Suppose now that $H_{**}(A)$ is finitely generated as a bigraded Γ-algebra. By the minimality of the Tate resolution X of A, [3], this means that $F_r X = X$ for some $r>0$. If $r = 1$ or 2, then A is a graded complete intersection by I and II of § 1. So assume that $r>2$. Proceeding as in the proof of Theorem 1 (implication (5) $\Rightarrow$ (2)), we get $X = F\langle W_1, \ldots, W_t\rangle$ for a finite number of variables $W_1, \ldots, W_t$. By Lemma 2 we infer that $q(F)<\infty$. But $H_{**}(F) = \mathrm{Tor}^{A''}_{**}(K, A)$, and hence A has finite projective dimension over A''.

Proposition 2. *If a minimal generating set of the ideal $\mathfrak{A}'$ (notation as in Theorem 2) is concentrated in a single degree and the field K is infinite, then $\mathfrak{A}'$ has the property* (# #).

Proof. Suppose that $\mathfrak{A}'$ contains a homogeneous non-zero divisor. We will prove that such an element can be chosen from some minimal set of generators of $\mathfrak{A}'$. Let V be a K-vector space generated by some fixed minimal set of generators of the ideal $\mathfrak{A}'$. If every homogeneous element from $\mathfrak{A}'\setminus I'\mathfrak{A}'$ is a zero-divisor in A', then $V'\subset\mathfrak{z}(A') = P_1'\cup\ldots\cup P_k'$ because of the assumption that V' is concentrated in a single degree. This implies $V' = \bigcup V'\cap P_i$ and consequently $V' = V'\cap P_i$ for some i since a finite-dimensional vector space over an infinite field cannot be a set-theoretic union of a finite number of its proper subspaces. Thus $\mathfrak{A}'\subset P_i$ and we get a contradiction of the fact that $\mathfrak{A}'$ contains a homogeneous non-zero divisor.

We complete the proof by induction with respect to the length of a minimal regular sequence in $\mathfrak{A}'$.

Added in proof. When the paper had been submitted for publication I learned that the following result follows from Lemma 3.7 of [6].

Proposition 3. *If A is a free finitely generated K-algebra generated by elements of the same even degree and the field K is infinite, then every non-zero homogeneous ideal of A' has the property* (# #).

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The André-Quillen homology of commutative graded algebras

by

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Abstract. The paper presents an extension of André-Quillen (co-)homology theory of commutative rings to graded commutative algebras.

Basic definitions and properties are given and characterization of free algebras and graded complete intersections is obtained in terms of the André-Quillen (co-)homology.

§ 1. Introduction. The aim of this paper is to present an extension of the André-Quillen (co-)homology theory of commutative rings (see [1], [2], [8]) to commutative graded algebras.

We consider only algebras graded by natural numbers and such an algebra $A = \{A^p\}_{p=0,1,\ldots}$ is said to be *commutative* if

$$x\cdot y = (-1)^{\deg x\cdot\deg y}y\cdot x\,.$$
$$x^2 = 0 \text{ when } \deg x \text{ is odd}\,,$$

for homogeneous elements x, y in A.

§ 2 contains the definitions and basic properties of graded derivations and differentials.

In § 3 we give the definition of homology and cohomology modules for graded algebras and state their main general properties.

§ 4 is of an auxiliary nature and introduces some results on classical Tor-homology needed in the sequel.

In § 5 we compute low-dimensional (co-)homology modules and Theorem (5.5) gives the basic relation between multiplication in Tor-homology and the second André-Quillen homology module.

In § 6 we prove the Vanishing Theorem (6.1), characterizing regular sequences in a commutative graded algebra in terms of the André-Quillen (co-)homology and its consequences.

§ 2. Graded derivations and differentials. Let K be a commutative ring with an identity. We denote by K-Alg the category of graded (by natural numbers 0, 1, ...) commutative algebras over K and their homomorphisms of degree zero. Objects of K-Alg will be called "graded K-algebras" for short. For a graded

K-algebra B we denote by B-Mod the category of graded (by integers) left B-modules and their homomorphisms of degree zero. Objects of B-Mod will be called "graded B-modules" for short.

(2.1) DEFINITION. Let B be a graded K-algebra and let M be a graded B-module. We define a *covariant functor*

$$\mathrm{Hom}_B(M, -)\colon B\text{-Mod} \to B\text{-Mod}$$

as follows:

$$\mathrm{Hom}_B(M, Y) = \{\mathrm{Hom}_B^p(M, Y)\}_{p\in Z} \quad \text{for any object } Y \text{ of } B\text{-Mod};$$

f belongs to $\mathrm{Hom}_B^p(M, Y)$ if f is a K-homomorphism $f\colon M\to Y$ satisfying

1^0 $\deg f = p$, i.e., $f(M^k)\subset Y^{k+p}$ for any k,

2^0 $f(bm) = (-1)^{\deg b \deg f} bf(m)$ for any homogeneous b in B, m in M.

The B-module structure on $\mathrm{Hom}_B(M, Y)$ comes from that on Y and morphisms induced by $\mathrm{Hom}_B(M, -)$ are defined in a standard manner.

(2.2) DEFINITION. Let A, B be graded K-algebras and let $\varphi\colon A\to B$ be a morphism in K-Alg. We define a *covariant functor*

$$\mathrm{Der}(\varphi, -)\colon B\text{-Mod} \to B\text{-Mod}$$

as follows:

$$\mathrm{Der}(\varphi, Y) = \{\mathrm{Der}^p(\varphi, Y)\}_{p\in Z} \quad \text{for any object } Y \text{ of } B\text{-Mod};$$

δ belongs to $\mathrm{Der}^p(\varphi, Y)$ if δ is a mapping $\delta\colon B\to Y$ satisfying

1^0 $\delta\in \mathrm{Hom}_A^p(B, Y)$ (B is regarded as a graded A-module via φ),

2^0 $\delta(bb') = \delta(b)b' + (-1)^{\deg b \deg \delta} b\delta(b')$

for any homogeneous b, b' in B.

The B-module structure on $\mathrm{Der}(\varphi, Y)$ comes from that on Y and the behaviour of $\mathrm{Der}(\varphi, -)$ on morphisms is determined by saying that $\mathrm{Der}(\varphi, -)$ is a subfunctor of $\mathrm{Hom}_A(B, -)$.

