It is easy to check that DC does not hold for Φ and that Φ is a Σ_3^1-formula. Hence Σ_3^1-DC fails and therefore so does Π_2^1-DC. ■

Final remarks. In the paper we have shown that in second order arithmetic DC ≡ **DC**, AC ↮ **AC**, and AC ↮ DC.

The only remaining question is whether AC→**DC**. This problem has been answered negatively by S. G. Simpson ([Si]). Simpson's proof, however is not known to the author of this paper.

References

[Fe] U. Felgner, *Models of* ZF-*Set Theory*, 1971.

[Le] A. Lévy, *Definability in the Axiomatic Set Theory II*, in: Mathematical Logic and Foundations of Set Theory, 1970.

[Si] S. G. Simpson, *Choice Schemata in Second Order Arithmetic*, Notices AMS, 1973.

MATHEMATISCH INSTITUUT
KATHOLIEKE UNIVERSITEIT
Nijmegen, The Netherlands

Accepté par la Rédaction le 14. 10. 1974

Movability and shape-connectivity

by

G. Kozlowski and J. Segal * (Seattle, Wash.)

Abstract. Theorem 1. *If (X, x) is a uniformly movable pointed continuum with $\underline{\pi}_n(X, x) = 0$ for all n, then (X, x) has trivial shape.* From this and the fact that a metric continuum X is approximately 1-connected if and only if it is the inverse limit of a sequence of simply connected ANR's, one obtains the corollary: *An approximately* 1-*connected movable metric continuum X with $\underline{\pi}_n(X) = 0$, for all n, has the shape of a point.* Another corollary is that the concept of uniform movability introduced in [12] is stronger than movability.

Introduction. In this paper we obtain a special case of a shape version of the Whitehead Theorem without a dimension restriction.

Theorem 1. *If (X, x) is a uniformly movable pointed continuum with $\underline{\pi}_n(X, x) = 0$, for all n, then (X, x) has trivial shape.*

Uniform movability here is taken in the sense of Kozlowski–Segal [12] which is a generalization of the concept of uniform movability defined by M. Moszyńska in [17] and which coincides for metric compacta with K. Borsuk's concept of movability [3]. As a corollary of Theorem 1 we show in Section 3 that a certain compact connected topological group is movable but not uniformly movable. This example is inspired by and heavily depends on the work of J. Keesling. As another application we have

Corollary. *An approximately* 1-*connected movable metric continuum X with $\underline{\pi}_n(X) = 0$, for all n, has the shape of a point.*

In this paper a compactum means a compact Hausdorff space, continuum means a connected compactum. All ANR's are understood to be compact. As a reference for the ANR-system approach to shape see [15]. *We assume that when we deal with a continuum the* ANR-*system associated with it is composed of connected* ANR'*s*. As a reference for the shape groups $\underline{\pi}_n$ see [16] where their isomorphism with the limit homotopy groups is established. Here we deal with only the latter groups which we accordingly take as the definition of the $\underline{\pi}_n$'s: if the ANR-system $\{(X_\alpha, x_\alpha), p_{\alpha\alpha'}, \mathscr{A}\}$ is associated with (X, x), then $\underline{\pi}_n(X, x)$ is defined to be $\underline{\lim}\{\pi_n(X_\alpha, x_\alpha), p_{\alpha\alpha'\#}, \mathscr{A}\}$. *In dealing with maps between* ANR'*s and their induced homomorphisms between homotopy groups we shall omit reference to base-points.*

* The second named author was partially supported by NSF grant GP-34058.

1. Movable pointed compacta. First we give a brief description of the natural transformation approach to shape theory and movability. For further details see [12].

In Section 1 it is assumed that *all* spaces are equipped with base-points and *all* maps and homotopies are base-point preserving. Because of this we find it notationally simpler to suppress further reference to base-points in this section.

Let $\boldsymbol{P}$ be the category of (compact) ANR's and homotopy classes of continuous maps between them. If X is a compactum, then Π_X is the functor from $\boldsymbol{P}$ to the category of sets and functions which assigns to an ANR P the set $\Pi_X(P) = [X; P]$ of all homotopy classes of maps of X into P and which assigns to any homotopy class $h: P\to Q$ between ANR's the induced function $h_{\#}: [X; P]\to[X; Q]$ which maps the homotopy class $\varphi: X\to P$ into the composition $h\varphi = h_{\#}(\varphi)$ of the homotopy classes of h and φ. A natural transformation G of the functor Π_X into the functor Π_Y assigns to each homotopy class $\varphi: X\to P$ a homotopy class $G(\varphi)$: $Y\to P$ in such a way that for all homotopy classes $\varphi: X\to P$, $\psi: X\to Q$, and $h: P\to Q$ such that $h\varphi = \psi$ we have $hG(\varphi) = G(\psi)$. If $f: X\to Y$ is a map, then there is a natural transformation $f^{\#}: \Pi_Y\to\Pi_X$ which assigns to the homotopy class $h: Y\to P$ the composition $h[f] = f^{\#}(h)$ of the homotopy class $[f]$ of f with h. (The natural transformations from Π_Y to Π_X correspond to the fundamental classes from X to Y in Borsuk's theory of shape.)

