The partition Q defined in Example 1 now becomes an α^+ partition. An argument similar to the one used in Example 1 now shows that Q does not admit a selector of multiplicative class α. Thus, an α^+ partition need not, in general, admit a selector of multiplicative class α.

EXAMPLE 8. Here is an example of an analytic partition of a Polish space which does not admit an analytic selector. The example is related to Sierpiński's example of a planar Borel set which cannot be uniformized by an analytic set ([9], p. 138).

Let f be a continuous function on Σ onto an analytic non-Borel subset Y of $[0, 1]$. Denote by Q the partition of Σ induced by f, i.e., $Q = \{f^{-1}(\{y\}): y \in Y\}$. As is easily checked, Q is an analytic partition. Suppose, by way of contradiction, that S is an analytic selector for Q. Define a function $g: \Sigma \to \Sigma$ by: $g(\sigma)$ = the unique element of $S \cap f^{-1}(\{f(\sigma)\})$. We now verify that g is Borel measurable. First, note that, for any subset E of Σ, $g^{-1}(E) = f^{-1}(f(E \cap S))$. Hence, if E is any Borel subset of Σ, then $g^{-1}(E)$ is analytic. So, in particular, both $g^{-1}(E)$ and $g^{-1}(\Sigma - E)$ are analytic, whenever E is a Borel subset of Σ. It now follows by a well known result of Souslin ([5], p. 395) that $g^{-1}(E)$ is a Borel subset of Σ, whenever E is a Borel subset of Σ. Thus, g is Borel measurable. An immediate consequence of this is that S is a Borel subset of Σ, for $S = \{\sigma \in \Sigma: g(\sigma) = \sigma\}$. Now the restriction of f to S is one-one and $f(S) = Y$. Hence, Y is a Borel subset of $[0, 1]$ ([5], p. 397), which contradicts our assumption that Y is non-Borel.

We therefore conclude that Corollary 4.4 is the best possible result concerning selectors for analytic partitions.

References

[1] A. V. Arhangel'skiĭ, *Open and near open mappings. Connections between spaces*, Trudy Moskov. Mat. Obsc. 15 (1966), pp. 181–223 (in Russian), Trans. Moscow Math. Soc., pp. 204–250 (English translation).

[2] N. Bourbaki, *General Topology*, Part 2, Paris 1966.

[3] R. Engelking, R. W. Heath and E. Michael, *Topological well-ordering and continuous selections*, Inventiones Math. 6 (1968), pp. 150–158.

[4] F. Hausdorff, *Set Theory*, New York 1957.

[5] K. Kuratowski, *Topologie*, Vol. 1, Warszawa 1958.

[6] — *On set-valued B-measurable mappings and a theorem of Hausdorff*, Theory of Sets and Topology (pp. 355–362), Berlin 1972.

[7] — and A. Maitra, *Some theorems on selectors and their applications to semi-continuous decompositions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), pp. 877–881.

[8] S. Mazurkiewicz, *Sur une propriété des ensembles C(A)*, Fund. Math. 10 (1927), pp. 172–174.

[9] W. Sierpiński, *Sur l'uniformisation des ensembles measurables* (B), Fund. Math. 16 (1930), pp. 136–139.

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Remarks on Cartesian products

by

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Abstract. In this paper we consider Cartesian products of topological spaces. Using the method of A. M. Gleason, we give simple proofs of a few well-known theorems and some their strengthenings. We give an answer to a question of R. Engelking and M. Karłowicz and prove that the defined by A. H. Stone closed subsets of $N^{\aleph_1}$ cannot be separated by G_δ-sets. We also show that the realcompact G_δ-subspaces of $N^{\mathfrak{m}}$ are homeomorphic with $N^{\mathfrak{m}}$ for $\mathfrak{m} > \aleph_0$.

This paper is devoted to some problems connected with Cartesian products of topological spaces. Using the method of A. M. Gleason, we give simple proofs of a few important and well-known theorems about products as well as their strengthenings (Section 2 and 3). In particular, we give an answer to a question raised by R. Engelking and M. Karłowicz [11]. Next, on the grounds of the results of Sections 2 and 3 and the well-known facts about Σ-products, we discuss properties of the products of natural numbers $N^{\mathfrak{m}}$ (Section 4), of the functionally closed subsets of products of metrizable spaces (Section 5) and of the realcompact subspaces of products of first-countable spaces (Section 6). We show, among other things, that the closed subsets of $N^{\aleph_1}$ defined by A. H. Stone cannot be separated by G_δ-sets and that the realcompact G_δ-subspaces of the space $N^{\mathfrak{m}}$ are homeomorphic with $N^{\mathfrak{m}}$ (for $\mathfrak{m} > \aleph_0$).