Elements of $\mathrm{Der}^p(\varphi, Y)$ are called *derivations* of B with values in Y of degree p. We also use the notation $\mathrm{Der}(B/A, -)$ instead of $\mathrm{Der}(\varphi, -)$ when it is more convenient and there is no danger of confusion.

(2.3) PROPOSITION. *Let $\varphi\colon A\to B$ be a morphism in K-Alg. The functor $\mathrm{Der}(\varphi, -)$ is representable in the category B-Mod, i.e., there exists a graded B-module $\mathrm{Dif}(\varphi)$ such that the functors $\mathrm{Der}(\varphi, -)$ and $\mathrm{Hom}_B(\mathrm{Dif}(\varphi), -)$ are equivalent.*

Proof. Let J be the kernel of the map $B\otimes_A B\to B$ induced by the multiplication in B. J is a homogeneous ideal in $B\otimes_A B$ and inherits the structure of a two-sided graded B-module. On the factor module J/J^2 these two structures coincide and we put $\mathrm{Dif}(\varphi) = J/J^2$.

For a given Y in B-Mod we define two maps,

$$\gamma\colon \mathrm{Hom}_B(\mathrm{Dif}(\varphi), Y)\to \mathrm{Der}(\varphi, Y),$$
$$\alpha\colon \mathrm{Der}(\varphi, Y)\to \mathrm{Hom}_B(\mathrm{Dif}(\varphi), Y),$$

which appear to be inverse to each other.

For homogeneous f in $\mathrm{Hom}_B(\mathrm{Dif}(\varphi), Y)$ we put $\gamma(f) = fd$ where $d\colon B\to \mathrm{Dif}(\varphi)$ is the canonical derivation of degree zero, $d(b) = (b\otimes 1 - 1\otimes b) \bmod J^2$.

For a derivation δ from $\mathrm{Der}(\varphi, Y)$ we put

$$\alpha(\delta)\left(\sum b_i\otimes b_i'\right) = -\sum (-1)^{\deg b_i \deg \delta} b_i\delta(b_i').$$

The graded B-module $\mathrm{Dif}(\varphi)$ is called a *graded module of differentials of* φ. We sometimes use notation $\mathrm{Dif}(B/A)$ instead of $\mathrm{Dif}(\varphi)$ when there is no danger of confusion.

(2.4) DEFINITION. Let V be a K-module graded by natural numbers. We denote by $S_K(V)$ (or simply by $S(V)$) the graded symmetric K-algebra on V. It is defined as an algebra isomorphic to $E(V^-)\otimes P(V^+)$ where $E(V^-)$ denotes the exterior algebra generated by the odd part V^- of V and $P(V^+)$ denotes the polynomial algebra generated by the even part V^+ of V.

(2.5) LEMMA. *Graded $S(V)$-modules $\mathrm{Dif}(S(V)/K)$ and $S(V)\otimes_K V$ are isomorphic.*

Proof. Let M be a graded $S(V)$-module. $\mathrm{Hom}_K(V, M)$ can be endowed with the structure of a graded $S(V)$-module by putting

$$(uf)(v) = uf(v)$$

for homogeneous u in $S(V)$, v in V, f in $\mathrm{Hom}_K(V, M)$.

At first we will prove that the functors $\mathrm{Der}(S(V)/K, -)$ and $\mathrm{Hom}_K(V, -)$ on $S(V)$-Mod are equivalent. We define the right action of $S(V)$ on M by the formula

$$m\cdot x = (-1)^{\deg m \deg x} xm, \quad x\in S(V),\ m\in M.$$

For $\delta\in \mathrm{Der}(S(V)/K, M)$ we put $T(\delta) = \delta\omega$ where $\omega\colon V\to S(V)$ is the canonical map. This gives us the map

$$T\colon \mathrm{Der}(S(V)/K, M)\to \mathrm{Hom}_K(V, M).$$

On the other hand, we define a map

$$F\colon \mathrm{Hom}_K(V, M)\to \mathrm{Der}(S(V)/K, M),$$

$$F(g)(v_1 v_2 \dots v_n) = \sum_{i=1}^{n} (-1)^{\left(\sum_1^{i-1} \deg v_k\right)\deg g} (v_1 \dots v_{i-1})\, g(v_i)\cdot(v_{i+1}\dots v_n)$$

for $g\in \mathrm{Hom}_K(V, M)$, $v_i\in V$.

A straightforward calculation shows that F and T are inverse to each other. Since $\mathrm{Hom}_K(V, -)\simeq \mathrm{Hom}_{S(V)}(S(V)\otimes_K V, -)$ on $S(V)$-Mod, we obtain by Proposition (2.3) the required equivalence.

(2.6) LEMMA. *Let A, C be graded K-algebras. We have the following isomorphisms of graded $A\otimes C$-modules:*

a) $\mathrm{Der}(A\otimes C/A, Y)\simeq \mathrm{Der}(C/K, Y)$ *for any $A\otimes C$-module Y,*

b) $\mathrm{Dif}(A\otimes C/A)\simeq A\otimes \mathrm{Dif}(C/K)$.

The proof is routine and we omit it.

(2.7) Definition. Let A be a graded K-algebra and V a graded K-module. A graded K-algebra $A \underset{K}{\otimes} S(V)$ is called the *symmetric A-algebra* on V and is denoted by $S_A(V)$. If V is a free K-module, then $S_A(V)$ is called a *free A-algebra*.

(2.8) Corollary. *Graded $S_A(V)$-modules* $\mathrm{Dif}(S_A(V)/A)$ *and* $S_A(V) \underset{K}{\otimes} V$ *are isomorphic.*

(2.9) Corollary. *If $S_A(V)$ is a free A-algebra, then* $\mathrm{Dif}(S_A(V)/A)$ *is a free graded $S_A(V)$-module.*

(2.10) Definition. For a morphism $\varphi: A \to B$ in K-Alg we define a *covariant functor*

$$\mathrm{Dif}(\varphi, -): B\text{-Mod} \to B\text{-Mod}$$

by putting $\mathrm{Dif}(\varphi, M) = \mathrm{Dif}(\varphi) \underset{B}{\otimes} M$ and appropriately on morphisms. We write also $\mathrm{Dif}(B/A, M)$ instead of $\mathrm{Dif}(\varphi, M)$ when there is no danger of confusion.