Given two compacta X and Y we say that the shape of X dominates the shape of Y if and only if there are natural transformations $F: \Pi_Y\to\Pi_X$ and $G: \Pi_X\to\Pi_Y$ such that $GF = 1_Y^{\#}$. If in addition $FG = 1_X^{\#}$, then X and Y are said to be of the same shape. In other words, X and Y have the same shape if and only if there is an invertible natural transformation (i.e., a natural equivalence) of the functors Π_X and Π_Y. Lastly, X has trivial shape if it has the shape of a point.

Remark 1. If $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$, is any inverse system such that $X = \varprojlim \underline{X}$, then by Theorems 4 and 3 of [13], we have that, for any ANR P, $\Pi_X(P)$ can be represented as the direct limit of the system $\{\Pi_{X_\alpha}(P), p_{\alpha\alpha'}^{\#}, \mathscr{A}\}$ by means of the functions $p_\alpha^{\#}: \Pi_{X_\alpha}(P)\to\Pi_X(P)$.

DEFINITION 1. A compactum X is said to be *uniformly movable provided*, that for each map $f: X\to P$ of X into an ANR P, there exist an ANR Q and natural transformations $\Phi: \Pi_X\to\Pi_Q$, $\Psi: \Pi_Q\to\Pi_X$ such that $\Psi\Phi[f] = [f]$.

Remark 2. Since any ANR is dominated by a polyhedron, for any map $f: X\to P$ into an ANR there exists a polyhedron K and maps $g: X\to K$ and $\theta: K\to P$ such that $f\simeq\theta g$. Using this it is not hard to see that in Definition 1 "ANR" may be replaced by "compact polyhedron". It then follows that Definition 1 is equivalent for compacta to the definition of uniform movability given in [12].

LEMMA 1. *Let the inverse system* $\{Y_\beta, q_{\beta\beta'}, \mathscr{B}\}$ *of* ANR's *be associated with* Y. (1) *If* $\varphi_\beta: X\to Y_\beta$ *are maps satisfying* $q_{\beta\beta'}\varphi_{\beta'}\simeq\varphi_\beta$, *then there is a unique natural transformation* $\Phi: \Pi_Y\to\Pi_X$ *such that* $\Phi[q_\beta] = [\varphi_\beta]$. (2) *If* $\Phi: \Pi_Y\to\Pi_X$ *is a natural transformation and* $\varphi_\beta: X\to Y_\beta$ *satisfy* $[\varphi_\beta] = \Phi[q_\beta]$ *for all* $\beta\in\mathscr{B}$, *then for any factorization* $f\simeq f_\beta q_\beta$ *of a map* $f: Y\to P$ *into an* ANR P, $\Phi[f] = [f_\beta\varphi_\beta]$.

Proof. If $f: Y\to P$ is any map into an ANR, then since $\Pi_Y(P)$ is represented as the direct limit of $\{\Pi_{Y_\beta}(P), q_{\beta\beta'}^{\#}, \mathscr{B}\}$ there is an index β and a map $f_\beta: Y_\beta\to P$ such that $f_\beta q_\beta\simeq f$. Define $\Phi: \Pi_Y\to\Pi_X$ by $\Phi[f] = [f_\beta\varphi_\beta]$. This is well-defined, because if there exist β' and $f_{\beta'}: X_{\beta'}\to P$ such that $f_{\beta'}q_{\beta'}\simeq f$, then by the direct limit representation, there is a $\beta''\geqslant\beta, \beta'$ such that

$$f_{\beta'}q_{\beta'\beta''}\simeq f_\beta q_{\beta\beta''},$$

and thus

$$f_{\beta'}\varphi_{\beta'}\simeq f_{\beta'}q_{\beta'\beta''}\varphi_{\beta'}\simeq f_\beta q_{\beta\beta''}\varphi_{\beta''}\simeq f_\beta\varphi_\beta.$$

It is not hard to show along the same lines that Φ is a natural transformation. The uniqueness of Φ and the second assertion of the lemma are obtained from the following computation.

$$\Phi[f] = \Phi[f_\beta q_\beta] = \Phi f_{\beta\#}[q_\beta] = f_{\beta\#}\Phi[q_\beta] = f_{\beta\#}[\varphi_\beta] = [f_\beta\varphi_\beta].$$

LEMMA 2. *Let the inverse system* $\{X_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$ *of* ANR's *be associated with the compactum* X. *If* X *is uniformly movable, then for any map* $f: X\to P$ *of* X *into an* ANR P *there exist* $\alpha\in\mathscr{A}$ *and* $\Phi: \Pi_X\to\Pi_{X_\alpha}$ *such that* $p_\alpha^{\#}\Phi[f] = [f]$. *In fact,* X *is uniformly movable if and only if for any* $\alpha\in\mathscr{A}$ *there exists* $\alpha'\in\mathscr{A}$ *and* $\Phi: \Pi_X\to\Pi_{X_{\alpha'}}$ *such that* $\Phi[p_\alpha] = [p_{\alpha\alpha'}]$.

Proof. Given $f: X\to P$ we choose a polyhedron Q, a map $g: X\to Q$ and a natural transformation $\Psi: \Pi_X\to\Pi_Q$ such that $g^{\#}\Psi[f] = [f]$. Making use of the representation (Remark 1) of $\Pi_X(Q)$ as the direct limit of $\{\Pi_{X_\alpha}(Q), p_{\alpha\alpha'}^{\#}, \mathscr{A}\}$ there exist $\alpha\in\mathscr{A}$, and a map $g_\alpha: X_\alpha\to Q$ such that $g\simeq g_\alpha p_\alpha$. Then $\Phi = g_\alpha^{\#}\Psi$ is the desired natural transformation.