1. Notation and terminology. We adopt the notation and terminology of [10]. In particular, $w(X)$ denotes the weight, $\chi(X)$ the character, $d(X)$ the density and $c(X)$ the Souslin number of a topological space X. The Cartesian product of a family of sets $\{X_s\}_{s \in S}$ is denoted by $\underset{s\in S}{\boldsymbol{P}} X_s$; for $T \subset S$, the symbol $p_T: \underset{s\in S}{\boldsymbol{P}} X_s \to \underset{s\in T}{\boldsymbol{P}} X_s$ denotes the projection. By a cube in the product $\underset{s\in T}{\boldsymbol{P}} X_s$ we mean a subset of the form $K = \underset{s\in S}{\boldsymbol{P}} K_s$, where $K_s \subset X_s$, for $s \in S$. The set K_s will be called the *s-th face* of the cube K and the set $D(K) = \{s \in S: K_s \neq X_s\}$ will be called the set of its distinguished indices. If $\overline{\overline{D(K)}} \leqslant \mathfrak{m}$, we shall say that K is an $\mathfrak{m}$-*cube*. Besides the Tychonoff topology in the product $\underset{s\in S}{\boldsymbol{P}} X_s = X$ (the space X with this topology will be denoted simply by $\underset{s\in S}{\boldsymbol{P}} X_s$) we shall consider the $\mathfrak{m}$-*box topology*, generated by a base which consists of all cubes with open faces having the set of distinguished indices of cardinality less than $\mathfrak{m}$. The set $\underset{s\in S}{\boldsymbol{P}} X_s$ with the $\mathfrak{m}$-box topology will be denoted by $(\underset{s\in S}{\boldsymbol{P}} X_s)_{\mathfrak{m}}$. For a topological space X the $\mathfrak{m}$-modification of X is the topological space with the same underlying set and the topology consisting of the unions of intersections of fewer than $\mathfrak{m}$ open subsets of X. A subset of topological space X which is an intersection of $\mathfrak{m}\geqslant\aleph_0$ open sets will be called a $G_\delta^{\mathfrak{m}}$-set and a union of an arbitrary number of $G_\delta^{\mathfrak{m}}$-sets will be called a $G_{\delta\Sigma}^{\mathfrak{m}}$-set. The *$\Sigma$-product* of a family of topological spaces $\{X_s\}_{s\in S}$ with the base point $x=(x_s)\in\underset{s\in S}{\boldsymbol{P}}X_s$ is the subspace

$$\Sigma(x)=\{y=(y_s)\colon \overline{\overline{\{s\in S\colon x_s\neq y_s\}}}\leqslant\aleph_0\}$$

of the product $\underset{s\in S}{\boldsymbol{P}}X_s$. The symbol $f\colon X\to Y$ and the word *mapping* always denote a continuous function; N denotes the natural numbers, R — the reals, I — the unit interval and $D(\mathfrak{m})$ the discrete space of cardinality $\mathfrak{m}$. For a family of sets $\mathscr{A}$ the union of $\mathscr{A}$ will be denoted by $\bigcup\mathscr{A}$. By a cardinal number $\mathfrak{m}$ we always mean an infinite cardinal and $\mathfrak{m}^+$ denotes the successor of $\mathfrak{m}$.

2. General Theorems. In the sequel, the following theorem will be of primary importance:

THEOREM 1. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces with $w(X_s)\leqslant\mathfrak{w}$ and $\mathscr{A}$ a family of $\mathfrak{m}$-cubes in $\underset{s\in S}{\boldsymbol{P}}X_s$. Then*

(i) *there exists a subfamily $\mathscr{B}\subset\mathscr{A}$ of cardinality $\overline{\overline{\mathscr{B}}}\leqslant\mathfrak{m}\cdot\mathfrak{w}$ such that the union $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$,*

(ii) *for any cardinal number $\mathfrak{n}$ there exists a subfamily $\mathscr{B}\subset\mathscr{A}$ of cardinality $\overline{\overline{\mathscr{B}}}\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ such that the union $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$ with respect to the $\mathfrak{n}^+$-box topology.*

Proof. (ii) Let λ be the initial ordinal of cardinality $\mathfrak{n}^+$. Using transfinite induction, we shall define two sequences:

(1) $\emptyset\neq S_0\subset S_1\subset\ldots\subset S_\xi\subset\ldots\subset S$, where $\overline{\overline{S}}_\xi\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$, for $\xi<\lambda$,

(2) $\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_\xi,\ldots\subset\mathscr{A}$, where $\overline{\overline{\mathscr{A}}}_\xi<(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$, for $\xi<\lambda$,

such that

(3) $S_\xi=\bigcup\{D(K)\colon K\in\bigcup_{\alpha<\xi}\mathscr{A}_\alpha\}\cup\bigcup_{\alpha<\xi}S_\alpha$ for $1\leqslant\xi<\lambda$,

(4) $p_{S_\xi}(\bigcup\mathscr{A}_\xi)$ is dense in $p_{S_\xi}(\bigcup\mathscr{A})$ in the $\mathfrak{n}^+$-box topology (in $(\underset{s\in S_\xi}{\boldsymbol{P}}X_s)_{\mathfrak{n}^+}$) for $\xi\geqslant1$.

For $\xi=0$ take $S_0=\{s\}$, where $s\in S$, and $\mathscr{A}_0=\emptyset$. Suppose that $\mathscr{A}_\alpha$ and S_α are defined for $\alpha<\xi\geqslant1$. The set S_ξ defined in (3) has cardinality $\overline{\overline{S}}_\xi\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$. It is easy to see that $w((\underset{s\in S_\xi}{\boldsymbol{P}}X_s)_{\mathfrak{n}^+})\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$, so that we can choose a subset $A\subset\bigcup\mathscr{A}$ of cardinality $\overline{\overline{A}}\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ such that $p_{S_\xi}(A)$ is dense with respect to the $\mathfrak{n}^+$-box topology in $p_{S_\xi}(\bigcup\mathscr{A})$. Now, we finish the construction by taking $\mathscr{A}_\xi\subset\mathscr{A}$ such that $A\subset\bigcup\mathscr{A}_\xi$ and $\overline{\overline{\mathscr{A}}}_\xi\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$.

We claim that

(5) for $\mathscr{B}=\bigcup_{\xi<\lambda}\mathscr{A}_\xi$ the union $\bigcup\mathscr{B}$ is dense with respect to the $\mathfrak{n}^+$-box topology in $\bigcup\mathscr{A}$.