Observe that Dif and Der functorially depend on both arguments. This means in particular that a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow[\varphi']{} & B' \end{array}$$

and morphisms in K-Mod over β: $f: M \to M'$, $g: M' \to M$, induce maps

$$\mathrm{Dif}(\beta/\alpha, f): \mathrm{Dif}(\varphi, M) \to \mathrm{Dif}(\varphi', M'),$$
$$\mathrm{Der}(\beta/\alpha, g): \mathrm{Der}(\varphi', M') \to \mathrm{Der}(\varphi, M),$$

and they are natural.

(2.11) Definition. If A is an object in K-Alg we denote by A-Alg the category of all objects of K-Alg under A. Its objects are called *graded A-algebras*.

§ 3. Homology and cohomology; definitions and general properties. We now define the André-Quillen homology and cohomology of graded commutative algebras as functors obtained from Dif and Der functors by using a general procedure described by André in [1]. The general definition applied to our situation is as follows.

Let $\varphi: A \to B$ be a morphism in K-Alg and let M be a graded B-module. Free A-algebras will serve as models in our theory.

We define a complex of graded B-modules $T_\bullet(\varphi, M)$ as follows:

$$T_n(\varphi, M) = \bigoplus_{(\alpha_0, \ldots, \alpha_n) \in I_n} \mathrm{Dif}(\alpha_0 \cdot \ldots \cdot \alpha_{n-1} \cdot \alpha_n, M);$$

$(\alpha_0, \ldots, \alpha_n)$ means here a sequence of morphisms

$$A_{i_n} \xrightarrow{\alpha_n} A_{i_{n-1}} \xrightarrow{\alpha_{n-1}} \ldots \to A_{i_0} \xrightarrow{\alpha_0} B \quad \text{in } A\text{-Alg}$$

where A_{i_p} are models and the set I_n consists of all such sequences. The A_{i_n}-module structure on M is determined by the composite map $\alpha_0 \cdot \ldots \cdot \alpha_{n-1} \cdot \alpha_n$.

The differential $d_n: T_n(\varphi, M) \to T_{n-1}(\varphi, M)$ is defined as an alternating sum $d_n = \sum_{i=0}^{n} (-1)^i d_n^i$ where

$$d_n^i[\alpha_0, \ldots, \alpha_n] = [\alpha_0, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_n], \quad 0 \leqslant i < n,$$
$$d_n^n[\alpha_0, \ldots, \alpha_n] = [\alpha_0, \ldots, \alpha_{n-1}] \cdot \mathrm{Dif}(\alpha_n/1, 1)$$

and $[\alpha_0, \ldots, \alpha_n]$ means the canonical map of the direct summand corresponding to $(\alpha_0, \ldots, \alpha_n)$.

(3.1) Definition. The graded B-module

$$D_n(\varphi, M) = H_n(T_\bullet(\varphi, M)), \quad n \geqslant 0,$$

is called the *n-th André-Quillen homology module* of $\varphi: A \to B$ with coefficients in the graded B-module M. We also use the notation $D_n(B/A, M)$ instead of $D_n(\varphi, M)$ and speak about the nth homology module of the graded A-algebra B with coefficients in M.

To define cohomology we consider a complex of graded B-modules $T^\bullet(\varphi, M)$,

$$T^n(\varphi, M) = \prod_{(\alpha_0, \ldots, \alpha_n) \in I_n} \mathrm{Der}(\alpha_0 \cdot \ldots \cdot \alpha_{n-1} \cdot \alpha_n, M),$$

the product being taken over the same set of indices I_n as above. The differential $d^n: T^n(\varphi, M) \to T^{n+1}(\varphi, M)$ is defined as an alternating sum

$$d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n$$

where

$$(d_i^n f)(\alpha_0, \ldots, \alpha_{n+1}) = f(\alpha_0, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_{n+1}), \quad 0 \leqslant i \leqslant n,$$
$$(d_{n+1}^n f)(\alpha_0, \ldots, \alpha_{n+1}) = \mathrm{Der}(\alpha_{n+1}/1, 1)(f(\alpha_0, \ldots, \alpha_n)),$$

for $f \in T^n(\varphi, M)$.

(3.2) Definition. The graded B-module

$$D^n(\varphi, M) = H^n(T^\bullet(\varphi, M)), \quad n \geqslant 0,$$

is called the *n-th André-Quillen cohomology module* of $\varphi: A \to B$ with coefficients in the graded B-module M. We also use the notation $D^n(B/A, M)$ instead of $D^n(\varphi, M)$ and speak about the nth cohomology module of the graded A-algebra B with coefficients in M.

D_n and D^n functorially depend on both arguments in the same manner as Dif and Der, respectively.

The functors D_n and D^n can also be obtained as derived functors, i.e., can be computed by means of some special kind of resolutions which we are now going to describe more closely.

(3.3) Definition. A *free simplicial resolution of a graded A-algebra B* is a pair (X, ε) consisting of a simplicial graded A-algebra X (i.e., a simplicial object in the category A-Alg) and a morphism $\varepsilon\colon X \to \bar{B}$ of simplicial graded A-algebras (where $\bar{B}$ denotes the constant simplicial A-algebra equal in each dimension to B) satisfying the following conditions:

1° a graded A-algebra X_n is a model, i.e., a free graded A-algebra for each $n \geqslant 0$,

2° the map ε induces an isomorphism in homology, i.e., $H_0(X) \simeq B$, $H_i(X) = 0$ for $i > 0$.

The step-by-step construction presented in [1; Ch. I, § 6] furnishes us with a proof of the existence of a free simplicial resolution for any graded A-algebra.

Proceeding as in [2; Ch. V] one can prove the following:

(3.4) Theorem. *Let X be a free simplicial resolution of a graded A-algebra B. Then there exist isomorphisms of graded B-modules*

$$D_n(B/A, M) \simeq H_n(\mathrm{Dif}(X/A, M)) = H_n(\mathrm{Dif}(X/A, B) \otimes_B M),$$

$$D^n(B/A, M) \simeq H^n(\mathrm{Der}(X/A, M)) = H^n(\mathrm{Hom}_B(\mathrm{Dif}(X/A, B), M)).$$

(3.5) Remark. Observe that by Corollary (2.9) $\mathrm{Dif}(X/A, B)$ is a complex of free graded B-modules.