For the second assertion of the lemma we assume X is uniformly movable and start with $p_\alpha: X\to X_\alpha$. Choose by the first assertion $\bar{\alpha}\in\mathscr{A}$ and $\Phi': \Pi_X\to\Pi_{X_{\bar{\alpha}}}$ such that $p_{\bar{\alpha}}^{\#}\Phi'[p_\alpha] = [p_\alpha]$. Let $\varphi: X_{\bar{\alpha}}\to X_\alpha$ satisfy $[\varphi] = \Phi'[p_\alpha]$. Since $\varphi p_{\bar{\alpha}}\simeq p_\alpha = p_{\alpha\bar{\alpha}}p_{\bar{\alpha}}$, it follows from the representation of $\Pi_X(X_\alpha)$ as the direct limit of the system $\{\Pi_{X_{\alpha'}}, p_{\alpha'\alpha''}^{\#}, \mathscr{A}\}$ that there is an $\alpha'\geqslant\bar{\alpha}$ such that $\varphi p_{\bar{\alpha}\alpha'}\simeq p_{\alpha\bar{\alpha}}p_{\bar{\alpha}\alpha'} = p_{\alpha\alpha'}$. Take $\Phi = p_{\bar{\alpha}\alpha'}^{\#}\Phi'$.

Conversely, assume the condition is satisfied and consider a map $f: X\to P$ into an ANR. By the direct limit representation of $\Pi_X(P)$ there exist an $\alpha\in\mathscr{A}$ and a map $f_\alpha: X_\alpha\to P$ such that $f_\alpha p_\alpha\simeq f$; also there is an $\alpha'\in\mathscr{A}$ and a natural transformation $\Phi: \Pi_X\to\Pi_{X_{\alpha'}}$ such that $\Phi[p_\alpha] = [p_{\alpha\alpha'}]$. Then

$$\begin{aligned} p_{\alpha'}^{\#}\Phi[f] &= p_{\alpha'}^{\#}\Phi f_{\alpha\#}[p_\alpha] = p_{\alpha'}^{\#}f_{\alpha\#}\Phi[p_\alpha] \\ &= p_{\alpha'}^{\#}f_{\alpha\#}[p_{\alpha\alpha'}] = [f_\alpha p_{\alpha\alpha'}p_{\alpha'}] = [f]. \end{aligned}$$

Remark 3. Lemmas 1 and 2 and their proofs hold verbatim in the unbased version of shape.

THEOREM 1. *Let* X *be a uniformly movable continuum with* $\underline{\pi}_n(X) = 0$, *for all* n, *then* X *has trivial shape.*

Proof. Let $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$ be an ANR-system associated with X. We first show (I) that for any $\alpha \in \mathscr{A}$, there exists an α' such that for any map $\theta: S^n \to X_{\alpha'}$ the composition $p_{\alpha\alpha'}\theta$ is nullhomotopic (α' is independent of n.) Given $\alpha \in \mathscr{A}$, we choose $\alpha' \in \mathscr{A}$ as in Lemma 2 to obtain a natural transformation $\Phi: \Pi_X \to \Pi_{X_{\alpha'}}$ such that $\Phi[p_\alpha] = [p_{\alpha\alpha'}]$. To see that α' works let $\varphi_{\bar\alpha} = \Phi[p_{\bar\alpha}]$ for all $\bar\alpha \in \mathscr{A}$. If $\theta: S^n \to X_{\alpha'}$ is any map of S^n into $X_{\alpha'}$, then the family of maps $\varphi_{\bar\alpha}\theta$ defines an element of the limit homotopy group $\underline{\pi}_n(X)$. Since this group is assumed to be zero, $\varphi_{\bar\alpha}\theta$ is nullhomotopic for all $\bar\alpha \in \mathscr{A}$. Applying Lemma 1 (2) to the factorization $p_\alpha = p_{\alpha\alpha'}p_{\alpha'}$, we get

(1) $$p_{\alpha\alpha'} \simeq p_{\alpha\alpha'}\varphi_{\alpha'}\,.$$

Hence we have

$$p_{\alpha\alpha'}\theta \simeq p_{\alpha\alpha'}\varphi_{\alpha'}\theta \simeq 0\,.$$

(II) For each n and for any $\alpha \in \mathscr{A}$, there is an $\alpha' \in \mathscr{A}$ such that for any map $\theta: Q \to X_{\alpha'}$ defined on a space Q dominated by an n-dimensional complex the composition $p_{\alpha\alpha'}\theta$ is nullhomotopic. To see this let $\alpha = \alpha_{n+1}$ and by (I) choose $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \ldots \geqslant \alpha_n$ such that for any map $\theta: S^k \to X_{\alpha_k}$ the composition $p_{\alpha_{k+1}\alpha_k}\theta$ is nullhomotopic. Then $\alpha' = \alpha_1$ is the desired index. To show this it suffices to consider the case in which Q is an n-dimensional polyhedron. Assume a triangulation of Q is given and let the k-skeleton of this triangulation be denoted by Q^k ($k = 0, 1, \ldots, n$). Since X_{α_1} is connected, θ is homotopic to a map $\theta_0: Q \to X_{\alpha_1}$ in which $\theta_0(Q^0)$ is a point. Assume that we have obtained $\theta_{k-1}: Q \to X_{\alpha_k}$ such that $\theta_{k-1}(Q^{k-1})$ is a point and $\theta_{k-1} \simeq p_{\alpha_k\alpha_1}\theta$. Since, for each k-simplex s of Q^k, θ_{k-1} maps the boundary ∂s of s to a point, there is by (I) a nullhomotopy relative to ∂s of $p_{\alpha_{k+1}\alpha_k}\theta_{k-1}|s$; hence $p_{\alpha_{k+1}\alpha_k}\theta_{k-1}$ is homotopic to a map $\theta_k: Q \to X_{k+1}$ such that $\theta_k(Q^k)$ is a point, and $\theta_k \simeq p_{\alpha_{k+1}\alpha_1}\theta$. It follows inductively that $p_{\alpha\alpha'}\theta$ is homotopic to a map $\theta_n: Q \to X_\alpha$ such that $\theta_n(Q) = \theta_n(Q^n)$ is a point. (Note that α' depends on n.)

