Spaces with increment of dimension n

by

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Abstract. Results include (i) a generalization to arbitrary uniform spaces of a result of Smirnov characterizing intrinsically the covering dimension of the increment of a space satisfying the bicompact axiom of countability (ii) a new intrinsic characterization of the covering dimension of the increment of a uniform space complete in the sense of Čech, and (iii) the first example of a semicompact space every increment of which is a normal space of covering dimension $\geqslant n$.

1. Introduction. It was first established by Freudenthal [5] that a separable metric space X has a compactification with increment of covering dimension (dim) $\leqslant 0$ if and only if X is semicompact, i.e., whenever $x \in G$, where G is open in X, there is an open set H of X with $x \in H \subset \overline{H} \subset G$ and $\overline{H} - H$ compact. It is now known [8, 11] that this result is valid for all spaces satisfying the bicompact axiom of countability, i.e., those spaces one (and hence every) increment of which is Lindelöf. Skljarenko [8] gives an example of a semicompact space every increment of which is non-normal, and hence has dim>0. As the Čech increment of this space is of $\dim^* = 0$, where dim* is defined the same way as dim except that we replace "open set" by "cozero set", Skljarenko remarks that the question of extending Freudenthal's result still further is open. In Example 2, Section 3, we construct a space X_n every increment of which is normal with $\dim \geqslant n$, $n = 1, 2, 3, \ldots, \infty$. In the opposite direction, Smirnov [9] gave an example of non-semicompact Tychonoff space whose Čech increment has ind $= 0$ (but dim>0). To the best of my knowledge, we have no example of a non-semicompact Tychonoff space some increment of which has dim $= 0$ (¹).

A brief history of the problem of generalizing Freudenthal's result to higher dimensions can be found in [11, 12]. We are mainly interested in the following result of Smirnov's. Let Y be the compactification of a Tychonoff space X corresponding to a precompact uniformity $\mathscr{U}$ on X. If X is normally adjoined to its increment $Y-X$, i.e., every two disjoint closed sets of $Y-X$ are separated by disjoint open sets of Y, then $\dim(Y-X) = \dim^\infty X$. The definition of $\dim^\infty$ is as follows. A finite collection $\{G_1, G_2, \ldots, G_k\}$ of open sets is called an extendable fringe of $(X, \mathscr{U})$ if $X - \bigcup_{i=1}^{k} G_i$ is compact, and for every open neighbourhood G_0

(¹) Added in proof. For such a space see J. R. Isbell, *Uniform Space*, p. 132.

of $X - \bigcup_{i=1}^{k} G_i \{G_0, G_1, \ldots, G_k\}$ is a uniform cover of $(X, \mathscr{U})$. $\dim^\infty X \leqslant n$ if and only if every extendable fringe of X is refined by an extendable fringe of order $\leqslant n$. We introduce a dimension function $\mathscr{U}$-$\dim^\infty$, and show that $\mathscr{U}$-$\dim^\infty X = \mathscr{U}$-$\dim(Y-X)$, where $\mathscr{U}$-dim is the dimension function studied in [1]. For spaces satisfying the bicompact axiom of countability, and are hence normally adjoined to their increments, [11], $\mathscr{U}$-$\dim^\infty = \dim^\infty$, and as $\dim = \mathscr{U}$-dim for Lindelöf spaces [1], Smirnov's result and ours coincide. Our Proposition 4 gives another intrinsic characterization of $\dim(Y-X)$. In the final section, we establish that if Y is the Freudenthal compactification of X, then $\dim Y \leqslant \dim X + 1$.

2. Dimension of increments. A subset of a uniform space $(X, \mathscr{U})$ is called *$\mathscr{U}$-open* (*$\mathscr{U}$-closed*) [1] if it is the inverse image of an open (closed) set of R, the space of real numbers, under a bounded, uniformly continuous function. In hyphenated words where $\mathscr{U}_Z$ the relativisation of $\mathscr{U}$ to a subset Z of X, occurs, the suffix "Z" is dropped. Thus, for example, "$\mathscr{U}_Z$-open" becomes "$\mathscr{U}$-open in Z". The collection of all $\mathscr{U}$-open sets of X is closed with respect to countable unions and finite intersections, and a set is $\mathscr{U}$-open in Z if and only if it is of the form $Z \cap G$ with G $\mathscr{U}$-open in X. $\mathscr{U}$-$\dim X = -1$, $\mathscr{U}$-$\operatorname{Ind} X = -1$, or $\mathscr{U}$-$\operatorname{ind} X = -1$ if and only if X is empty. $\mathscr{U}$-$\dim X \leqslant n$ if every finite $\mathscr{U}$-cover of X, i.e., cover of X consisting of $\mathscr{U}$-open sets, is refined by a finite $\mathscr{U}$-cover of order $\leqslant n$. $\mathscr{U}$-$\operatorname{Ind} X \leqslant n$ if whenever E_1, E_2 are disjoint $\mathscr{U}$-closed sets of X, there are disjoint $\mathscr{U}$-open sets G_1, G_2 of X with $E_1 \subset G_1$, $E_2 \subset G_2$ and $\mathscr{U}$-$\operatorname{Ind}(X - G_1 \cup G_2) \leqslant n-1$. $\mathscr{U}$-$\operatorname{ind} X \leqslant n$ if whenever $x \in G$ with G open in X, there is a $\mathscr{U}$-open set H and a $\mathscr{U}$-closed set F of X with $x \in H \subset F \subset G$ and $\mathscr{U}$-$\operatorname{ind}(F-H) \leqslant n-1$.