Let $x\in\bigcup\mathscr{A}$ and $U=\underset{s\in S}{\boldsymbol{P}}U_s$ be an $\mathfrak{n}$-cube with open faces and $x\in U$. Let $T=\bigcup_{\xi<\lambda}S_\xi$. Since $\overline{\overline{D(U)\cap T}}\leqslant\mathfrak{n}$, there exists a $\xi<\lambda$ such that $D(U)\cap T\subset S_\xi$. By (4) there exists a $y'=(y'_s)\in\underset{s\in S}{\boldsymbol{P}}X_s$ such that $y'\in\bigcup\mathscr{A}_\xi$ and $y'_s\in U_s$ for $s\in T$. Now, let K be a cube such that $y'\in K\in\mathscr{A}_\xi$. Take $y=(y_s)\in\underset{s\in S}{\boldsymbol{P}}X_s$ with $y_s=y'_s$ for $s\in T$ and $y_s=x_s$ for $s\in S\setminus T$. Since, by (3), $D(K)\subset S_{\xi+1}\subset T$, we have $y\in K$ and thus $y\in K\cap U$, which proves (5). This concludes the proof, as $\overline{\overline{\mathscr{B}}}\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}\cdot\mathfrak{n}^+=(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$.

(i) The proof is similar to although simpler than the proof of (ii). In analogy to (1) and (2) one defines two sequences S_n, $\mathscr{A}_n$ for $n<\omega_0$ such that $\overline{\overline{S}}_n\leqslant\mathfrak{m}\cdot\mathfrak{w}$, $\overline{\overline{\mathscr{A}}}_n\leqslant\mathfrak{m}\cdot\mathfrak{w}$ and (3) and (4) are fulfilled, where instead of the box topology the Tychonoff topology is considered. Then (5) defines the required family $\mathscr{B}$.

COROLLARY 1. *Let $\{X_s\}_{s\in S}$ and $\mathscr{A}$ be as in Theorem 1. Then for an arbitrary, number $\mathfrak{n}$ there exists a subfamily $\mathscr{B}\subset\mathscr{A}$ of cardinality $\overline{\overline{\mathscr{B}}}\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ such that the union $\bigcup\mathscr{B}$ is dense in $\bigcup\mathscr{A}$ with respect to the topology of the $\mathfrak{n}^+$-modification of the product $\underset{s\in S}{\boldsymbol{P}}X_s$.*

Proof. Let X'_s be the $\mathfrak{n}^+$-modification of X_s. Then $w(X'_s)\leqslant\mathfrak{w}^{\mathfrak{n}}$. By Theorem 1(ii) let us choose a family $\mathscr{B}$ of cardinality $\overline{\overline{\mathscr{B}}}\leqslant(\mathfrak{m}\cdot\mathfrak{w}^{\mathfrak{n}})^{\mathfrak{n}}=(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ dense with respect to the $\mathfrak{n}^+$-box topology in $\bigcup\mathscr{A}\subset\underset{s\in S}{\boldsymbol{P}}X'_s$ and observe that the $\mathfrak{n}^+$-box topology in $\underset{s\in S}{\boldsymbol{P}}X'_s$ coincides with the $\mathfrak{n}^+$-modification of the product $\underset{s\in S}{\boldsymbol{P}}X_s$.

Theorem 1 is closely related to the results of Engelking and Karłowicz [7], [11]. From the theorems in those papers one can derive particular cases of Theorem 1; it seems to us that applying the methods used in those papers one could also obtain the full strength of Theorem 1. However, the method applied here, which follows an idea of A. Gleason (see [13], the proof of Theorem 19, VII, or [17], the proof of Theorem 2, § 41, IX), seems simpler.

Remark 1. W. Comfort and S. Negrepontis [4] proved (Theorem 2.3) that *if $\mathfrak{n}<\mathfrak{m}$ and $\mathfrak{m}$ is regular and strongly $\mathfrak{n}$-inaccesible (i.e., if $\mathfrak{k}<\mathfrak{m}$ and $\mathfrak{l}<\mathfrak{n}$ then $\mathfrak{k}^{\mathfrak{l}}<\mathfrak{m}$) then for a family $\{X_s\}_{s\in S}$ of topological spaces such that $d(X_s)<\mathfrak{m}$ the Souslin number $c((\underset{s\in S}{\boldsymbol{P}}X_s)_{\mathfrak{n}})\leqslant\mathfrak{m}$.*

This theorem is a simple consequence of the following theorem:

If $\mathfrak{n}<\mathfrak{m}$ are cardinals such that $\mathfrak{m}$ is regular and strongly $\mathfrak{n}$-inaccesible, then for an arbitrary family $\mathscr{A}$ of cubes K with $\overline{\overline{D(K)}}<\mathfrak{n}$ in the product $\underset{s\in S}{\boldsymbol{P}}X_s$ of spaces of weight $w(X_s)<\mathfrak{m}$ there exists a subfamily $\mathscr{B}\subset\mathscr{A}$ of cardinality $\overline{\overline{\mathscr{B}}}\leqslant\mathfrak{m}$ such that $\bigcup\mathscr{B}$ is dense with respect to the $\mathfrak{n}$-box topology in $\bigcup\mathscr{A}$ (if, moreover, $\mathfrak{n}$ is regular, then we can have $\overline{\overline{\mathscr{B}}}<\mathfrak{m}$ (cf. [4], Corollary 2.4)).

This theorem can be proved in the same way as Theorem 1(ii). Such a proof is much simpler than the one given by Comfort and Negrepontis, who used the Erdös–Rado theorem (see also [18], Theorem 1; [7], Theorem 6; [14], 4.7; [19], XII, 3.1).