We now show that X has trivial shape by showing (III) that any map $f: X \to P$ into an ANR P is nullhomotopic. By Lemma 2 there exist $\alpha \in \mathscr{A}$ and $\Phi: \Pi_X \to \Pi_{X_\alpha}$ such that $p_\alpha^{\#}\Phi[f] = [f]$. Let $f_\alpha: X_\alpha \to P$ satisfy $[f_\alpha] = \Phi[f]$, and let $\varphi_{\bar\alpha}: X_\alpha \to X_{\bar\alpha}$ satisfy $[\varphi_{\bar\alpha}] = \Phi[p_{\bar\alpha}]$. It follows from Corollary 6.2 of [7, p. 211] that X_α is dominated by the nerve of one of its finite open covers. Hence there is an integer n such that X_α is dominated by an n-dimensional polyhedron. We now apply (II) with this choice of n to obtain the desired index $\alpha' \in \mathscr{A}$. Since $f \simeq f_\alpha p_\alpha$, Lemma 1 implies that $[f_\alpha\varphi_\alpha] = \Phi[f] = [f_\alpha]$; hence $f \simeq f_\alpha\varphi_\alpha p_\alpha$. Because Φ is a natural transformation, $\varphi_\alpha \simeq p_{\alpha\alpha'}\varphi_{\alpha'}$, and by (II), $p_{\alpha\alpha'}\varphi_{\alpha'} \simeq 0$. Thus f is nullhomotopic. (It has been brought to our attention by the referee that Borsuk has proved a metric version of Theorem 1 in *Some remarks on shape properties of compacta*, Fund. Math. 85 (1974), pp. 185–195.)

2. Approximately 1-connected continua. In [2] Borsuk shows a finite dimensional metric continuum X which is movable, approximately 1-connected and with all shape groups $\underline{\pi}_n(X) = 0$, has the shape of a point. He asks in Problem (6.5) if this result remains true when the finite dimensionality hypothesis is deleted. In this section we show the answer to this question is affirmative. The following definition is easily shown to be equivalent to Borsuk's for metric continua.

DEFINITION 2. A metric continuum X in the Hilbert Cube I^∞ is said to be *approximately* 1-*connected* if every neighborhood V of X contains a neighborhood V_0 of X such that every map of the 1-sphere S^1 into V_0 is nullhomotopic in V.

LEMMA 3. *A metric continuum X is approximately* 1-*connected if and only if for any* ANR-*sequence associated with X there is a subsequence $\{X_i, p_{i,i+1}\}$ such that $p_{i,i+1\#}: \pi_1(X_{i+1}) \to \pi_1(X_i)$ is zero for $i = 1, 2, \ldots$*

Proof. Assume X is approximately 1-connected and embedded in the Hilbert Cube. First we consider an inclusion ANR-sequence $\underline{X} = \{X_i, p_{i,i+1}\}$ associated with X, i.e., $X = \varprojlim \underline{X}$ where each X_i is a neighborhood of X in the Hilbert Cube which is a connected ANR and each $p_{i,i+1}: X_{i+1} \to X_i$ is an inclusion map (see [14]). Then from Definition 2 we have for the neighborhood X_i a neighborhood $X_{i'}$ such that every map of S^1 into $X_{i'}$ is nullhomotopic in X_i. We delete terms and renumber our sequence so that $i' = i+1$. Then we have $p_{i,i+1\#}: \pi_1(X_{i+1}) \to \pi_1(X_i)$ is zero for $i = 1, 2, \ldots$

We now establish the following assertion: if $\underline{X} = \{X_i, p_{i,i+1}\}$ is any ANR-sequence associated with X having the property that $p_{i,i+1\#}: \pi_1(X_{i+1}) \to \pi_1(X_i)$ is zero for $i = 1, 2, \ldots$ and if $\underline{Y} = \{Y_i, q_{i,i+1}\}$ is another ANR-system associated with X, then $\underline{Y}$ has a subsequence having the same property. Since $\underline{X}$ and $\underline{Y}$ are of the same homotopy type (see [15]) we have maps of systems $\underline{f}: \underline{X} \to \underline{Y}$ and $\underline{g}: \underline{Y} \to \underline{X}$ such that $\underline{fg} \simeq \underline{1}_Y$ and $\underline{gf} \simeq \underline{1}_X$. Since $\underline{g}$ is a map of systems we have for each positive integer i an integer $i' \geqslant i$ such that