LEMMA 1. *If H_1, H_2 are disjoint $\mathscr{U}$-closed sets of a subset of X, there are disjoint $\mathscr{U}$-open sets G_1, G_2 of X with $H_1 \subset G_1$ and $H_2 \subset G_2$.*

Proof. This is Lemma 7 of [1]. If H_1, H_2 are taken to be disjoint $\mathscr{U}$-closed sets of X, it merely expresses the normality of the lattice of all $\mathscr{U}$-open sets of X.

LEMMA 2. *Let $\{G_1, \ldots, G_k\}$ be a $\mathscr{U}$-cover of X. Then there are $\mathscr{U}$-closed sets F_i, $i = 1, \ldots, k$, of X with $F_i \subset G_i$ and $\bigcup_{i=1}^{k} F_i = X$.*

Proof. Let $G = \bigcup_{i=1}^{k-1} G_i$. Then there is a $\mathscr{U}$-closed set F_k and a $\mathscr{U}$-open set H of X with $X - G \subset H \subset F_k \subset G_k$ (Lemma 1). By an obvious induction hypothesis, there are $\mathscr{U}$-closed sets E_i, $i = 1, \ldots, k-1$, of G with $E_i \subset G_i$ and $\bigcup_{i=1}^{k-1} E_i = G$. $F_i = E_i - H$, $i = 1, \ldots, k-1$, is $\mathscr{U}$-closed in $X - H$ (which is $\mathscr{U}$-closed in X) and hence in X. Clearly $F_i \subset G_i$ for each i and $\bigcup_{i=1}^{k} F_i = X$.

$\{B_s\}_{s \in S}$ is called a *swelling* of $\{A_s\}_{s \in S}$ if $A_s \subset B_s$ for each s, and if $\bigcap_{s \in T} B_s = \emptyset$ if and only if $\bigcap_{s \in T} A_s = \emptyset$ for each finite subset T of S [4].

LEMMA 3. *A $\mathscr{U}$-cover $\{H_1, \ldots, H_k\}$ of a subset of X has a swelling $\{G_1, \ldots, G_k\}$ consisting of $\mathscr{U}$-open sets of X.*

Proof. We may suppose $H_1 \cap \ldots \cap H_k = \emptyset$. Then there are disjoint $\mathscr{U}$-open sets Q, G_{kk} of X with $H_1 \cap \ldots \cap H_{k-1} \subset Q$ and $H_k \subset G_{kk}$ (Lemma 1). Assuming that the result holds for dimension $k-1$, there are swellings $\{P_1, \ldots, P_{k-1}\}$ of $\{H_1 - Q, \ldots, H_{k-1} - Q\}$, and $\{G_{ij}\}_{j \neq i}$ of $\{H_j\}_{j \neq i}$, $i = 1, \ldots, k$, consisting of $\mathscr{U}$-open sets of X. Let $G_{ii} = P_i \cup Q$, $i = 1, \ldots, k-1$, and $G_i = \bigcap_{j=1}^{k} G_{ji}$, $i = 1, \ldots, k$.

The same argument yields

LEMMA 4. *A finite collection of $\mathscr{U}$-closed sets of a subset of X has a swelling consisting of $\mathscr{U}$-open sets of X.*

In the sequel all spaces will be assumed to be Tychonoff.

LEMMA 5. *Let $F_1, F_2, \ldots$ be zero sets of X. Then the following statements are equivalent.*

(i) *$\beta X - X \subset \bigcup_{n=1}^{\infty} \bar{F}_n$, where βX is the Stone-Čech compactification of X.*

(ii) *Whenever $\{E_\alpha\}$ is a collection of zero sets of X with the finite intersection property, and such that for each i there is an α with $E_\alpha \cap F_i = \emptyset$ then $\bigcap E_\alpha \neq \emptyset$.*

Proof. (i) → (ii): For zero sets E, F of X, $E \cap F = \emptyset$ implies $\bar{E} \cap \bar{F} = \emptyset$. Hence $\bar{F}_n \cap \bigcap \bar{E}_\alpha = \emptyset$, and since $\beta X - X \subset \bigcup \bar{F}_n$, $\bigcap \bar{E}_\alpha \subset X$. Hence $\bigcap E_\alpha = \bigcap \bar{E}_\alpha$. Since $\{E_\alpha\}$ has the finite intersection property, so does $\{\bar{E}_\alpha\}$, and hence $\bigcap E_\alpha = \bigcap \bar{E}_\alpha \neq \emptyset$.

(ii) → (i): Suppose $x \in \beta X - X$. For each closed set A_α of βX with $x \notin A_\alpha$, let P_α be a cozero and Q_α a zero set of βX with $x \in P_\alpha \subset Q_\alpha \subset \beta X - A_\alpha$. If $x \notin \bigcup \bar{F}_n$, then $\{Q_\alpha \cap X\}$ satisfies (ii) but $\bigcap Q_\alpha \cap X = \emptyset$.