We list below some basic general properties of the functors D_n (dual formulations give them for D^n); the word "general" means here that they are valid also for commutative (ungraded) algebras and their proofs (see [2]) extend without essential changes to the graded case.

Let $\varphi\colon A \to B$ be a morphism in K-Alg.

I. If a graded A-algebra B is a model, i.e., $B \simeq S_A(V)$ for a free graded K-module V, then $D_n(\varphi, M) = 0$ for $n \geqslant 1$ and an arbitrary graded B-module M.

II. A short exact sequence of graded B-modules $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence

$$\ldots \to D_n(\varphi, M') \to D_n(\varphi, M) \to D_n(\varphi, M'') \to D_{n-1}(\varphi, M') \to \ldots$$

III. A sequence of graded K-algebra morphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ induces for any graded C-module M a long exact sequence

$$\ldots \to D_n(\varphi, M) \to D_n(\psi\varphi, M) \to D_n(\psi, M) \to D_{n-1}(\varphi, M) \to \ldots$$

IV. For two A-algebras B and C, and for a graded $B \otimes_A C$-module M the canonical homomorphism

$$D_n(B/A, M) \oplus D_n(C/A, M) \to D_n(B \otimes_A C/A, M)$$

is an isomorphism if $\mathrm{Tor}_i^A(B, C) = 0$ for $i = 1, 2, \ldots, n$.

From III and IV formally follows

V. For two A-algebras B and C, and for a graded $B \otimes_A C$-module M the canonical homomorphism

$$D_n(B/A, M) \to D_n(B \otimes_A C/C, M)$$

is an isomorphism if $\mathrm{Tor}_i^A(B, C) = 0$ for $i = 1, 2, \ldots, n$.

§ 4. Classical homology (revision). We recall and prove in this section some auxiliary results needed in §§ 5, 6.

(4.1) Definition [6]. Let A be a graded K-algebra. A homogeneous element x in A is called *regular* if

$$\mathrm{Ann}(x) = 0 \quad \text{for} \quad \deg x \text{ even},$$
$$\mathrm{Ann}(x) = (x) \quad \text{for} \quad \deg x \text{ odd}.$$

Let x be a homogeneous element in a graded K-algebra A and let $S_A(X) = S_A(V)$ where V is a free graded K-module generated by a single element X, $\deg X = \deg x$. Denote by A', A'' a K-algebra A regarded as a graded $S_A(X)$-module via homomorphisms $S_A(X) \to A$ where $X \mapsto 0$, $X \mapsto x$, respectively.

(4.2) Lemma. *If x is regular in A, then*

$$\mathrm{Tor}_p^{S_A(X)}(A', A'') = 0 \quad \textit{for} \quad p > 0.$$

Proof. Consider two cases:

1) $\deg x$ is even; the sequence

$$\ldots \to 0 \to 0 \to S_A(X) \xrightarrow{X} S_A(X)$$

provides a free resolution of a graded $S_A(X)$-module A'. Consequently

$$\mathrm{Tor}_p^{S_A(X)}(A', A'') = 0 \quad \text{for} \quad p > 1$$

and

$$\mathrm{Tor}_1^{S_A(X)}(A', A'') = \mathrm{Ker}(A'' \xrightarrow{x} A'') = 0$$

because x is regular.

2) $\deg x$ is odd; here a free resolution of A' over $S_A(X)$ is given by the infinite sequence

$$\ldots \to S_A(X) \xrightarrow{X} S_A(X) \xrightarrow{X} S_A(X).$$

Thus

$$\mathrm{Tor}_p^{S_A(X)}(A', A'') = \mathrm{Ker}(A'' \xrightarrow{x} A'')/\mathrm{Im}(A'' \xrightarrow{x} A'') = 0$$

by the definition of a regular element.

Let $\varphi\colon A \to B$ be an epimorphism of graded K-algebras. We recall that

$$\mathrm{Tor}^A(B, B) = \{\mathrm{Tor}_{pq}^A(B, B)\}_{p,q=0,1,\ldots}$$

admits the canonical structure of a bigraded commutative B-algebra [4; Ch. XI]. This means that for $x \in \mathrm{Tor}^A_{pq}(B, B)$, $y \in \mathrm{Tor}^A_{rs}(B, B)$

$$x \cdot y = (-1)^{pr+qs} y \cdot x\,,$$
$$x^2 = 0 \quad \text{if} \quad p+q \text{ is odd}\,.$$

Furthermore, for any graded B-module M, $\mathrm{Tor}^A(B, M)$ admits the canonical structure of a bigraded module over $\mathrm{Tor}^A(B, B)$. This action is natural with respect to the pair (A, B).

(4.3) DEFINITION [6]. A sequence of homogeneous elements $x_1, \ldots, x_n$ in a graded K-algebra A is called *regular* if the image of x_i in $A/(x_1, \ldots, x_{i-1})A$ is regular, $1 \leqslant i \leqslant n$.

(4.4) PROPOSITION. *Let A be a graded K-algebra over a field K of characteristic different from 2 and let I be an ideal generated by the sequence $x_1, \ldots, x_n$. Put $B = A/I$. Then $x_1, \ldots, x_n$ is regular in A if and only if*

$$\mathrm{Tor}^A_2(B, B) = \mathrm{Tor}^A_1(B, B) \cdot \mathrm{Tor}^A_1(B, B)$$

and I/I^2 is a free graded B-module on cosets $x_1 + I^2, \ldots, x_n + I^2$.

Proof. Let V be a vector space over K generated by $x_1, \ldots, x_n$. We recall that a Koszul complex E on $x_1, \ldots, x_n$ is defined as a differential bigraded A-algebra $E = A \otimes \Gamma(U^-) \otimes E(U^+)$ where Γ denotes the divided power algebra functor and E — the exterior algebra functor, U^-, U^+ are copies of V^-, V^+, respectively, with homological degree shifted by one, i.e., the bi-degree of $u \in U$ corresponding to $v \in V$ is $(1, \deg v)$ (in the notation of [6] $E = F_1 X$ where X is the Tate resolution of B over A).