COROLLARY 2. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces with $\chi(X_s)\leqslant\mathfrak{m}$. Then the closure $\bar{A}$ of a subset $A\subset\mathop{\boldsymbol{P}}_{s\in S}X_s$ which is the union of $\mathfrak{m}$-cubes is also the union of $\mathfrak{m}$-cubes.*

Proof. Let $x=(x_s)\in\bar{A}$. Let X_s' be a topological space defined on the set X_s by assuming as the topology the family consisting of all neighbourhoods of the point x_s (in the topology of X_s) and of the empty set. Thus $w(X_s')\leqslant\mathfrak{m}$. Let $A=\bigcup\mathscr{A}$, where $\mathscr{A}$ is a family of $\mathfrak{m}$-cubes in $X=\mathop{\boldsymbol{P}}_{s\in S}X_s$ and thus also in $X'=\mathop{\boldsymbol{P}}_{s\in S}X_s'$. From Theorem 1(i) follows the existence of a family $\mathscr{B}\subset\mathscr{A}$ of cardinality $\leqslant\mathfrak{m}$ the union $B=\bigcup\mathscr{B}$ of which is dense with respect to the topology of X' in the set A. Let us assume $T=\bigcup\{D(K)\colon K\in\mathscr{B}\}$. We thus have $\bar{\bar{T}}\leqslant\mathfrak{m}$, $p_T(x)\in\overline{p_T(B)}$ and $p_T^{-1}p_T(B)=B\subset A$. Since the projection is open, we obtain $p_T^{-1}p_T(x)\subset\overline{p_T^{-1}p_T(B)}\subset\bar{A}$, which completes the proof.

The following lemma will be useful in the sequel.

LEMMA 1. *Let $\{X_s\}_{s\in S}$ be a family of arbitrary topological spaces and let $A\subset\mathop{\boldsymbol{P}}_{s\in S}X_s$ be a union of $\mathfrak{m}$-cubes. If $F_1\subset F_2\subset\ldots$ is a sequence of closed subsets of the product $\mathop{\boldsymbol{P}}_{s\in S}X_s$ such that $A\subset\bigcup_i F_i$, then for $F_i^*=\bigcup\{K\subset F_i\colon K$ is an $\mathfrak{m}$-cube$\}$ we have $A\subset\bigcup_i\overline{F_i^*}$.*

Proof. Let us assume, on the contrary, that the set $A\setminus\bigcup_i\overline{F_i^*}$ is nonempty. By induction we define a sequence of $\mathfrak{m}$-cubes $K_0\supset K_1\supset\ldots$, whose faces with distinguished indices are one-point sets such that

$$K_0\subset A\setminus\bigcup_i\overline{F_i^*}\quad\text{and}\quad K_i\cap F_i=\emptyset\quad\text{for}\quad i=1,2,\ldots$$

The existence of K_0 follows from the simple observation that $A\setminus\bigcup_i\overline{F_i^*}$, as the intersection of a union of $\mathfrak{m}$-cubes with a G_δ-set (and thus with the union of $\aleph_0$-cubes), is the union of $\mathfrak{m}$-cubes. Let us assume that the cubes K_j are already defined for $j<i\geqslant 1$. We have $K_{i-1}\setminus F_i\neq\emptyset$, or else we would have $K_{i-1}\subset F_i^*$, contrary to our choice. Thus the set $K_{i-1}\setminus F_i$ is a nonempty union of $\mathfrak{m}$-cubes and we can pick $K_i\subset K_{i-1}\setminus F_i$ satisfying our assumption. One can easily observe that the decreasing sequence of cubes $K_0\supset K_1\supset\ldots$ such that their faces with distinguished indices are one-point sets has a nonempty intersection. We have obtained

$$\emptyset\neq\bigcap_i K_i\subset A\setminus\bigcup_i F_i=\emptyset,$$

which is a contradiction.

Note that Lemma 1 can also be derived from the fact that in the product of discrete spaces the Baire Theorem holds (see [10], Exercise 3.9.C(a)).

3. Corollaries. From Theorem 1 one can easily derive a number of theorems, which have been demonstrated by R. Engelking and M. Karłowicz in [7] and [11] by using other methods. Namely: Theorem 6 of [11] (by Corollary 1, where $\mathfrak{m}=\mathfrak{n}$), Theorem 7 of [11] and Theorem 3 of [7] (by Theorem 1(i), where $\mathfrak{m}=\mathfrak{w}$), Theorem 5 of [7] (1) (by Theorem 1(i)) and also Theorem 1 of [7] (by the Hewitt–Marczewski–Pondiczery Theorem and Theorem 1(i)). The derivation of the following corollary is less obvious.

COROLLARY 3 ([11], Theorem 4). *Let $\{X_s\}_{s\in S}$ be an arbitrary family of topological spaces such that $w(X_s)\leqslant\mathfrak{w}$ for $s\in S$ and let $U,V\subset\mathop{\boldsymbol{P}}_{s\in S}X_s$ be $G_{\delta\Sigma}^{\mathfrak{n}}$- and $G_{\delta\Sigma}^{\mathfrak{m}}$-sets respectively with $\mathfrak{n}\leqslant\mathfrak{m}$. If $U\cap V=\emptyset$, then there exists a set $T\subset S$ such that $\bar{\bar{T}}\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ and $p_T(U)\cap p_T(V)=\emptyset$.*