A compact subset F of X will be called *accessible* if there are zero sets $F_1, F_2, \ldots$ of X satisfying (ii) of Lemma 5 and $F_n \cap F = 0$, $n = 1, 2, \ldots$

LEMMA 6. *Let Y be a compactification of X. Then a compact subset F of X is accessible if and only if $Y - X$ is contained in a σ-compact subset of Y disjoint from F.*

Proof. Let $f\colon \beta X \to Y$ be the extension of the inclusion $X \to Y$. If F is accessible, by Lemma 5, there are closed sets F_n, $n = 1, 2, \ldots$, of X with $F \cap F_n = F \cap \bar{F}_n = \emptyset$ and $\beta X - X \subset \bigcup \bar{F}_n$. Then $Y - X \subset \bigcup f(\bar{F}_n)$, and each $f(\bar{F}_n)$ is compact and disjoint from F. Conversely, suppose $Y - X \subset \bigcup E_n$ where E_n is compact and $E_n \cap F = \emptyset$ for each n. Then $F_n = f^{-1}(E_n)$ is closed in βX, $F_n \cap F = \emptyset$ and $\beta X - X \subset \bigcup F_n$. Let P_n be a cozero and Q_n a zero set of βX with $F_n \subset P_n \subset Q_n \subset \beta X - F$. Then $F_n \subset \overline{Q_n \cap X}$, and Lemma 5 implies that F is accessible.

COROLLARY 1. *A compact subset F of a space X satisfying the bicompact axiom of countability is accessible.*

Proof. $\beta X - X$ is Lindelöf, and hence there are open sets G_n and closed sets F_n of βX, $n = 1, 2, \ldots$, with $G_n \subset F_n$, $F \cap F_n = \emptyset$, and $\beta X - X \subset \bigcup_{n=1}^{\infty} G_n$.

In the sequel, Y will invariably denote the compactification of a space X corresponding to a pre-compact uniformity $\mathcal{U}$ on Y. A $\mathcal{U}$-cover $\{G_1, \ldots, G_k\}$ of X will be called a *$\mathcal{U}$-fringe* if $F = X - \bigcup_{i=1}^{k} G_i$ is accessible and for every open neighbourhood G_0 of F, $\{G_0, G_1, \ldots, G_k\}$ is a uniform cover of X. Every $\mathcal{U}$-fringe is extendable, but not conversely (Example 1). $\mathcal{U}\text{-}\dim^\infty X \leqslant n$ if and only if every $\mathcal{U}$-fringe of X is refined by a $\mathcal{U}$-fringe of order $\leqslant n$. For an open set G of X, $\operatorname{Ex} G$ will denote the complement in Y of the closure of $X - G$ in Y. $\operatorname{Ex} G$ is the largest open set of Y whose intersection with X is G, $\operatorname{Ex} G_1 \cup \operatorname{Ex} G_2 \subset \operatorname{Ex}(G_1 \cup G_2)$, $\operatorname{Ex}(G_1 \cap G_2) = \operatorname{Ex} G_1 \cap \operatorname{Ex} G_2$, and hence $\{\operatorname{Ex} G_1, \ldots, \operatorname{Ex} G_k\}$ is a swelling of $\{G_1, \ldots, G_k\}$ [4, 8, 11, 12].

Lemma 7. *A collection $\{G_1, \ldots, G_k\}$ of open sets of X is an extendable fringe of X if and only if $Y - X \subset \bigcup_{i=1}^{k} \operatorname{Ex} G_i$.*

Proof. This is Lemma 1 of [11].

Example 1. Let $Z = R \cup \{\infty\}$ be the one-point compactification of the set of real numbers R with the discrete topology, I the unit interval with the usual topology, $Y = I \times Z$, Q the set of rationals, $F = \{1\} \times (Q \cup \{\infty\})$, $G = [0, 1) \times Z$, and $X = F \cup G$. Then G is $\mathcal{U}$-open with $Y - X \subset Y - F = \operatorname{Ex} G$, and hence $\{G\}$ is an extendable fringe of X. If F_n is a compact set of Y disjoint from F, its complement contains $\{1\} \times \{\infty\}$, and hence F_n contains only a finite number of points of the uncountable set $Y - X$. It follows that F is not accessible and $\{G\}$ is not a $\mathcal{U}$-fringe.

Lemmas 6 and 7 imply

Corollary 2. *If $H_1, \ldots, H_k$ are cozero (and hence $\mathcal{U}$-open) sets of Y with $Y - X \subset \bigcup_{i=1}^{k} H_i$, then $\{H_1 \cap X, \ldots, H_k \cap X\}$ is a $\mathcal{U}$-fringe of X.*

Lemma 8. *Let $\{G_1, \ldots, G_k\}$ be an extendable fringe of X with $X - \bigcup_{i=1}^{k} G_i$ accessible. Then there are $\mathcal{U}$-open sets $H_1, \ldots, H_k$ of Y with $Y - X \subset \bigcup_{i=1}^{k} H_i$ and $H_i \cap X \subset G_i$.*

Proof. By Lemmas 6 and 7, there is a σ-compact set Z with $Y - X \subset Z \subset \bigcup_{i=1}^{k} \operatorname{Ex} G_i$. Then there are cozero sets P_n, $n = 1, 2, \ldots$, of Y with $Z \subset \bigcup_{n=1}^{\infty} P_n$ and $P_n \subset \operatorname{Ex} G_{i(n)}$. Let $H_i = \bigcup (P_n \colon P_n \subset \operatorname{Ex} G_i)$.

Proposition 1. *Let X satisfy the bicompact axiom of countability. Then $\mathcal{U}\text{-}\dim^\infty X = \dim^\infty X$.*

Proof. Suppose $\dim^\infty X \leqslant n$. Then a $\mathcal{U}$-fringe $\{G_i\}$ of X is refined by an extendable fringe $\{H'_j\}$ of order $\leqslant n$. By Corollary 1 and Lemma 8, there are $\mathcal{U}$-open sets H_j of Y with $Y - X \subset \bigcup H_j$ and $H_j \cap X \subset H'_j$. $\{H_j \cap X\}$ is a $\mathcal{U}$-fringe of order $\leqslant n$ refining $\{G_i\}$ (Corollary 2). Conversely, suppose $\mathcal{U}\text{-}\dim^\infty X \leqslant n$, and let $\{P_i\}$ be an extendable fringe of X. By Lemma 8 and Corollaries 1 and 2, there is a $\mathcal{U}$-fringe $\{Q_i\}$ of X with $Q_i \subset P_i$. Then $\{Q_i\}$, and hence $\{P_i\}$, is refined by a $\mathcal{U}$-fringe of order $\leqslant n$.