From Proposition (5.1) in [6] it follows that if $x_1, \ldots, x_n$ is regular in A, then the Koszul complex provides a free resolution of B over A, i.e.,

$$\mathrm{Tor}^A(B, B) = H(E \underset{A}{\otimes} B) = E \underset{A}{\otimes} B = B \otimes \Gamma(U^-) \otimes E(U^+)$$

as bigraded B-algebras. Hence if $\mathrm{char}\, K \neq 2$ we obtain our conclusion because obviously I/I^2 is free over B.

To prove the converse we recall that starting with the Koszul complex and continuing by the procedure of killing nonbounding cycles [6; § 4] one can build a free A algebra resolution X of B (called the Tate resolution in [6]) which contains E.

If $x_1, \ldots, x_n$ is not regular, then again by Proposition (5.1) in [6] $H_1(E) \neq 0$. Let cycles $s_1, \ldots, s_p$, $p \geqslant 1$, be representatives of a minimal set of generators of $H_1(E)$. Denote by $F = E\langle S_1, \ldots, S_p; dS_i = s_i\rangle$ the subalgebra of X obtained from E by the adjunction of the variables S_i which kill cycles s_i. Each S_i has homological degree 2.

The assumption that I/I^2 is free implies $s_i \in IE$ and therefore $S_i \otimes 1$ is a cycle in $X \underset{A}{\otimes} B$. Since $H(X \underset{A}{\otimes} B) = \mathrm{Tor}^A(B, B)$, $S_i \otimes 1$ represents an element ξ_i of $\mathrm{Tor}^A_2(B, B)$ and we will prove that ξ_i does not belong to $(\mathrm{Tor}^A_1(B, B))^2$.

Indeed, the assumption $\xi_i \in (\mathrm{Tor}^A_1(B, B))^2$ would imply $S_i \otimes 1 = e \otimes 1 + dW \otimes 1$ in $X \underset{A}{\otimes} B$, where $e \in E$, $W \in X$, and consequently $S_i - e - dW \in IX$. But $dW = \sum_{j=1}^{r} \lambda_j S_j + e'$, $e' \in E$, and by the assumption that s_i are representatives of a minimal generating set of $H_1(E)$ we get $\lambda_i \in \mathfrak{M}$, $\mathfrak{M}$ being the maximal homogeneous ideal of A. This means that $S_i \in \mathfrak{M}X$, which contradicts the fact that S_i is a new variable attached to E to kill the cycle s_i and therefore $S_i \notin \mathfrak{M}X$.

(4.5) COROLLARY. *Let A be a graded K-algebra over a field K, $\mathrm{char}\, K \neq 2$. If $B = A/I$, I is generated by a regular sequence in A and M is a graded B-module, then*

$$\mathrm{Tor}^A_2(B, M) = \mathrm{Tor}^A_1(B, B) \cdot \mathrm{Tor}^A_1(B, M)\,.$$

§ 5. Low dimensions.

(5.1) PROPOSITION. *Let $\varphi\colon A \to B$ be a morphism in K-Alg. The following functors on B-Mod are equivalent:*

$$D_0(\varphi, -) \simeq \mathrm{Dif}(\varphi, -)\,, \qquad D^0(\varphi, -) \simeq \mathrm{Der}(\varphi, -)\,.$$

Proof. If X is a simplicial object, we denote by $\tilde{\varepsilon}^i_n$ and $\tilde{\eta}^i_n$ its face and degeneracy maps, respectively.

Consider the beginning

$$X_1 \underset{\tilde{\varepsilon}^1_1}{\overset{\tilde{\varepsilon}^0_1}{\rightrightarrows}} X_0 \xrightarrow{\tilde{\varepsilon}^0_0} B$$

of a free simplicial resolution X of a graded A-algebra B. We will prove that the sequence

(1) $$0 \to \mathrm{Der}(B/A, M) \xrightarrow{\tilde{\varepsilon}^0_0} \mathrm{Der}(X_0/A, M) \underset{\tilde{\varepsilon}^1_1}{\overset{\tilde{\varepsilon}^0_1}{\rightrightarrows}} \mathrm{Der}(X_1/A, M)$$

is exact for any graded B-module M.

Since $\tilde{\varepsilon}^0_0$ is an epimorphism, $\tilde{\varepsilon}^0_0$ is a monomorphism. If $(\tilde{\varepsilon}^0_1 - \tilde{\varepsilon}^1_1)(\delta) = 0$ for $\delta \in \mathrm{Der}(X_0/A, M)$, then there exists a graded derivation $\delta'\colon B \to M$ defined by $\delta'(\tilde{\varepsilon}^0_0 x_0) = \delta(x_0)$ and $\delta = \tilde{\varepsilon}^0_0(\delta')$. This means that $\mathrm{Im}\, \tilde{\varepsilon}^0_0 = \mathrm{Ker}(\tilde{\varepsilon}^0_1 - \tilde{\varepsilon}^1_1)$ and consequently, by Theorem (3.4), $D^0(B/A, M) \simeq \mathrm{Der}(B/A, M)$.

Now consider a sequence of graded B-modules

(2) $$\mathrm{Dif}(X_1/A) \underset{X_1}{\otimes} B \xrightarrow{(\tilde{\varepsilon}^0_1 - \tilde{\varepsilon}^1_1) \otimes 1} \mathrm{Dif}(X_0/A) \underset{X_0}{\otimes} B \xrightarrow{\tilde{\varepsilon}^0_0} \mathrm{Dif}(B/A) \to 0\,,$$

By applying the functor $\mathrm{Hom}_B(-, M)$ to (2) we get an exact sequence (1) for any graded B-module M. This implies that the initial sequence (2) is exact. From (2) tensored by M we obtain $D_0(B/A, M) \simeq \mathrm{Dif}(B/A, M)$.