Proof. For $T\subset S$ and $A\subset\mathop{\boldsymbol{P}}_{s\in T}X_s$ let $\bar{A}$ denote the closure of the set A in the $\mathfrak{n}^+$-modification of the product $\mathop{\boldsymbol{P}}_{s\in T}X_s$. The set V, as a $G_{\delta\Sigma}^{\mathfrak{m}}$-set, is the union of a family $\mathscr{A}$ of $\mathfrak{m}$-cubes. From Corollary 1 we see that there exists a subfamily $\mathscr{B}\subset\mathscr{A}$ of cardinality $\leqslant(\mathfrak{m}\cdot\mathfrak{w})^{\mathfrak{n}}$ such that $\overline{\bigcup\mathscr{B}}\supset\bigcup\mathscr{A}=V$. Let $T=\bigcup\{D(K)\colon K\in\mathscr{B}\}$; thus $\bar{\bar{T}}\leqslant(m\cdot\mathfrak{w})^{\mathfrak{n}}$ and for $B=\bigcup\mathscr{B}$ we have $p_T^{-1}p_T(B)=B\subset V$. Hence $p_T(U)\cap p_T(B)=\emptyset$, and since $p_T(U)$ is open in the $\mathfrak{n}^+$ modification of $\mathop{\boldsymbol{P}}_{s\in S}X_s$, also $\overline{p_T(U)\cap p_T(B)}=\emptyset$. Thus we obtain $p_T(U)\cap p_T(V)\subset p_T(U)\cap p_T(\bar{B})\subset p_T(U)\cap\overline{p_T(B)}=\emptyset$, which completes the proof.

The next theorem is a strengthening of Theorem 5 in [11] and (for $\mathfrak{m}=\mathfrak{w}=\aleph_0$) is an answer to a question raised by R. Engelking and M. Karłowicz ([11], Remark, p. 282).

THEOREM 2. *Let $\{X_s\}_{s\in S}$ be a family of topological spaces with $w(X_s)\leqslant\mathfrak{w}$ and let $U,V\subset\mathop{\boldsymbol{P}}_{s\in S}X_s$ be disjoint a G_δ-set and a $G_{\delta\Sigma}^{\mathfrak{m}}$-set, respectively. Then there exists a $T\subset S$ of cardinality $\bar{\bar{T}}\leqslant\mathfrak{m}\cdot\mathfrak{w}$ such that $p_T(U)\cap p_T(V)=\emptyset$.*

Proof. Let $F_1\subset F_2\subset\ldots$ be a sequence of closed sets such that $X\setminus U=\bigcup_i F_i$. Let $F_i^*=\bigcup\{K\subset F_i\colon K$ is an $\mathfrak{m}$-cube$\}$. As $V\subset\bigcup_i F_i$ and V is a union of $\mathfrak{m}$-cubes, by Lemma 1 we infer that $V\subset\bigcup_i\overline{F_i^*}$. For each $i=1,2,\ldots$ there exists by Theorem 1, (i) a family of $\mathfrak{m}$-cubes $\mathscr{F}_i$ such that $\bar{\bar{\mathscr{F}}}_i\leqslant\mathfrak{m}\cdot\mathfrak{w}$ and $\overline{\bigcup\mathscr{F}_i}=\overline{F_i^*}$. Let

$$T=\bigcup\{D(K)\colon K\in\mathscr{F}_i,\ i=1,2,\ldots\}.$$

Then for each i we have $p_T^{-1}p_T(\overline{F_i^*})=\overline{F_i^*}$ and $\bar{\bar{T}}\leqslant\mathfrak{m}\cdot\mathfrak{w}$ (see also [7], Theorem 5). We obtain $p_T(U)\cap p_T(V)\subset p_T(U)\cap p_T(\bigcup_i\overline{F_i^*})=p_T(U)\cap\bigcup_i p_T(\overline{F_i^*})=\emptyset$, since for $i=1,2,\ldots$ we have $U\cap p_T^{-1}p_T(\overline{F_i^*})=U\cap\overline{F_i^*}=\emptyset$.

Remark 2. *Let G be a G_δ-set in the product $\mathop{\boldsymbol{P}}_{s\in S}X_s=X$ with $w(X_s)\leqslant\mathfrak{m}$. Then*

(1) The proof given in [7] contains a gap, but the theorem itself can easily be derived from Theorems 7 and 1 of [7].

there exists a set $T_0 \subset S$ of cardinality $\leqslant \mathfrak{m}$ such that for each $T \supset T_0$ of cardinality $\leqslant \mathfrak{m}$ the projection $p_T(G)$ is a G_δ-set in the space $\boldsymbol{P}_{s\in S} X_s$.

Proof. Let $X \setminus G = \bigcup_i F_i$, where $F_1 \subset F_2 \subset \ldots$ are closed sets. Put

$$F_i^* = \bigcup \{K \subset F_i:\ K \text{ is an } \mathfrak{m}\text{-cube}\}.$$

By Theorem 1(i) (see also [7], Theorem 5) we infer that the sets F_i^* are closed and for some $T_0 \subset S$ of cardinality $\leqslant \mathfrak{m}$ we have $p_{T_0}^{-1} p_{T_0}(F_i^*) = F_i^*$. Let $T \supset T_0$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$. We shall show that for the closed sets $p_T(F_i^*) = K_i$ we have

$$\boldsymbol{P}_{s\in T} X_s \setminus p_T(G) = \bigcup_i K_i.$$

Since each $p_T^{-1}(K_i) = F_i^*$ is disjoint with G, the inclusion $\supset$ holds. Let $x \notin p_T(G)$. Then $p_T^{-1}(x)$ is an $\mathfrak{m}$-cube contained in $\bigcup_i F_i$; hence, by Lemma 1, we have $p_T^{-1}(x) \subset \bigcup_i F_i^*$, and thus $x \in \bigcup_i K_i$, which completes the proof.

From the above Remark and Theorem 2 it easily follows that

every two disjoint G_δ-sets in the product $\boldsymbol{P}_{s\in S} X_s$ with $w(X_s) \leqslant \mathfrak{m}$ can be separated by G_δ-sets depending on $\mathfrak{m}$ coordinates (²).