Proposition 1 may fail even if X is normally adjoined to its increment (Example 2).

Proposition 2. $\mathcal{U}\text{-}\dim^\infty X = \mathcal{U}\text{-}\dim(Y - X)$.

Proof. Suppose $\mathcal{U}\text{-}\dim^\infty X \leqslant n$, and let $\{G_i - X\}$ be a $\mathcal{U}$-cover of $Y - X$ with G_i $\mathcal{U}$-open in Y. Let E_i, T_i be $\mathcal{U}$-closed and S_i $\mathcal{U}$-open sets of $\bigcup G_i$ with $E_i \subset S_i \subset T_i \subset G_i$ and $\bigcup E_i = \bigcup G_i$ (Lemmas 1 and 2). By Corollary 2, $\{S_i \cap X\}$ is a $\mathcal{U}$-fringe of X, and hence it is refined by a $\mathcal{U}$-fringe $\{H_j\}$ of order $\leqslant n$. By Lemma 8, there are $\mathcal{U}$-open sets P_j of $\bigcup G_i$ with $Y - X \subset \bigcup P_j$, $P_j \cap X \subset H_j$ and hence order $\{P_j\} \leqslant n$. If $P_j - G_i \neq \emptyset$, then $P_j - T_i \neq \emptyset$, and hence $(P_j - T_i) \cap X \neq \emptyset$ and $(H_j - S_i) \cap X \neq \emptyset$. It follows that $\{P_j\}$ refines $\{G_i\}$, and $\{P_j - X\}$ is a $\mathcal{U}$-cover of $Y - X$ of order $\leqslant n$ refining $\{G_i - X\}$. Thus $\mathcal{U}\text{-}\dim Y - X \leqslant n$.

Conversely, suppose $\mathcal{U}\text{-}\dim Y - X \leqslant n$, and let $\{G_i\}$ be a $\mathcal{U}$-fringe of X. By Lemma 8, there are $\mathcal{U}$-open sets H_i of Y with $H_i \cap X \subset G_i$ and $Y - X \subset \bigcup H_i$. Since $\mathcal{U}\text{-}\dim(Y - X) \leqslant n$, there is $\mathcal{U}$-cover $\{S_i\}$ of $Y - X$ of order $\leqslant n$ with $S_i \subset H_i$. Let $\{T_i\}$ be a swelling of $\{S_i\}$ consisting of $\mathcal{U}$-open sets of Y (Lemma 3). Then $\{T_i \cap H_i \cap X\}$ is a $\mathcal{U}$-fringe of order $\leqslant n$, refining $\{G_i\}$ (Corollary 2). Hence $\mathcal{U}\text{-}\dim^\infty X \leqslant n$.

Consider the following conditions on X.

A_n: *Whenever E_i, F_i, $i = 1, \ldots, n+1$, are pairs of distant sets of X, there are pairs of disjoint $\mathcal{U}$-open sets G_i, H_i of X such that $E_i \subset G_i$, $F_i \subset H_i$ and $\{G_i, H_i\}$ is a $\mathcal{U}$-fringe of X.*

B_n: *Whenever E_i, F_i, $i = 1, \ldots, n+1$, are pairs of distant sets of Y, there are pairs of disjoint $\mathcal{U}$-open sets G_i, H_i of Y such that $E_i \subset G_i$, $F_i \subset H_i$ and $Y - X \subset \bigcup G_i \cup H_i$.*

Proposition 3. *A_n and B_n are equivalent.*

Proof. Suppose A_n holds, and let E_i, F_i, $i = 1, \ldots, n+1$, be distant sets of Y. Let P_i, S_i be $\mathcal{U}$-open and Q_i, T_i $\mathcal{U}$-closed sets of Y with $E_i \subset P_i \subset Q_i$, $F_i \subset S_i \subset T_i$ and $Q_i \cap T_i = \emptyset$. By A_n, there are disjoint $\mathcal{U}$-open sets U_i, V_i of X such that $Q_i \cap X \subset U_i$, $T_i \cap X \subset V_i$ and $\{U_i, V_i\}_i$ is a $\mathcal{U}$-fringe of X. By Lemma 8, there are $\mathcal{U}$-open sets L_i, M_i of Y with $L_i \cap X \subset U_i$, $M_i \cap X \subset V_i$ and $Y - X \subset \bigcup L_i \cup M_i$. Then $G_i = P_i \cup L_i$, $H_i = S_i \cup M_i$ satisfy B_n.

That B_n implies A_n follows from Corollary 2.

Lemma 9. *$\mathcal{U}\text{-}\dim(Y - X) \leqslant n$ implies B_n.*

Proof. Let E_i, F_i, $i = 1, \ldots, n+1$, be distant sets of Y. Take $\mathcal{U}$-open sets P_i, S_i and $\mathcal{U}$-closed sets Q_i, T_i with $E_i \subset P_i \subset Q_i$, $F_i \subset S_i \subset T_i$ and $Q_i \cap T_i = \emptyset$. If $\mathcal{U}\text{-}\dim(Y - X) \leqslant n$, by Proposition 5 of [1], there are disjoint $\mathcal{U}$-open sets U_i, V_i of $Y - X$ with $Q_i - X \subset U_i$, $T_i - X \subset V_i$ and $Y - X \subset \bigcup U_i \cup V_i$. In view of Lemma 3, we may suppose U_i, V_i are $\mathcal{U}$-open in Y. Then $G_i = P_i \cup (U_i - T_i)$, $H_i = S_i \cup (V_i - Q_i)$ satisfy B_n.