(5.2) PROPOSITION. *Let $\varphi\colon A \to B$ be a surjection in K-Alg with kernel I. Then the following functors on B-Mod are equivalent:*

$$D_1(\varphi, -) \simeq I/I^2 \underset{B}{\otimes} -\,, \qquad D^1(\varphi, -) \simeq \mathrm{Hom}_B(I/I^2, -)\,.$$

(5.3) Lemma. *Let Y be a free simplicial resolution of a graded A-algebra B and let a simplicial homogeneous ideal J be defined by an exact sequence*

(3) $$0\to J\to Y\underset{A}{\otimes} B\to \bar{B}\to 0$$

of simplicial graded B-modules. Then $\mathrm{Dif}(Y/A, B)\simeq J/J^2$ *as simplicial graded B-modules.*

Proof. By the proof of Proposition (2.3) $\mathrm{Dif}(Y_n/A) = J_n'/J_n'^2$ where

(4) $$0\to J_n'\to Y_n\underset{A}{\otimes} Y_n\to Y_n\to 0 \quad \text{is exact}.$$

Tensoring (4) over Y_n by B on the right gives

(5) $$0\to J_n'\underset{Y_n}{\otimes} B\to Y_n\underset{A}{\otimes} B\to B\to 0.$$

From (5) and (3) follows that $J = J'\underset{Y}{\otimes} B$ and consequently $J^2 = J'^2\underset{Y}{\otimes} B$. Since by Corollary (2.9) $J_n'/J_n'^2$ is free over Y_n, we get

$$\mathrm{Dif}(Y/A, B) = J'/J'^2\underset{Y}{\otimes} B = J'\underset{Y}{\otimes} B/J'^2\underset{Y}{\otimes} B\simeq J/J^2.$$

Proof of Proposition (5.2). Since $\varphi\colon A\to B$ is an epimorphism, one can find such free simplicial resolution Y of B over A that $Y_0 = A$. From (3) we get

$$H_n(J\underset{B}{\otimes} M)\simeq \mathrm{Tor}_n^A(B, M) \quad \text{for} \quad n>0,$$

and from Lemma (5.3) it follows that

$$H_n(J/J^2\underset{B}{\otimes} M)\simeq D_n(B/A, M) \quad \text{for} \quad n\geqslant 0.$$

Consider now an exact sequence of simplicial graded B-modules

(6) $$0\to J^2\underset{B}{\otimes} M\to J\underset{B}{\otimes} M\to J/J^2\underset{B}{\otimes} M\to 0.$$

The equality $X_0 = A$ implies $J_0 = 0$, i.e., $H_0(J^2\underset{B}{\otimes} M) = 0$ and Lemma (5.4) below states that $H_1(J^2\underset{B}{\otimes} M) = 0$. Thus from the long exact homology sequence induced by (6) we get

$$D_1(B/A, M) = H_1(J/J^2\underset{B}{\otimes} M)\simeq H_1(J\underset{B}{\otimes} M)\simeq \mathrm{Tor}_1^A(B, M)\simeq I/I^2\underset{B}{\otimes} M.$$

The proof for cohomology is similar.

(5.4) Lemma. *With the above notation* $H_1(J^2\underset{B}{\otimes} M) = 0$.

Proof. Since $J_0 = 0$, the equality

$$(\tilde{\varepsilon}_2^0-\tilde{\varepsilon}_2^1+\tilde{\varepsilon}_2^2)(\tilde{\eta}_1^0(x)\cdot\tilde{\eta}_1^1(y)) = -xy$$

for any homogeneous x, y in J_1 proves the Lemma.

(5.5) Theorem. *Let $\varphi\colon A\to B$ be a surjection in K-Alg with kernel I and let K be a field of characteristic different from 2. Then there exists an equivalence of functors on B-Mod*

$$D_2(\varphi, -)\simeq \mathrm{Tor}_2^A(B, -)/\mathrm{Tor}_1^A(B, B)\cdot\mathrm{Tor}_1^A(B, -)$$

where the dot in the denominator denotes the action of a bigraded B-algebra $\mathrm{Tor}^A(B, B)$ on $\mathrm{Tor}^A(B, -)$ as described in § 4.

Proof. The idea of the proof comes from [2].

There exists a free A-algebra $S_A(F)$ and a commutative diagram of graded K-algebra surjections

(7) $$\begin{array}{ccc} S_A(F) & \xrightarrow{\psi} & A \\ \alpha\downarrow & & \downarrow\varphi \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

such that $\alpha(F) = I$. Since the $\mathrm{Tor}^A(B, B)$-module structure on $\mathrm{Tor}^A(B, M)$ is natural with respect to the pair (A, B), we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_1^{S_A(F)}(A, A)\underset{A}{\otimes}\mathrm{Tor}_1^{S_A(F)}(A, M) & \to & \mathrm{Tor}_2^{S_A(F)}(A, M) \\ \downarrow & & \downarrow\eta \\ \mathrm{Tor}_1^A(B, B)\underset{B}{\otimes}\mathrm{Tor}_1^A(B, M) & \to & \mathrm{Tor}_2^A(B, M) \end{array}$$

By Corollary (4.5) the top horizontal map is an epimorphism. By Proposition (5.2) and the fact that $\alpha(F) = I$ the left vertical map is also an epimorphism. These together imply that $\mathrm{Im}\,\eta = \mathrm{Tor}_1^A(B, B)\cdot\mathrm{Tor}_1^A(B, M)$. Now Theorem (5.5) results from the following

(5.6) Lemma. *There is an exact sequence of graded B-modules*

$$\mathrm{Tor}_2^{S_A(F)}(A, M)\xrightarrow{\eta}\mathrm{Tor}_2^A(B, M)\to D_2(B/A, M)\to 0.$$

Proof. Let X be a free simplicial $S_A(F)$-algebra resolution of A (over ψ; see (7)) and let Y be a free simplicial A-algebra resolution of B (over φ). By the step-by-step construction, [1], one can choose X and Y so that

1) $X_0 = S_A(F)$, $Y_0 = A$,
2) $X_1\underset{S_A(F)}{\otimes} A\simeq Y_1$,
3) Y is a free simplicial graded $X\underset{S_A(F)}{\otimes} A$-algebra.

This ensures the existence of a map $X\underset{S_A(F)}{\otimes} A\to Y$ of simplicial A-algebras such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \to & X\underset{S_A(F)}{\otimes} B & \to & \bar{B} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & J & \to & Y\underset{A}{\otimes} B & \to & \bar{B} & \to & 0 \end{array}$$

with exact rows commutes.