Remark 3. By a little modification of an argument of R. Engelking ([7], the proof of Theorem 8; see also [6], the proof of Theorem 4) one can give on the strength of Theorem 1(i) (the proof of which is easy) a simple proof of the following

EFIMOV THEOREM. *If $f: \boldsymbol{P}_{s\in S} X_s \to X$ maps the product of compact spaces of weight $\leqslant \mathfrak{m}$ onto a Hausdorff space X containing a dense subset X_0 such that $\chi(x, X) \leqslant \mathfrak{m}$ for each $x \in X_0$, then $w(X) \leqslant \mathfrak{m}$.*

Proof. Choose for each $x \in X_0$ an $\mathfrak{m}$-cube $K(x) \subset f^{-1}(x)$. By Theorem 1(i) there exists a family $\mathscr{B} \subset \mathscr{A} = \{K(x):\ x \in X_0\}$ of cardinality $\leqslant \mathfrak{m}$ such that $\overline{\bigcup \mathscr{B}} = \overline{\bigcup \mathscr{A}}$. Let $T = \bigcup \{D(K):\ K \in \mathscr{B}\}$. Put $X_s' = X_s$ for $s \in T$ and $X_s' = \{a_s\}$ for $s \notin T$, where $a_s \in X_s$. Since the weight of $X' = \boldsymbol{P}_{s\in S} X_s'$ is not greater than $\mathfrak{m}$, and for $\mathscr{B}' = \{K \cap X':\ K \in \mathscr{B}\}$ we have

$$f(\overline{\bigcup \mathscr{B}'}) = \overline{f(\bigcup \mathscr{B}')} = \overline{f(\bigcup \mathscr{B})} = f(\overline{\bigcup \mathscr{B}}) = f(\overline{\bigcup \mathscr{A}}) = \overline{X}_0 = X,$$

we obtain $w(X) \leqslant w(X') \leqslant \mathfrak{m}$.

4. Remarks on $N^{\mathfrak{m}}$. The following theorem is an answer to a question communicated to the authors by K. Alster.

(²) A subset $A \subset \boldsymbol{P}_{s\in S} X^s$ depends on $\mathfrak{m}$ coordinates if $A = p_T^{-1} p_T(A)$ for some subset $T \subset S$ of cardinality $\leqslant \mathfrak{m}$.

PROPOSITION 1. *A. H. Stone's sets* (³) *F_0 and F_1 in $N^{\aleph_1}$ cannot be separated by disjoint G_δ-sets.*

Proof. In the opposite case, by Theorem 2 (for $\mathfrak{m} = \mathfrak{w} = \aleph_0$), the projections of F_0 and F_1 on some countable subproduct $N^{\aleph_0}$ would be disjoint, and this is impossible.

We shall now show that the separation of closed sets in $N^{\mathfrak{m}}$ by G_δ-sets is a weaker property than the separation by open sets.

EXAMPLE 1. *There exist disjoint closed subsets A and B of $N^{\mathfrak{c}}$ (where $\mathfrak{c} = 2^{\aleph_0}$) such that A is countable and discrete and B depends on countably many coordinates (and thus is a G_δ-set) and yet A and B cannot be separated by open sets.*

Choose a discrete, countable set $A' = \{a_1, a_2, \ldots\}$ in $N^{\mathfrak{c}}$ which is not C-embedded (⁴) in $N^{\mathfrak{c}}$. The existence of such a set can be proved directly by an application of the function f defined on the interval I by R. Engelking ([9], Theorem 2, [10], Exercise 3.1.M (a)), which embeds I as a discrete closed subspace of $N^{\mathfrak{c}}$; namely one can assume $A' = f(Q)$ where Q are rational numbers in I. Let $b_n \in N^{\aleph_0}$ for $n = 0, 1, \ldots$ be distinct points such that $b_0 = \lim b_n$. In $N^{\mathfrak{c}} = N^{\mathfrak{c}} \times N^{\aleph_0}$ define $A = \{(a_n, b_n):\ n = 1, 2, \ldots\}$ and $B = N^{\mathfrak{c}} \times \{b_0\}$. By using the fact that the closure of an open subset of $N^{\mathfrak{c}}$ depends on countably many coordinates ([10], Problem 2.7.12 (a)), it is not hard to verify that A and B cannot be separated by open sets.

As noticed by K. Alster, the property of separating closed sets by G_δ-sets is weaker than subparacompactness (see [1], Remark 1.3). Thus, Proposition 1 is a strengthening of Theorem 2 of [2], which states that $N^{\aleph_1}$ is not subparacompact. The space $N^{\aleph_1}$ is not even θ-refinable (for the definition see [26]). For the proof it suffices to combine H. Corson's conclusion that the closed subspace F_0, defined above, of the space $N^{\aleph_1}$ is collectionwise normal but not paracompact ([5], Theorem 4) with the Worrell and Wicke theorem, which states that a θ-refinable and collectionwise normal space is paracompact ([26], Theorem (iii), p. 825). This fact follows also from Example 2 below (since, as is easy to verify, a subset of a θ-refinable space which is a locally F_σ-set is an F_σ-set).

EXAMPLE 2. *There exists a subset E of $N^{\aleph_1}$ which is locally an F_σ-set at each point of $N^{\aleph_1}$ (i.e., for each $x \in N^{\aleph_1}$ and for some neighbourhood V of x, the intersection $V \cap E$ is an F_σ-set in V) but is not an F_σ-set.*

Our construction is related to the proof of Theorem 2 in [2]. Let $W(\xi)$ be the set of all ordinals less than ξ. By the Mycielski Theorem [21] there exists a closed discrete subspace $X = \{x_\xi:\ \xi < \omega_1\}$ of the space $N^{W(\omega_1)} = N^{\aleph_1}$. For each $\alpha < \omega_1$ choose a finite set $F_\alpha \subset W(\omega_1)$ such that

$$p_{F_\alpha}^{-1} p_{F_\alpha}(x_\alpha) \cap X = \{x_\alpha\}.$$

(³) A. H. Stone [24] proved that two closed subsets F_0 and F_1 of $N^{\aleph_1}$, where F_i, for $i = 0, 1$, is the set of $x \in N^{\aleph_1}$ such that for any integer $n \neq i$ at most one coordinate of x is equal to n, cannot be separated by disjoint open sets.