Proposition 4. *Let X be complete in the sense of Čech. Then* $\dim(Y-X)\leqslant n$ *if and only if* A_n.

Proof. $Y-X=\bigcup_{m=1}^{\infty} Z_m$ where each Z_m is compact, and $\dim Y-X = \mathscr{U}\text{-}\dim(Y-X)$ [1]. That $\dim(Y-X)\leqslant n$ implies A_n follows from Lemma 9 and Proposition 3. If A_n holds, then B_n and hence the following weaker statement holds.

Whenever E_i, F_i, $i=1,\dots,n+1$, are disjoint closed sets of Z_m, there are disjoint $\mathscr{U}$-open sets G_i, H_i of Z_m with $E_i\subset G_i$, $F_i\subset H_i$ and $Z_m=\bigcup G_i\cup H_i$.

This implies $\dim Z_m\leqslant n$ and hence, by the countable sum theorem for dim, $\dim Y-X\leqslant n$ [6].

Corollary 3. *If X is complete in the sense of Čech, then* $\dim\beta X-X\leqslant n$ *if and only if whenever E_i, F_i, $i=1,\dots,n+1$, are disjoint zero sets of X, there are disjoint cozero sets G_i, H_i of X with $E_i\subset G_i$, $F_i\subset H_i$ and $X-\bigcup G_i\cup H_i$ compact.*

Proof. If $\{P_i\}$ is a finite cover of X by cozero sets, then $\beta X=\operatorname{Ex}(\bigcup P_i)=\bigcup \operatorname{Ex}P_i$ [e.g. 4], and hence $\{P_i\}$ is a uniform cover of X with respect to the uniformity $\mathscr{U}$ induced by βX on X. It now follows from Corollary 1 that if X is complete in the sense of Čech and $Q_1,\dots,Q_k$ are cozero sets of X with $X-\bigcup Q_i$ compact then $\{Q_i\}$ is a $\mathscr{U}$-fringe of X.

Lemma 10. $\mathscr{U}$-$\operatorname{Ind}X\leqslant n$ *if and only if*

C_n: *Whenever E, F are distant sets of X, there are disjoint $\mathscr{U}$-open sets G, H of X with $E\subset G$, $F\subset H$ and* $\mathscr{U}$-$\operatorname{Ind}(X-G\cup H)\leqslant n-1$.

Proof. That $\mathscr{U}$-$\dim X\leqslant n$ implies C_n follows from the easily established fact that two distant sets can be separated by disjoined $\mathscr{U}$-closed sets. It is also easily established (from the fact that this holds in R) that if D is a $\mathscr{U}$-closed set then there are $\mathscr{U}$-closed sets D_k, $k=1,2,\dots$, such that $D=\bigcap D_k$ and D_{k+1} is distant from $X-D_k$.

Suppose C_n holds, and let E, F be disjoint $\mathscr{U}$-closed sets of X. Choose $\mathscr{U}$-closed sets E_k, F_k, $k=1,2,\dots$, of X such that $E=\bigcap E_k$, $F=\bigcap F_k$, and E_{k+1}, $X-E_k$ and F_{k+1}, $X-F_k$ are distant. Then $E-F_k$, $F\cup(X-E_k)$ are distant, and hence there are disjoint $\mathscr{U}$-open set G_k, H_k of X with $E-F_k\subset G_k$, $F\cup(X-E_k)\subset H_k$ and $\mathscr{U}$-$\operatorname{Ind}X-G_k\cup H_k\leqslant n-1$. Let $G=\bigcup G_k$ and $H=\bigcap H_k$. Then G is $\mathscr{U}$-open, $G\cap H=\emptyset$, $E\subset G$, $F\subset H$ and for each k,

$$H\subset \bigcup_i (X-E_i)\cap H\subset \bigcup_i (X-E_i)\bigcap_{j<i} H_j\subset H_k.$$

Hence $H=\bigcup_i (X-E_i)\bigcap_{j<i} H_j$ is $\mathscr{U}$-open. Also $X-G\cup H\subset Z=\bigcup X-G_k\cup H_k$ and by the subset and countable sum theorem for $\mathscr{U}$-Ind [2], $\mathscr{U}$-$\operatorname{Ind}X-G\cup H\leqslant\mathscr{U}$-$\operatorname{Ind}Z\leqslant n-1$. It follows that $\mathscr{U}$-$\operatorname{Ind}X\leqslant n$.

Corollary 4. *For a metric space X,* $\operatorname{Ind}X\leqslant n$ *if and only if whenever E, F are distant sets of X there are disjoint open sets G, H with $E\subset G$, $F\subset H$ and* $\operatorname{Ind}X-G\cup H\leqslant n-1$.

Proof. If $\mathscr{U}$ is induced by a metric, "$\mathscr{U}$-open" means "open" and $\mathscr{U}$-Ind $=$ Ind [2].

Proposition 5. $\mathscr{U}$-$\dim Y-X\leqslant 0$ *if and only if* A_0.