(1) $$g_{f(i)}q_{gf(i),i'} \simeq p_{f(i),f(i)+1}g_{f(i)+1}q_{g(f(i)+1),i'}\,.$$

Since $\underline{fg} \simeq \underline{1}_Y$ there is an $i'' \geqslant i$ such that

(2) $$q_{ii''} \simeq f_i g_{f(i)} q_{gf(i),i''}\,.$$

Let $\bar{i} = \max\{i', i''\}$ so that (1) and (2) hold for $\bar{i}$. Consider any map $\varphi: S^1 \to Y_{\bar{i}}$. From (2) and (1) we have

(3) $$q_{i\bar{i}}\varphi \simeq f_i g_{f(i)} q_{gf(i),\bar{i}}\varphi \simeq f_i p_{f(i),f(i)+1} g_{f(i)+1} q_{g(f(i)+1),\bar{i}}\varphi\,.$$

The latter map is nullhomotopic since $p_{f(i),f(i)+1}g_{f(i)+1}q_{g(f(i)+1),\bar{i}}\varphi$ is nullhomotopic being a map of $S^1 \to X_{f(i)+1} \to X_{f(i)}$. We delete terms in $\underline{Y}$ and renumber so that $\bar{i} = i+1$ to obtain the desired subsequence.

For the converse just consider an inclusion ANR-sequence associated with X and choose a subsequence $\{X_i, p_{i,i+1}\}$ as in the condition of the lemma. Then for any neighborhood V of X in the Hilbert Cube there is an $X_i \subset V$. Take $V_0 = X_{i+1}$ in the definition of approximately 1-connected.

LEMMA 4. *If Y is a connected* ANR *and $f: Y \to Z$ is a map such that $f_\#: \pi_1(Y) \to \pi_1(Z)$ is zero. Then there is a simply connected* ANR *Y' containing Y and a map $f': Y' \to Z$ extending f.*

Proof. Since Y is dominated by a finite complex $\pi_1(Y)$ is finitely generated. Let $g_i\colon S^1 \to Y$ be based maps for $i = 1, \ldots, n$ whose homotopy classes generate $\pi_1(Y)$. By the Borsuk–Whitehead Theorem the space Y' obtaining by adjoining 2-cells [18, p. 145] via the maps g_i $(i = 1, \ldots, n)$ is an ANR. By Theorem 3.8.10 of [18] Y' is simply-connected. For each i the map $fg_i\colon S^1 \to Z$ is nullhomotopic; hence it has an extension $B^2 \to Z$ of the 2-cell into Z. These extensions serve to define the map $f'\colon Y' \to Z$ extending f.

LEMMA 5. *If X is an approximately 1-connected metric continuum, then for any ANR-sequence associated with X there is a subsequence $\underline{X} = \{X_i, p_{i,i+1}\}$ and an ANR-sequence $\underline{X}' = \{X'_i, p'_{i,i+1}\}$ associated with X such that $X_i \subset X'_i$, $p'_{i,i+1}|X_i = p_{i,i+1}$ and each X'_i is simply connected.*

Proof. By Lemma 3 a given ANR-sequence has an ANR-subsequence $\underline{X} = \{X_i, p_{i,i+1}\}$ such that $p_{i,i+1\#}\colon \pi_1(X_{i+1}) \to \pi_1(X_i)$ is zero for $i = 1, 2, \ldots$ For technical reasons we take $\underline{X}$ to include X_0, a singleton, and $p_{0,1}\colon X_1 \to X_0$. Then for each $i = 0, 1, 2, \ldots$ we have by Lemma 4 a simply connected ANR X'_i containing X_i and a map $p'_{i,i+1}\colon X'_{i+1} \to X_i \subset X'_i$ extending $p_{i,i+1}$. Therefore the inverse sequence $\underline{X}' = \{X'_i, p'_{i,i+1}\}$ is the desired sequence associated with X.

THEOREM 2. *A metric continuum X is approximately 1-connected if and only if it is the inverse limit of a sequence of simply connected ANR's.*

Proof. If X is approximately 1-connected then Lemma 5 applies. Conversely, it follows from the assertion in the proof of Lemma 3 that any ANR-sequence associated with X has a subsequence $\underline{X} = \{X_i, p_{i,i+1}\}$ such that $p_{i,i+1\#}\colon \pi_1(X_{i+1}) \to \pi_1(X_i)$ is zero for $i = 1, 2, \ldots$

By Lemma 3 then X is approximately 1-connected. (The referee has pointed out that Trybulec has an unpublished result generalizing Theorem 2 to the approximately k-connected case.)

THEOREM 3. *If X is a uniformly movable continuum which has an inverse system of simply connected ANR's associated with it, then for any $x \in X$ the pointed continuum (X, x) is uniformly movable.*

Proof. First recall the well-known fact regarding simply connected ANR's: $(*)$ any homotopy class of a compactum A into a simply connected ANR M contains a base-point preserving map and any two base-point preserving maps in this homotopy class are based homotopic.

Assume we have an inverse system $\{X_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$ associated with X in which each X_α is a simply connected ANR. For each $\alpha \in \mathscr{A}$ let $x_\alpha = p_\alpha(x)$; hence $\{(X_\alpha, x_\alpha), p^{\cdot}_{\alpha\alpha'}, \mathscr{A}\}$ is associated with (X, x), where the dot indicates a based map.