Since by Lemma (5.3)

$$H_2(I \underset{B}{\otimes} M) = \operatorname{Tor}_2^{S_A(F)}(A, M),$$

$$H_2(J \underset{B}{\otimes} M) = \operatorname{Tor}_2^{A}(B, M),$$

$$H_2(J/J^2 \underset{B}{\otimes} M) = D_2(B/A, M),$$

and the diagram

$$(8)\qquad \begin{array}{ccccccc} H_2(I^2 \underset{B}{\otimes} M) & \xrightarrow{\beta} & H_2(I \underset{B}{\otimes} M) & \to & H_2(I/I^2 \underset{B}{\otimes} M) & & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\eta} & & \downarrow & & \\ H_2(J^2 \underset{B}{\otimes} M) & \to & H_2(J \underset{B}{\otimes} M) & \xrightarrow{\gamma} & H_2(J/J^2 \underset{B}{\otimes} M) & \to & H_1(J^2 \underset{B}{\otimes} M) \end{array}$$

with exact rows is commutative, we have to show that the sequence

$$H_2(I \underset{B}{\otimes} M) \xrightarrow{\eta} H_2(J \underset{B}{\otimes} M) \xrightarrow{\gamma} H_2(J/J^2 \underset{B}{\otimes} M) \to 0$$

is exact. To do so it is sufficient to prove that the maps α, β, γ as indicated in (8) are epimorphisms.

γ is an epimorphism by Lemma (5.4).

β is an epimorphism because $H_2(I/I^2 \underset{B}{\otimes} M) = D_2(A/S_A(F), M) = 0$.

Indeed, using a chain of K-algebra morphisms $A \to S_A(F) \to A$ and an induced long exact sequence III, § 3, we get the required result because $S_A(F)$ is a model.

To prove that also α is an epimorphism we consider the commutative diagram

$$\begin{array}{ccccc} I_3^2 \otimes M & \xrightarrow{d_3} & I_2^2 \otimes M & \xrightarrow{d_2} & I_1^2 \otimes M \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ J_3^2 \otimes M & \xrightarrow[d_3]{} & J_2^2 \otimes M & \xrightarrow[d_2]{} & J_1^2 \otimes M . \end{array}$$

Since by Lemma (5.4) d_2's are epimorphisms, $d_2((\tilde{\eta}_1^0 J_1 \cdot \tilde{\eta}_1^1 J_1) \otimes M) = J_1^2 \otimes M$ and p_1 is an isomorphism by construction, we infer that $(\tilde{\eta}_1^0 J_1 \cdot \tilde{\eta}_1^1 J_1) \otimes M \subset \operatorname{Im} p_2$. On the other hand, straightforward calculation shows that $J_2^2 = d_3 J_3^2 + \tilde{\eta}_1^0 J_1 \cdot \tilde{\eta}_1^1 J_1$; hence $J_2^2 \otimes M = \operatorname{Im} d_3 + \operatorname{Im} p_2$. Since p_1 is an isomorphism, this implies that α is an epimorphism.

§ 6. Vanishing Theorem. In this section A always denotes a graded connected K-algebra over a field K. The main purpose is to prove the following

(6.1) THEOREM. *Suppose A is a graded K-algebra over a field K of characteristic different from 2, I an ideal in A and $B = A/I$. The following are equivalent:*

(i) *an ideal I admits a regular sequence of generators;*

(ii) *$D_2(B/A, B) = 0$ and I/I^2 is a free graded B-module;*

(iii) *$D_n(B/A, B) = 0$ for $n \geqslant 2$ and I/I^2 is a free graded B-module;*

(iv) *$D_2(B/A, M) = 0$ for any graded B-module M;*

(v) *$D_n(B/A, M) = 0$ for $n \geqslant 2$ and any graded B-module M;*

(vi) *$D^2(B/A, M) = 0$ for any graded B-module M;*

(vii) *$D^n(B/A, M) = 0$ for $n \geqslant 2$ and any graded B-module M.*

(6.2) LEMMA. *Let A be a graded K-algebra, $I = xA$ an ideal generated by a homogeneous element x and $B = A/I$. If x is regular in A, then $D_n(B/A, M) = 0$ for all $n \geqslant 2$ and any graded B-module M, and I/I^2 is a free graded B-module.*

Proof. Using the results and notations of Lemma (4.2) we have $\operatorname{Tor}_p^{S_A(X)}(A', A'') = 0$ for $p > 0$ since x is regular in A. By property V, § 3 we get

$$D_n(A'/S_A(X), M) \simeq D_n(A' \underset{S_A(X)}{\otimes} A''/A'', M) \simeq D_n(B/A, M) \quad \text{for} \quad n \geqslant 1 .$$

Since $S_A(X)$ is a model, we infer by property III, § 3 applied to the sequence $A \to S_A(X) \to A$ that $D_n(A'/S_A(X), M) = 0$ for $n \geqslant 2$ as required. I/I^2 is obviously free over B.

(6.3) Remark. Lemma (6.2) can also be proved by using a special free simplicial resolution of the A-algebra B as in [5].

Proof of Theorem (6.1). (i) $\Rightarrow$ (iii). Let $x_1, \ldots, x_n$ be a regular sequence of generators of I. It is straightforward that I/I^2 is free over B. Let $A_p = A/I_p$ for $p = 0, 1, \ldots, n$, where $I_p = (x_1, \ldots, x_p)A$; in particular, $A_0 = A$, $A_n = B$. We will prove by induction on p that $D_s(A_p/A, A_p) = 0$ for $s \geqslant 2$.

The case $p = 1$ follows from Lemma (6.2).