Proof. In view of Lemma 9 and Proposition 3, we need only prove A_0 implies $\mathscr{U}$-$\dim Y-X\leqslant 0$. It follows from Proposition 3 and Lemma 10 that A_0 implies $\mathscr{U}$-$\operatorname{Ind}Y-X\leqslant 0$. Finally, $\mathscr{U}$-$\operatorname{Ind}\leqslant 0$ and $\mathscr{U}$-$\dim\leqslant 0$ are equivalent [2].

Corollary 5. *If* $\mathscr{U}$-$\dim(Y-X)\leqslant 0$, *then X is semicompact.*

The converse is false (Example 3).

Corollary 6. *If $\beta X-X$ is C^*-imbedded in βX, then* $\dim^*\beta X-X\leqslant 0$ *if and only if whenever E, F are disjoint zero sets of X, there are disjoint cozero sets G, H of X with $E\subset G$, $F\subset H$ and $X-G\cup H$ accessible.*

Proof. If $\beta X-X$ is C^*-imbedded in βX, then $\dim^*\beta X-X=\mathscr{U}$-$\dim(\beta X-X)$ [1], where $\mathscr{U}$ denotes the uniformity induced by βX on $\beta X-X$.

Corollary 7. *If $Y-X$ has the monotonicity property relative to* dim [12], *then A_0 implies* $\dim(Y-X)\leqslant 0$.

Proof. There is a compactification Z of $Y-X$ with $\dim Z=\mathscr{U}$-$\dim(Y-X)$ [1, Proposition 8].

Corollary 8. *If X satisfies the bicompact axiom of countability, then* $\dim Y-X\leqslant 0$ *if and only if X satisfies A_0.*

Corollary 9. *If X satisfies the bicompact axiom of countability, then* $\dim\beta X-X\leqslant 0$ *if and only if whenever E, F are disjoint zero sets of X, there are disjoint cozero sets G, H of X with $E\subset G$, $F\subset H$ and $X-G\cup H$ compact.*

Smirnov [11] calls X proximally semibicompact if whenever E, F are distant sets of X, there are disjoint open sets G, H of X such that $E\subset G$, $F\subset H$, $X-G\cup H$ is compact and for every open neighbourhood P of $X-G\cup H$, $G-P$ is distant from $H-P$. The last condition simply means that $\{G,H\}$ is an extendable fringe [11, Lemma 1]. Thus A_0 implies proximal semibicompactness, and for spaces satisfying the bicompact axiom of countability the two conditions are equivalent since if X is also proximally semibicompact, then $\dim Y-X=\dim^{\infty}X=0$ [11, Theorem 3]. For arbitrary spaces, however, proximal semibicompactness does not imply A_0 (Example 3). Corollaries 8 and 9 are equivalent to Smirnov's [11] Theorem 3 and its Corollary 2, respectively.

3. Examples.

Example 2. For each ordinal $\alpha\leqslant\omega_1$, the first uncountable ordinal, let I^n_α be a subset of I^n, $n=1,2,\dots,\infty$, such that $\dim I^n_\alpha=0$, $I^n_\alpha\subset I^n_\beta$ for $\alpha\leqslant\beta$, and $I^n=I^n_{\omega_1}=\bigcup_{\alpha<\omega_1} I^n_\alpha$ [7, Theorem 13–15]. $M_n=\bigcup_{\alpha<\omega_1}\{\alpha\}\times I^n_\alpha$ and $K_n=\bigcup_{\alpha\leqslant\omega_1}\{\alpha\}\times I^n_\alpha$ are given the subspace topology induced by $[0, \omega_1]\times I^n$, and N_n is obtained from K_n by identifying all the points of $\{\omega_1\}\times I^n$. Then M_n, K_n, N_n are normal with $\operatorname{ind} M_n = \operatorname{ind} K_n = \operatorname{ind} N_n = \dim N_n = 0$ and $\dim M_n = \dim K_n = n$. These spaces are due to Smirnov [10], who generalizes an example of Dowker's [3].

Let ω be an ordinal greater than the weight of $Z_n = \beta N_n \times\{0, 1, 1/2, \ldots \ldots, 1/m, \ldots\}$. Let $Y_n = [0, \omega]\times Z_n$, $\mathscr{U}$ the unique uniformity on the compact space Y_n, and $X_n = Y_n - \{\omega\}\times M_n \times \{1, 1/2, \ldots, 1/m, \ldots\}$. Then since $\dim \beta N_n = \dim N_n = 0$, $\mathscr{U}\text{-}\dim Y_n = \dim Y_n = \dim Y_n = 0$, and by the subset theorem for $\mathscr{U}$-dim [1] $\mathscr{U}\text{-}\dim(Y_n - X_n) = 0$. Hence X_n is semicompact (Corollary 5). The choice of ω ensures that every real valued continuous function of X_n can be extended to Y_n, and hence $\beta X_n = Y_n$.