We now verify the condition of Lemma 2 for uniform movability. If $\alpha \in \mathscr{A}$ is given, the unbased form of Lemma 2 implies that there exist $\alpha' \in \mathscr{A}$ and a natural transformation $\Phi\colon \Pi_{X_{\alpha'}} \to \Pi_X$ such that $\Phi[p_\alpha] = [p_{\alpha\alpha'}]$, where the homotopy classes are unbased. Let $\varphi_{\bar\alpha}\colon X_{\alpha'} \to X_{\bar\alpha}$ satisfy $[\varphi_{\bar\alpha}] = \Phi[p_{\bar\alpha}]$ for all $\bar\alpha \in \mathscr{A}$. By $(*)$ we may assume that $\varphi_{\bar\alpha}$ defines a base-point preserving map

$$\varphi^{\cdot}_{\bar\alpha}\colon (X_{\alpha'}, x_{\alpha'}) \to (X_{\bar\alpha}, x_{\bar\alpha})$$

and (since $p_{\bar\alpha\hat\alpha}\varphi_{\hat\alpha} \simeq \varphi_{\bar\alpha}$) that there are base-point preserving homotopies

$$p^{\cdot}_{\bar\alpha\hat\alpha}\varphi^{\cdot}_{\hat\alpha} \simeq \varphi^{\cdot}_{\bar\alpha}, \quad \text{whenever} \quad \bar\alpha \leqslant \hat\alpha .$$

By Lemma 1 there is a natural transformation $\Phi^{\cdot}\colon \Pi_{(X,x)} \to \Pi_{(X_{\alpha'}, x_{\alpha'})}$ such that $\Phi^{\cdot}[p^{\cdot}_{\bar\alpha}] = [\varphi^{\cdot}_{\bar\alpha}]$. Since $[\varphi_\alpha] = \Phi[p_\alpha] = [p_{\alpha\alpha'}]$, it follows from $(*)$ that $[\varphi^{\cdot}_\alpha] = [p^{\cdot}_{\alpha\alpha'}]$; hence

$$\Phi^{\cdot}[p^{\cdot}_\alpha] = [p^{\cdot}_{\alpha\alpha'}] .$$

Remark 4. Borsuk [1] has shown that if (X, x) is a pointed movable metric continuum, then $\mathrm{Sh}(X, x) = \mathrm{Sh}(X, x')$ for any point x' of X. Hence for such X the choice of the base-point in computing $\underline{\pi}_n$ is immaterial and consequently suppressed.

COROLLARY 1. *If X is an approximately 1-connected movable metric continuum such that $\underline{\pi}_n(X) = 0$, for all n, then X has the shape of a point.*

Proof. Choose $x \in X$. By Theorem 2 X is the inverse limit of a sequence of simply connected ANR's and consequently by Theorem 3 (X, x) is movable. Then by Theorem 1 (X, x) has trivial shape, and hence X has trivial shape.

3. Movable topological groups. The purpose of this section is to establish certain movability properties and to give an example of a movable continuum which is not uniformly movable. For the example we use the fact that for the class of uniformly movable compact connected abelian groups A, $\underline{\pi}_1(A) = 0$ is equivalent to $A = 0$, which is established in Theorem 5. Of course, the class of such groups is extensive because all products of circle groups belong to it. This follows from Theorem 4 which shows uniform movability is preserved under taking arbitrary products.

THEOREM 4. *If $\{X^j \mid j \in \mathscr{J}\}$ is a family of uniformly movable compacta, then the product $X = \Pi\{X^j \mid j \in \mathscr{J}\}$ is uniformly movable.*

Proof. First we consider the case of the product of two uniformly movable compacta Y and Z. If $\{Y_\beta, q_{\beta\beta'}, \mathscr{B}\}$ and $\{Z_\gamma, r_{\gamma\gamma'}, \mathscr{C}\}$ are ANR-systems associated with Y and Z respectively, then $\{Y_\beta \times Z_\gamma, q_{\beta\beta'} \times r_{\gamma\gamma'}, \mathscr{B} \times \mathscr{C}\}$ is an ANR-system associated with $Y \times Z$, where $\mathscr{B} \times \mathscr{C}$ has the product order: $(\beta', \gamma') \geqslant (\beta, \gamma)$ provided $\beta' \geqslant \beta$, $\gamma' \geqslant \gamma$. If $f\colon Y \times Z \to P$ is any map into an ANR, then by Remark 1 there exist $\beta \in \mathscr{B}$, $\gamma \in \mathscr{C}$ and a map $f_{\beta\gamma}\colon Y_\beta \times Z_\gamma \to P$ such that

$$f \simeq f_{\beta\gamma}(q_\beta \times r_\gamma) .$$

By the definition of uniform movability there exist ANR's Q, R, maps $g\colon Y \to Q$, $h\colon Z \to R$, and natural transformations $\Phi\colon \Pi_Y \to \Pi_Q$, $\Psi\colon \Pi_Z \to \Pi_R$ such that $g^{\#}\Phi[q_\beta] = [q_\beta]$ and $h^{\#}\Psi[r_\gamma] = [r_\gamma]$. Let $\varphi_{\bar\beta}\colon Q \to Y_{\bar\beta}$ and $\psi_{\bar\gamma}\colon R \to Z_{\bar\gamma}$ be maps for all $\bar\beta \in \mathscr{B}$ and $\bar\gamma \in \mathscr{C}$ satisfying

$$[\varphi_{\bar\beta}] = \Phi[q_{\bar\beta}] \quad \text{and} \quad [\psi_{\bar\gamma}] = \Psi[r_{\bar\gamma}] .$$