(⁴) For this notion see for example [12].

Let

$$T_\alpha = W(\beta+1), \quad \text{where} \quad W(\beta) \supset \bigcup_{\xi<\alpha} T_\xi \cup F_\alpha \cup W(\alpha),$$

$$E_\alpha = p_{T_\alpha}^{-1} p_{T_\alpha}(x_\alpha) \quad \text{and} \quad E = \bigcup_{\alpha<\omega_1} E_\alpha .$$

For $x \in N^{\aleph_1}$ we can find an ordinal $\mu<\omega_1$ and a finite set $F \subset W(\mu_1)$ such that the neighbourhood $V = p_F^{-1} p_F(x)$ does not contain the points x_ξ for $\xi \geqslant \mu$. Thus for $\xi \geqslant \mu$ we have $F \subset T_\xi$; hence $p_F(E_\xi) = p_F(x_\xi) \neq p_F(x)$ and thus $V \cap E_\xi = \emptyset$. So $V \cap E = V \cap \bigcup_{\xi<\mu} E_\xi$ is an F_σ-set.

Suppose now that E is an F_σ-set. Then, since E is a $G_{\delta\Sigma}^{\aleph_0}$-set, by Theorem 2 ($\mathfrak{m} = \mathfrak{w} = \aleph_0$) there would exist an $\alpha<\omega_1$ such that

$$E = p_{W(\alpha)}^{-1} p_{W(\alpha)}(E).$$

For some β we have $T_{\alpha+1} = W(\beta+1)$; then $T_\alpha \subset W(\beta)$. Choose $y \in N^{\aleph_1}$ such that $p_\xi(y) = p_\xi(x_{\alpha+1})$ for $\xi \neq \beta$ and $p_\beta(y) \neq p_\beta(x_{\alpha+1})$. Take $\xi<\omega_1$. If $\xi<\alpha+1$ then $F_\xi \subset T_\xi \subset W(\beta)$ and $p_{F_\xi}(y) = p_{F_\xi}(x_{\alpha+1}) \neq p_{F_\xi}(x_\xi)$; hence $y \notin E_\xi$; if $\xi = \alpha+1$ we have $p_{T_\xi}(y) \neq p_{T_\xi}(x_\xi)$ and $y \notin E_\xi$; finally for $\xi>\alpha+1$ we have $F_{\alpha+1} \subset T_{\alpha+1} \subset T_\xi$ and $p_{F_{\alpha+1}}(y) = p_{F_{\alpha+1}}(x_{\alpha+1}) \neq p_{F_{\alpha+1}}(x_\xi)$ and so $y \notin E_\xi$. Thus $y \notin E$, but at the same time $p_{W(\alpha)}(y) = p_{W(\alpha)}(x_{\alpha+1}) \in p_{W(\alpha)}(E)$, and we have a contradiction.

5. Remarks on products of metrizable spaces. Let $X = \mathop{\boldsymbol{P}}_{s\in S} X_s$ be a product of spaces satisfying one of the following conditions:

(i) *each X_s is metrizable and separable,*

(ii) *each X_s is completely metrizable,*

(iii) *each X_s is a Lindelöf and Čech-complete space of countable tightness* ([5]).

Then for each $x \in X$ we have

(6) *the Σ-product $\Sigma(x)$ is normal and C-embedded in X.*

The first part of (6) in the case (i) was proved by A. Kombarov and V. Malyhin [16], in the case (ii) by H. Corson [5] and in the case (iii) by A. Kombarov [15]. The second part of (6) follows from a theorem of M. Ulmer ([25], Theorem 2.2) in the cases (i) and (ii), and from a theorem of R. Engelking ([8], Theorem 1) in the case (iii) (see also [10], Problems 2.7.13, 2.7.14, 4.5.12).

LEMMA 2. *Let E be a subset of the product $\mathop{\boldsymbol{P}}_{s\in S} X_s$ which is a union of $\aleph_0$-cubes and let $\Sigma = \Sigma(x)$ be a Σ-product with an arbitrary base point $x = (x_s)$. Then $E \subset \overline{E \cap \Sigma}$.*

Proof. Let $a = (a_s) \in E$ and let $U = \mathop{\boldsymbol{P}}_{s\in S} U_s$ be an open neighbourhood of a. There exists an $\aleph_0$-cube K such that $a \in K \subset E$. Take $T = D(K) \cup D(U)$; hence $\overline{\overline{T}} \leqslant \aleph_0$. For $y = (y_s)$ such that $y_s = a_s$ if $s \in T$ and $y_s = x_s$ if $s \notin T$ we have $y \in U \cap K \cap \Sigma$ which completes the proof.

([5]) The *tightness* of a space X is countable if for any set $A \subset X$ and a point $x \in \overline{A}$ there exists a countable set $A' \subset A$ with $x \in \overline{A'}$ (see also [10], Problem 1.7.13).