Lemma 11. *Let f: $Y_n \to S$ be the extension of the inclusion $X_n \to S$, where S is a compactification of X_n, and suppose E_1, E_2 are disjoint closed sets of M_n. Let $E_{i,m} = \{\omega\}\times E_i \times \{1/m\}$, $i = 1, 2$, $m = 1, 2, \ldots$ Then there is an integer k such that $f(E_{1,m}) \cap f(E_{2,m}) = \emptyset$ whenever $m > k$.*

Proof. Let $p_m = (\omega, \alpha_m, x_m, 1/m) \in E_{1,m}$, and $q_m = (\omega, \beta_m, y_m, 1/m) \in E_{2,m}$. Then there is an ordinal α with $\alpha_m, \beta_m < \alpha < \omega_1$ for each m. For each j, the image of $\{\omega\}\times \beta N_n \times \{1/j\}$ under f is a closed set of S containing only a finite number of points of $\{f(p_m), f(q_m)\}$. If F_i is the closure of $E_i \cap ([0, \alpha]\times I^n)$ in βN_n, it follows that the limit points of $\{f(p_m)\}$, $\{f(q_m)\}$ belong to $\{\omega\}\times F_1 \times \{0\}$, $\{\omega\}\times F_2\times\{0\}$, respectively. Finally, $F_1 \cap F_2 = \emptyset$, and if $f(p_m) = f(q_m)$ for infinitely many m's, then since S is compact $\{f(p_m)\}$, $\{f(q_m)\}$ have a common limit point belonging to $\{\omega\}\times(F_1 \cap F_2)\times\{0\} = \emptyset$.

If $\dim(S - X_n) = r - 1 < n$, let E_i, F_i, $i = 1, 2, \ldots, r$, be disjoint closed sets of M_n. Since the restriction of f to $Y_n - X_n$ is closed, then for some integer m $f(E_{i,m})$, $f(F_{i,m})$, $i = 1, 2, \ldots, r$, are disjoint closed sets of $S - X_n$, and hence there are disjoint open sets G_i, H_i of $S - X_n$ with $f(E_{i,m}) \subset G_i$, $f(E_{i,m}) \subset H_i$ and $S - X_n = \bigcup G_i \cup H_i$. Then $f^{-1}(G_i)$, $f^{-1}(H_i)$ are disjoint open sets of $Y_n - X_n$ with $E_{i,m} \subset f^{-1}(G_i)$, $F_{i,m} \subset f^{-1}(H_i)$ and $\{\omega\}\times M_n \times \{1/m\} \subset \bigcup f^{-1}(G_i) \cup f^{-1}(H_i)$. This is readily seen to imply $\dim M_n \leqslant r - 1 < n$. Hence $\dim(S - X_n) \geqslant n$. Moreover, the countable sum theorem for dim [6] implies $\dim(Y_n - X_n) = n$.

Let G_1, G_2 be disjoint open sets of $Y_n - X_n$. Since $Y_n - X_n$ and hence G_1, G_2 are open in $\{\omega\}\times N_n \times \{0, 1, 1/2, \ldots, 1/m, \ldots\}$, then, in $\{\omega\}\times Z_n$, $\operatorname{Ex} G_1$, $\operatorname{Ex} G_2$ are open and disjoint with $G_1 \subset \operatorname{Ex} G_1$ and $G_2 \subset \operatorname{Ex} G_2$. It follows from this observation that X_n is normally adjoined to $Y_n - X_n$, and hence to every one of its increments. Thus X is semicompact, every increment of X is normal with $\dim \geqslant n$, $\dim^\infty X_n = \dim(Y_n - X_n) = n$ [11], while $\mathscr{U}\text{-}\dim^\infty X_n = \mathscr{U}\text{-}\dim Y_n - X_n = 0$ (Proposition 2). A space with $\dim^\infty < \mathscr{U}\text{-}\dim^\infty$ is given in Example 4.

Example 3. A semicompact space every increment of which has $\mathscr{U}\text{-}\dim > 0$ and $\mathscr{U}\text{-}\operatorname{ind} > 0$.

Let L be the space obtained from $[0, \omega_1]$ by inserting a copy of Q, the rationals in $[0, 1]$, between any two consecutive ordinals α_1, α_2, and identifying 0 with α_1 and 1 with α_2. Let $X_n = L \times I_n$ where $I_n = I^n - \{0\}$. It follows from the fact that $\dim L = 0$ and I_n is semicompact that X_n is semicompact. Hence X_n is proximally semibicompact in the proximity induced by its Freudenthal compactification.

A G_δ set of X_n intersecting $\{\omega_1\}\times I_n$ contains a closed subset homeomorphic with Q, and is not therefore compact. Let $\mathscr{U}$ be the uniformity on X_n induced by a compactification Y. Let E_i, F_i, $i = 1, \ldots, n$ be disjoint closed sets of a compact subset of $\{\omega_1\}\times I_n$, and suppose G_i, H_i are disjoint $\mathscr{U}$-open sets of X with $E_i \subset G_i$, $F_i \subset H_i$ and $F = X - \bigcup G_i \cup H_i$ compact. Since F is also a G_δ set of X, $\{\omega_1\}\times \times I_n \subset \bigcup G_i \cup H_i$. This implies that every compact subset of I_n has $\dim \leqslant n-1$. It follows that X_n does not satisfy A_{n-1}, and $\mathscr{U}\text{-}\dim(Y - X_n) \geqslant n$ (Lemma 9 and Proposition 3).

Let ∞ be a limit point of $\{\omega_1\}\times I_n$ in Y, and E a non-empty compact subset of $\{\omega_1\}\times I_n$. Choose a $\mathscr{U}$-open set H and $\mathscr{U}$-closed set F of Y with $E \subset H \subset F \subset Y - \{\infty\}$. If $\mathscr{U}\text{-}\operatorname{ind}(Y - X_n) \leqslant 0$, in view of Lemma 1, there are disjoint $\mathscr{U}$-open sets G_1, G_2 of Y with $\infty \in G_1$, $F - X_n \subset G_2$ and $Y - X_n \subset G_1 \cup G_2$. Then $P_1 = G_1 - F$, $P_2 = G_2 \cup H$ are disjoint $\mathscr{U}$-open sets of Y intersecting $\{\omega_1\}\times I_n$ with $Y - X_n \subset P_1 \cup P_2$. Then $X_n - P_1 \cup P_2$ is a compact G_δ set of X_n and $\{\omega_1\}\times I_n \subset P_1 \cup P_2$. Since this implies that I_n is disconnected, $\mathscr{U}\text{-}\operatorname{ind}(Y - X_n) > 0$. We recall that if Y is the Freudenthal compactification of X, then $\operatorname{ind}(Y - X) \leqslant 0$ [8, 11].

