(f^{u/2})'(f^{u/2}(x))+(f^{u/2})'(f^{u/2}(y))}{2} \tag{27a}$$

or

$$(f^{u/2^n})'\left(\frac{x+y}{2}\right) > \sqrt{\varkappa} \frac{(f^{u/2})'(x)+(f^{u/2})'(y)}{2} \tag{27b}$$

can be obtained. Inductively, by a method similar to that employed in the preceding case we construct two convergent sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$(f^{u/2})'\left(\frac{x_n+y_n}{2}\right) > \sqrt[2^n]{\varkappa} \frac{(f^{u/2^n})'(x_n)+(f^{u/2^n})'(y)}{2}, \quad n = 1, 2, \ldots \tag{28}$$

whence, substracting the unity from both sides of (28) and dividing by $u/2^n$, we get

$$\frac{(f^{u/2^n})'\left(\frac{x_n+y_n}{2}\right)-1}{u/2^n} > \varkappa^{2^{-n}}\left(\frac{\frac{(f^{u/2^n})'(x_n)-1}{u/2^n}+\frac{(f^{u/2^n})'(y_n)-1}{u/2^n}}{2}\right)+\frac{1}{u}\cdot\frac{\varkappa^{2^{-n}}-1}{2^{-n}}, \quad n = 1, 2, \ldots \tag{29}$$

Since $\frac{f^v(x)-x}{v}$, $x \in (a, b)$, are clearly convex for all positive v, Lemma 3 and (29) lead to

$$h'\left(\frac{x_0+y_0}{2}\right) \geqslant \frac{h'(x_0)+h'(y_0)}{2}+\frac{1}{u}\ln\varkappa > \frac{h'(x_0)+h'(y_0)}{2}, \tag{30}$$

where x_0 and y_0 denote the limits of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, respectively. However, (30) contradicts the 2-convexity of h and the proof is completed.

§ 4. Observe that the principal iteration group of an absolutely monotonic function f is absolutely monotonic if and only if it is p-convex for every positive integer p. This, however, by Theorem 2, implies that the principal iteration group of f is absolutely monotonic if and only if the function h given by (18) is p-convex for all positive integers p. Thus the main result of [11] has just been derived from ours.

The example of an absolutely monotonic iteration group given in [11] yields simultaneously an example of a p-convex iteration group for an arbitrary p. But, of course, f may happen to possess a p-convex iteration group which is not $(p+1)$-convex, as can be seen from the following

Example. Take $f(x) := sx+x^2$, $s \in (0, 1)$, $x \in (0, 1-s)$ and consider the function which is given by (18) where $\sigma(x) = \lim_{n\to\infty} f^n(x)/f^n(x_0)$, $x_0 \in (0, 1-s)$. Clearly, σ is an absolutely monotonic solution of the Schröder equation (S) and so also $\bar{\sigma}$ is defined by the formula

$$\bar{\sigma}(x) := \lim_{n\to\infty} \frac{1}{s^n} f^n(x) \quad \text{for} \quad x \in (0, 1-s).$$

Moreover, on account of Koenig's theorem [4, page 140], we have

$$\bar{\sigma}'(0) = 1 \tag{31}$$

and

$$\sigma(x) = \eta\bar{\sigma}(x) \quad \text{for} \quad x \in (0, 1-s). \tag{32}$$

Evidently, η is non-negative since both of the functions σ and $\bar{\sigma}$ are non-negative. However, $\sigma(x_0) = 1$, whence the condition

$$\eta > 0 \tag{33}$$

results immediately from (32). We also have

$$\sigma'(x) = \eta\bar{\sigma}'(x) \quad \text{for} \quad x \in (0, 1-s),$$

which, by virtue of (31) and (33), implies

$$\sigma'(x) \geqslant \eta > 0 \quad \text{for} \quad x \in [0, 1-s). \tag{34}$$

Consequently, the formula

$$h(x) = (\ln s)\frac{\sigma(x)}{\sigma'(x)}$$

defines an analytic function on the interval $[0, 1-s)$. One can easily check that this function yields a solution of the functional equation

$$h(sx+x^2) = (s+2x)h(x) \quad \text{for} \quad x \in [0, 1-s) \tag{35}$$

such that $h(0) = 0$ and $h'(0) = \ln s$. Suppose that

$$h(x) = (\ln s)x + c_2x^2 + c_3x^3 + \ldots \quad \text{for} \quad x \in [0, 1-s)$$

yields a solution of (35). A simple calculation shows that

$$c_2 = \frac{\ln s}{s^2 - s} \quad \text{and} \quad c_3 = \frac{-2\ln s}{s(s^3 - s)}.$$

Thus we get the existence of a positive $b < 1-s$ such that $h'' > 0$ and $h''' < 0$ on $(0, b)$, i.e., h is convex but not 2-convex on $(0, b)$. On account of Theorem 2, f possesses a convex but not 2-convex iteration group.

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Accepté par la Rédaction le 17. 5. 1974

Mappings onto circle-like continua

by

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Abstract. The main object of the present paper is to give a characterization of continua which can be mapped onto non-planar circle-like curves. This result is then applied to show that certain classes of continua cannot be mapped onto such curves. These results extend several well-known facts in this field.

The term compactum is used to mean a compact metric space. A connected compactum is called a continuum. By a curve we mean a one-dimensional continuum. The terms map and mapping will be used interchangingly to mean a continuous function. A map $f\colon X \to Y$ is said to be an ε-mapping, $\varepsilon > 0$, provided $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in Y$. Throughout the paper we denote by S the unit circle in the complex plane and by I the unit interval $[0, 1]$ of reals. A continuum X is called *circle-like* (*snake-like*) if for every $\varepsilon > 0$ there exists an ε-mapping of X onto S (onto I, respectively). Clearly, any circle-like or snake-like continuum is a curve. The above classes of curves have been extensively studied by several authors. Known results show an important difference between the class of circle-like curves which can be embedded in the plane and the others. This difference will also be underlined by the results of this paper. Our main result gives a characterization of continua which can be mapped onto non-planar circle-like curves. This result solves a problem raised by Henderson in [7], and extends his result in this direction. We obtain also generalizations of the results of Ingram [8].

1. Some remarks on Abelian groups. Let G be an Abelian group. Denote by N the set of natural numbers, $N = \{1, 2, \ldots\}$. We say that $g \in G$ is *divisible* by a natural number n, notation: n/g, if $g = n \cdot g'$ for some $g' \in G$. For every $g \in G$ we define

$$d(g) = \sup\{n \in N\colon\ n/g\}.$$

Clearly, $d(0) = \infty$. If $d(g) < \infty$, then we say that g is *finitely divisible*; otherwise g is called *infinitely divisible*. If every element of G different from the neutral element 0 is finitely divisible, then we simply say that G is finitely divisible. Notice that every free Abelian group is finitely divisible.

1.1. *If $m, n \in N$ are relatively prime, $g \in G$, m/g and n/g, then $m \cdot n/g$.*