By Lemma 1 the maps

$$\varphi_{\bar\beta} \times \psi_{\bar\gamma}\colon Q \times R \to Y_{\bar\beta} \times Z_{\bar\gamma}$$

define a unique natural transformation
$$\Phi\times\Psi\colon \Pi_{Y\times Z}\to\Pi_{Q\times R}$$
such that
$$\Phi\times\Psi[q_{\bar\beta}\times r_{\bar\gamma}] = [\varphi_{\bar\beta}\times\psi_{\bar\gamma}]\,.$$
Since $\varphi_\beta g\simeq q_\beta$ and $\psi_\gamma h\simeq r_\gamma$,
$$(g\times h)^{\#}(\Phi\times\Psi)[q_\beta\times r_\gamma] = [\varphi_\beta g\times\psi_\gamma h] = [q_\beta\times r_\gamma]\,.$$
It follows easily that
$$(g\times h)^{\#}(\Phi\times\Psi)[f] = [f]\,.$$
Therefore, the product of two uniformly movable compacta is uniformly movable, and by induction it follows that the product of any finite number of uniformly movable compacta is uniformly movable.

For the general case consider $X = \Pi\{X^j|\ j\in\mathscr{I}\}$ as the inverse limit of the system of finite products on $\mathscr{I}$: $X = \varprojlim\{X_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$, where $\mathscr{A}$ is the collection of finite nonempty subsets of $\mathscr{I}$ with $\alpha\leqslant\alpha'$ meaning $\alpha\subset\alpha'$ and $X_\alpha = \Pi\{X^j|\ j\in\alpha\}$ with $p_{\alpha\alpha'}$ being the projection. If $f\colon X\to P$ is any map into an ANR, then by Remark 1 there exist an $\alpha\in\mathscr{A}$ and a map $f_\alpha\colon X_\alpha\to P$ such that $f_\alpha p_\alpha\simeq f$. Define a map $s\colon X_\alpha\to X$ by choosing a point $z_j\in X_j$ for each $j\notin\alpha$ and specifying that the jth coordinate of $s(x)$ is the jth coordinate of x if $j\in\alpha$ and is z_j otherwise. Since $p_\alpha s$ is the identity map on X_α and $f_\alpha p_\alpha\simeq f$, $fs\simeq f_\alpha$. Since X_α is uniformly movable by the first part of the proof, there exist an ANR Q, a map $g\colon X_\alpha\to Q$, and a natural transformation $\Phi\colon \Pi_{X_\alpha}\to\Pi_Q$ such that $g^{\#}\Phi[f_\alpha] = [f_\alpha]$. The natural transformation $\Phi s^{\#}$ and the map gp_α satisfy
$$(gp_\alpha)^{\#}\Phi s^{\#}[f] = p_\alpha^{\#}g^{\#}\Phi[f_\alpha] = p_\alpha^{\#}[f_\alpha] = [f]\,.$$
Hence X is uniformly movable.

LEMMA 5. *If a compact connected abelian group A is uniformly movable, then $(A, 0)$ is uniformly movable, where 0 is the zero element of A.*

Proof. Recall the well-known fact that there is an inverse system $\{T_\alpha, p_{\alpha\alpha'}, \mathscr{A}\}$ of tori T_α (a finite product of circle groups) and continuous homomorphisms $p_{\alpha\alpha'}$ associated with A. For the projection $p_\alpha\colon A\to T_\alpha$ Lemma 2 (unbased) implies that there exist $\alpha'\in\mathscr{A}$ and a natural transformation $\Phi\colon \Pi_A\to\Pi_{T_{\alpha'}}$ such that $\Phi[p_\alpha] = [p_{\alpha\alpha'}]$. Then by Theorem 1.2 of [9] there is a unique continuous homomorphism $\varphi\colon T_{\alpha'}\to A$ such that $\varphi^{\#} = \Phi$. But since homomorphisms preserve 0, Lemma 2 implies the definition of pointed uniform movability also holds. Hence $(A, 0)$ is uniformly movable.

THEOREM 5. *The only uniformly movable compact connected abelian group X with $\underline{\pi}_1(X) = 0$ is 0.*

Proof. Using the toroidal inverse limit representation for X we have $\underline{\pi}_n(X) = 0$ for all $n>1$ (since all the higher homotopy groups of any torus vanish). Thus with our hypothesis that $\underline{\pi}_1(X) = 0$ we have $\underline{\pi}_n(X) = 0$ for all n. Now Lemma 5 allows us to apply Theorem 1 to get that X has the shape of a point. Then [9, Corollary 1.3] implies X is a point.

COROLLARY 2. *If X is a uniformly movable torus-like continuum with $\underline{\pi}_1(X) = 0$, then X has the shape of a point.*

Proof. A torus-like continuum X has the same shape as the compact connected abelian group $A = \operatorname{char} H^1(X)$ (see [8]). So A is uniformly movable and $\underline{\pi}_1(A) = 0$. Therefore the theorem applies and A has the shape of a point.