Example 4. In Example 2 of [2], we give a space $(R^n, \mathscr{E})$ with $\mathscr{E}\text{-}\dim R^n = n - 1$ and $\dim R^n = 0$. Let Z_n be a compactification of R^n (as a topological space) such that $\mathscr{U}\text{-}\dim R^n = \mathscr{E}\text{-}\dim R^n$ [1, Proposition 8], where $\mathscr{U}$ is the unique uniformity on Z_n. Let $Y_n = [0, \omega_1]\times Z_n$ and $X_n = Y_n - \{\omega_1\}\times R^n$. Then as in Example 2, X_n is normally adjoined to its increment $Y_n - X_n$, and hence $\dim^\infty X_n = \dim(Y_n - X_n) = \dim R^n = 0$ [11], while $\mathscr{U}\text{-}\dim^\infty X_n = \mathscr{U}\text{-}\dim R^n = n - 1$ (Proposition 2).

4. Dimension of compactifications. Answering the question of Alexandroff whether a semicompact space has a compactification of the same dimension with zero-dimensional increment, Skljarenko [8] gives a semicompact metric separable space with dim = 1 every compactification with zero-dimensional increment of which is of $\dim \geqslant 2$. We show that, in fact, any compactification with zero-dimensional increment of a semicompact space X satisfying the bicompact axiom of countability is of $\dim \leqslant \dim X + 1$. The definition of a π-compactification of a (semicompact) space can be found in [8]. Here we only use the following consequence of Skljarento's Lemmas 8 and 9: If E, F are disjoint closed sets of a π-compactification Y of X, there are disjoint open sets G, H of Y with $E \subset G$, $F \subset H$ and $Y - G \cup H \subset X$.

Proposition 6. *Let Y be a π-compactification of X with $\dim F \leqslant m$ for every compact set F of X. Then $\dim Y \leqslant m+1$.*

Proof. Let E_i, F_i, $i = 1, \ldots, m+2$, be disjoint $\mathscr{U}$-closed sets of Y. Let P_i, S_i be $\mathscr{U}$-open and Q_i, T_i be $\mathscr{U}$-closed sets of Y with $E_i \subset P_i \subset Q_i$, $F_i \subset S_i \subset T_i$ and $Q_i \cap T_i = \emptyset$. Take disjoint open sets G_{m+2}, H_{m+2} with $Q_{m+2} \subset G_{m+2}$, T_{m+2}

$\subset H_{m+2}$ and $F = Y - G_{m+2} \cup H_{m+2} \subset X$ (Y is a π-compactification). By hypothesis $\mathscr{U}$-$\dim F = \dim F \leqslant m$, and hence there are disjoint $\mathscr{U}$-open sets U_i, V_i of F, $i = 1, \ldots, m+1$, with $Q_i \cap F \subset U_i$, $T_i \cap F \subset V_i$ and $F \subset \bigcup U_i \cup V_i$ [1]. In view of Lemma 1, U_i, V_i may be assumed to be $\mathscr{U}$-open in Y. For $i \leqslant m+1$, let $G_i = P_i \cup (U_i - T_i)$, $H_i = S_i \cup (V_i - Q_i)$. Then G_i, H_i, $i = 1, \ldots, m+2$, are disjoint open sets of Y with $E_i \subset G_i$, $F_i \subset H_i$ and $Y = \bigcup G_i \cup H_i$. This implies $\dim Y \leqslant m+1$ [6].

Since the Freudenthal compactification of a semicompact space, and any compactification with zero-dimensional increment of a space satisfying the bicompact axiom of countability is a π-compactification [8].

COROLLARY 10. *If Y is the Freudenthal compactification of a semicompact space X, then* $\dim Y \leqslant \dim X + 1$.

COROLLARY 11. *If Y is a compactification of a space X satisfying the bicompact axiom of countability and* $\operatorname{ind} Y - X \leqslant 0$, *then* $\dim Y \leqslant \dim X + 1$.

In Proposition 6 and its corollaries, dim may be replaced by ind or Ind. The proofs of the following results are the obvious modifications of the proof of Proposition 6.

PROPOSITION 7. *If* $\dim F \leqslant m$ *for every accessible G_δ set F of X, then* $\dim Y \leqslant \mathscr{U}\text{-}\dim(T - X) + m + 1$.

COROLLARY 12. *If* $\dim F \leqslant m$ *for every accessible G_δ set F of X, then* $\dim Y \leqslant \mathscr{U}\text{-}\dim^\infty X + m + 1$.

COROLLARY 13. *If* $\dim F \leqslant m$ *for every compact G_δ set F of a space X satisfying the bicompact axiom of countability, then* $\dim Y \leqslant \dim^\infty X + m + 1$.

PROPOSITION 8. *If* $\dim F \leqslant m$ *for every accessible G_δ subset F of $Y - X$, then* $\dim Y \leqslant \mathscr{U}\text{-}\dim X + m + 1$.

PROPOSITION 9. *If* $\dim F \leqslant m$ *for every compact subset F of X, and $Y - X$ is normal, then* $\dim Y \leqslant \dim(Y - X) + m + 1$.

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