Assume that $p \geqslant 1$ and $D_s(A_p/A, A_p) = 0$ for $s \geqslant 2$. A long exact sequence induced by the sequence $A \to A_p \to A_{p+1}$ of graded K-algebra homomorphisms (property III, § 3) gives us an isomorphism

$$D_s(A_{p+1}/A, A_{p+1}) \simeq D_s(A_p/A, A_{p+1}) \quad \text{for} \quad s \geqslant 2$$

because $D_s(A_{p+1}/A_p, A_{p+1}) = 0$ for $s \geqslant 2$ by Lemma (6.2). Now consider two cases:

a) $\deg x_{p+1}$ is even; the exact sequence of graded A_p-modules $0 \to A_p \xrightarrow{x_{p+1}} A_p \to A_{p+1} \to 0$ induces (property II, § 3) the long exact sequence

$$\ldots \to D_s(A_p/A, A_p) \to D_s(A_p/A, A_p) \to D_s(A_p/A, A_{p+1}) \to \ldots$$

$$\ldots \to D_2(A_p/A, A_p) \to D_2(A_p/A, A_{p+1}) \to D_1(A_p/A, A_p) \xrightarrow{x_{p+1}} D_1(A_p/A, A_p) \to \ldots$$

Using the induction hypothesis, we obtain $D_s(A_p/A, A_{p+1}) = 0$ for $s > 2$. Since by Proposition (5.2) $D_1(A_p/A, A_p) = I_p/I_p^2$ and I_p/I_p^2 is free over A_p, the map $D_1(A_p/A, A_p) \xrightarrow{x_{p+1}} D_1(A_p/A, A_p)$ is injective and we get also $D_2(A_p/A, A_{p+1}) = 0$.

b) $\deg x_{p+1}$ is odd; the exact sequence of graded A_p-modules $0 \to A_{p+1} \xrightarrow{x_{p+1}} A_p \to A_{p+1} \to 0$ induces (property II, § 3) the long exact sequence

$$\ldots \to D_s(A_p/A, A_{p+1}) \to D_s(A_p/A, A_p) \to D_s(A_p/A, A_{p+1}) \to \ldots$$

$$\ldots \to D_2(A_p/A, A_p) \to D_2(A_p/A, A_{p+1}) \to D_1(A_p/A, A_{p+1}) \xrightarrow{x_{p+1}} D_1(A_p/A, A_p) \to \ldots$$

Since by the induction hypothesis $D_s(A_p/A, A_p) = 0$ for $s \geq 2$, this gives us $D_s(A_p/A, A_{p+1}) \simeq D_2(A_p/A, A_{p+1})$ for any $s \geq 2$. Observe further that the map $D_1(A_p/A, A_{p+1}) \xrightarrow{x_{p+1}} D_1(A_p/A, A_p)$ is injective since it comes from the injection $A_{p+1} \xrightarrow{x_{p+1}} A_p$ by tensoring over A_p by the graded free A_p-module I_p/I_p^2. This proves $D_2(A_p/A, A_{p+1}) = 0$ and consequently $D_s(A_p/A, A_{p+1}) = 0$ for any $s \geq 2$.

(iii) ⇒ (v). Since $B = A/I$, we can choose such free simplicial resolution X of a graded A-algebra B that $X_0 = A$. Therefore we have $L_0 = 0$ where $L = \mathrm{Dif}(X/A, B)$. The assumption $D_n(B/A, B) = 0$ for $n \geq 2$ and Theorem (3.4) imply that the sequence of graded B-modules

$$(1) \qquad \ldots \to L_n \to \ldots \to L_2 \to L_1 \to D_1(B/A, B) \to 0$$

is exact. But $D_1(B/A, B) = I/I^2$ is free over B, and so (1) splits. This gives $D_n(B/A, M) = H_n(L \otimes_B M) = 0$ for $n \geq 2$ and any graded B-module M.

(iii) ⇒ (vii). Since sequence (1) splits, we have as above $D^n(B/A, M) = H^n(\mathrm{Hom}_B(L, M)) = 0$ for $n \geq 2$ and an arbitrary graded B-module M.

(v) ⇒ (iv). Trivial.

(vii) ⇒ (vi). Trivial.

(vi ⇒ (ii). Consider the sequence

$$(2) \qquad L_3 \xrightarrow{d_3} L_2 \xrightarrow{d_2'} \mathrm{Im}\, d_2 \to 0 .$$

Since $D^2(B/A, M) = 0$ for any M, the sequence

$$0 \to \mathrm{Hom}_B(\mathrm{Im}\, d_2, M) \xrightarrow{(d_2')^*} \mathrm{Hom}_B(L_2, M) \xrightarrow{d_3^*} \mathrm{Hom}_B(L_3, M)$$

is exact for an arbitrary graded B-module M. But this means [3; § 2, Th. 1] that (2) is exact, i.e., $D_2(B/A, B) = 0$.

From property II, § 3 and the assumption $D^2(B/A, -) = 0$ it follows that the functor $D^1(B/A, -)$ is right exact. But by Proposition (5.2) $D^1(B/A, -) \simeq \mathrm{Hom}_B(I/I^2, -)$ and we infer that I/I^2 is projective and hence free over B.

(iv) ⇒ (ii). As before we infer that I/I^2 is flat and hence free over B.

(ii) ⇒ (i). By Theorem (5.5) we know that $\mathrm{Tor}_2^A(B, B) = (\mathrm{Tor}_1^A(B, B))^2$, but this together with the assumption that I/I^2 is free over B on free generators $x_1 + I^2, \ldots, x_n + I^2$, gives us by Proposition (4.4) that the sequence $x_1, \ldots, x_n$ is regular in A.

(6.4) Corollary. *If A is a Noetherian graded connected K-algebra over a field K of characteristic different from 2, then the following are equivalent:*

(i) *A is a free K-algebra (see Definition (2.4));*
(ii) $D_2(K/A, K) = 0$;
(iii) $D_n(K/A, K) = 0$ *for* $n \geq 2$;
(iv) $D^2(K/A, K) = 0$;
(v) $D^n(K/A, K) = 0$ *for* $n \geq 2$.

Proof. The corollary follows immediately from Theorem (6.1) and Theorem (2.6) in [6], stating that A is a free K-algebra if and only if the augmentation ideal of A is generated by a regular sequence in A.

We recall from [7] the following

(6.5) Definition. A graded K-algebra A is called a *graded complete intersection* if A is a quotient of a free K-algebra A' by an ideal generated by a regular set in A'.

(6.6) Corollary. *If A is a Noetherian graded connected K-algebra over a field K of characteristic different from 2, then the following are equivalent:*

(i) *A is a graded complete intersection;*
(ii) $D_3(K/A, K) = 0$;
(iii) $D_n(K/A, K) = 0$ *for* $n \geq 3$;
(iv) $D^3(K/A, K) = 0$;
(v) $D^n(K/A, K) = 0$ *for* $n \geq 3$.

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