EXAMPLE OF A MOVABLE CONTINUUM NOT UNIFORMLY MOVABLE. The following example is based on the work of Keesling [11] who has shown how such examples can be obtained from certain algebraic data. Because no such examples have actually been given, we give one for the sake of definiteness and make no claim of originality. According to Keesling one starts with a discrete abelian group having certain properties. Such a group was suggested to us by R. J. Nunke who also outlined the proofs for the algebra.

Let $G = \boldsymbol{Z}^{\boldsymbol{R}}$ ($\boldsymbol{Z}$ = the integers, $\boldsymbol{R}$ = the reals) be the group consisting of all functions $g\colon \boldsymbol{R}\to\boldsymbol{Z}$ with coordinatewise addition, and let H be the subgroup consisting of all $g\in G$ such that $g(r) = 0$ for all but countably many $r\in\boldsymbol{R}$. The desired discrete abelian group is G/H, which is obviously a nontrivial group. Let $\eta\colon G\to G/H$ be the natural homomorphism.

The two algebraic properties to be verified for G/H are: (1) any countable subgroup is free and (2) the only homomorphism of the group into $\boldsymbol{Z}$ is zero.

Proof of (1). If Γ is a countable subgroup of G/H then there is a countable subgroup C of G such that $\eta(C) = \Gamma$. Since $C\cap H$ is countable, there is a countable set J such that any two elements g_1, g_2 of C which differ on some element of $\boldsymbol{R}-J$ have distinct images under η. (In fact, J may be taken to be the set of all $r\in\boldsymbol{R}$ such that $g(r)\neq 0$ for some $g\in C\cap H$.) If $g\in G$, let $g^0\in G$ be the function which agrees with g on $\boldsymbol{R}-J$ and which is 0 elsewhere. Let C_0 be the countable subgroup of G consisting of all g^0 where $g\in C$. Since $\eta(g) = \eta(g^0)$ for every $g\in C$, $\eta(C_0) = \Gamma$, and by the choice of J it follows that the kernel of $\eta|C_0$ is zero. Hence $C_0\cong\Gamma$.

Since C_0 is countable, there is a countable subset $L\subset\boldsymbol{R}$ such that any two distinct elements of C_0 disagree on some member of L. Restriction of functions defines a homomorphism $\theta\colon G\to\boldsymbol{Z}^L$ which is therefore injective on C_0; hence $C_0\cong\theta(C_0)$. By Theorem 19.2 of [4] $\theta(C_0)$ is free; hence Γ is free.

Proof of (2). Let S be the subgroup consisting of all $g\in G$ for which $g(r) = 0$ for all but finitely many $r\in\boldsymbol{R}$. Since $\boldsymbol{Z}$ is slender by Proposition 94.2 of [5], and since $\boldsymbol{R}$ has nonmeasurable cardinal by Theorem 12.5 of [6], the only homomorphism $G\to\boldsymbol{Z}$ which maps S to 0 is the zero homomorphism by Theorem 94.4 of [5]. If $\varphi\colon G/H\to\boldsymbol{Z}$ is a homomorphism, then $\varphi\eta\colon G\to\boldsymbol{Z}$ maps H and therefore S to 0, hence $\varphi\eta = 0$ and thus $\varphi = 0$.

Keesling first claimed the existence of an abelian group satisfying properties (1) and (2) in [11, Proposition 3.1]. Then by Theorem 3.3 of [11] the character group of such a group is a non-trivial compact connected abelian group which is movable and $\underline{\pi}_1 = 0$ and by Theorem 5 is not uniformly movable.

References

[1] K. Borsuk, *Some remarks concerning the shape of pointed compacta*, Fund. Math. 67 (1970), pp. 221–240.

[2] — *A note on the theory of shape of compacta*, Fund. Math. 67 (1970), pp. 265–278.

[3] — *On movable compacta*, Fund. Math. 66 (1969), pp. 137–146.

[4] L. Fuchs, *Infinite Abelian Groups*, I, New York 1970.

[5] — *Infinite Abelian Groups*, II, New York 1973.

[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Princeton 1960.

[7] S. T. Hu, *Theory of Retracts*, Detroit 1965.

[8] J. Keesling, *On the shape of torus-like continua and compact connected topological groups*, Proc. Amer. Math. Soc. 40 (1973), pp. 297–302.

[9] — *Shape theory and compact connected abelian topological groups*, Trans. Amer. Math. Soc. 194 (1974), pp. 349–358.

[10] — *An algebraic property of the Čech cohomology groups which prevents local connectivity and movability*, Trans. Amer. Math. Soc. 190 (1974), pp. 151–162.

[11] — *The Čech homology of compact connected abelian topological groups with application to shape theory*, Lecture Notes in Math. 438, Berlin 1975, pp. 325–331.

[12] G. Kozlowski and J. Segal, *Locally well-behaved paracompacta in shape theory* (to appear).

[13] C. N. Lee and F. Raymond, *Čech extensions of contravariant functors*, Trans. Amer. Math. Soc. 133 (1968), pp. 415–434.

[14] S. Mardešić and J. Segal, *Movable compacta and* ANR-*systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 649–654.

[15] — — *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

[16] M. Moszyńska, *Various approaches to the fundamental groups*, Fund. Math. 78 (1973), pp. 107–118.

[17] — *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.

[18] E. H. Spanier, *Algebraic Topology*, New York 1966.

THE UNIVERSITY OF WASHINGTON
Seattle, Washington

Accepté par la Rédaction le 28. 10. 1974