PROPOSITION 2. *Let $X = \mathop{\boldsymbol{P}}_{s\in S} X_s$ satisfies the condition (6). Then*

(i) *each subset A of X which is the closure of a union of $\aleph_0$-cubes is C-embedded in X*

(ii) *each closed G_δ-set A in X is functionally closed.*

Proof. (i) Let $f\colon A \to R$ be an arbitrary mapping. By the normality of Σ, there exists an extension $g\colon \Sigma \to R$ of the restriction $f|A \cap \Sigma$. By (6) we can extend g over the whole of the space X; denote this extension by $\bar{f}$. We have $\bar{f}|A \cap \Sigma = f|A \cap \Sigma$, which, by Lemma 2, gives $\bar{f}|A = f$, as $A = \overline{A \cap \Sigma}$.

(ii) Let $X \setminus A = \bigcup_i F_i$ where $F_1 \subset F_2 \subset \ldots$ are closed. By Lemma 1 we have $X \setminus A = \bigcup_i \overline{F_i^*}$, where $F_i^* = \bigcup\{K \subset F_i\colon K \text{ is an } \aleph_0\text{-cube}\}$. For each $A \cup \overline{F_i^*}$, by (i), there exists a mapping $f_i\colon X \to I$ such that $f_i^{-1}(0) \supset A$ and $f_i^{-1}(1) \supset \overline{F_i^*}$. For the map $f = \sum_i 2^{-i} f_i$ we have $f^{-1}(0) = A$.

For a product $\mathop{\boldsymbol{P}}_{s\in S} X_s$ of metrizable separable spaces even more can be proved; namely that Proposition 2(ii) holds for all sets A which are unions of $\aleph_0$-cubes (see [7], Corollary 1, p. 294). It may be interesting to find out whether or not a similar theorem holds in a more general case.

QUESTION 1. *Let $X = \mathop{\boldsymbol{P}}_{s\in S} X_s$ be a product of completely metrizable spaces.*

(a) *Is every regularly closed subset of X (i.e., a closure of an open set) a G_δ-set?*

(b) *Is every closed union of $\aleph_0$-cubes in X a G_δ-set?*

(c) *What is the answer to* (a) *or* (b) *in the case of $X = D(\aleph_1)^{\aleph_1}$?*

Remark 4. By Corollary 2 we have (b)$\Rightarrow$(a). Notice, that (a) is equivalent to the question whether X has the Bockstein Separation Property (i.e., whether disjoint open subsets can be separated by disjoint open F_σ-sets; see [3], also [10], Problem 2.7.12(b)); the equivalence easily follows from Proposition 2. Problem (b) for $X = D(\aleph_1)^{\aleph_1}$ is equivalent to the question whether the union $F = \bigcup_{\xi<\omega_1} F_\xi$ of an increasing sequence $F_1 \subset F_2 \subset \ldots \subset F_\xi \subset \ldots \subset X$, $\xi<\omega_1$ of closed G_δ-sets is a G_δ-set (from Lemma 4 it follows that F is closed).

QUESTION 2. *Let $X = \mathop{\boldsymbol{P}}_{s\in S} X_s$ be a product of arbitrary metrizable spaces. Is Proposition 2 true for X?*

Note that if for some $x \in X$ the Σ-product $\Sigma(x)$ is normal (hence (6) holds), then the answer is positive. The last problem was raised by H. Corson in [5] and, as far as we know, has not been solved.

6. Realcompact subspaces of products of first-countable spaces. In this section we shall assume that the spaces under consideration are completely regular. We shall investigate realcompact subspaces of

(7) $$X = \mathop{\boldsymbol{P}}_{s\in S} X_s \quad \text{with} \quad \chi(X_s) \leqslant \aleph_0 \quad \text{for} \quad s \in S .$$

It is convenient to consider a more general situation. For a topological space X the symbols $\mathbf{I}(A)$ and $\mathbf{C}(A)$ will denote respectively the interior and the closure of the set $A\subset X$ in the $\aleph_1$-modification of X. Having this topology in mind, we shall consider G_δ-open sets, G_δ-dense sets etc. in the space X. We shall be interested in the following property of the space X:

(*) $$\mathbf{I}(\overline{\mathbf{I}(A)}) = \overline{\mathbf{I}(A)} \quad \textit{for each } A\subset X,$$

which means that the closure of each G_δ-open set in X is G_δ-open. By Corollary 2 we infer that

(8) *if X is as in (7), then X satisfies* (*).

Proposition 3. *Let a space X satisfy* (*). *Then each $A\subset X$ which is dense in some G_δ-open subset of X (equivalently, if $A\subset\mathbf{I}(\bar{A})$) is C-embedded in its G_δ-closure $\mathbf{C}(A)$. If X is realcompact, then $\mathbf{C}(A)$ is the Hewitt-realcompactification of A.*

Proof. First let $B\supset A$ be an arbitrary G_δ-open subset of X. We claim that B is C-embedded in $B\cup\mathbf{C}(A) = B'$. Since each nonempty G_δ-open subset of B' intersects B, it is sufficient to show that B is C^*-embedded in B', or that the disjoint functionally closed subsets Z_0, Z_1 of B have disjoint closures in B' (see [12], Theorem 6.4 and 1.18). Let us assume, on the contrary, that there exists an $x\in\bar{Z}_0\cap\bar{Z}_1\cap\mathbf{C}(A)$. Since both Z_i are G_δ-open in B, they are also G_δ-open in X; hence by (*) the sets $\bar{Z}_i$ are G_δ-open. The set $\bar{Z}_0\cap\bar{Z}_1$ is thus a G_δ-open neighbourhood of $x\in\mathbf{C}(A)$; hence $\bar{Z}_0\cap\bar{Z}_1\cap A\neq\emptyset$. We obtain a contradiction:

$$\emptyset\neq\bar{Z}_0\cap\bar{Z}_1\cap B = Z_0\cap Z_1 = \emptyset .$$

Now let $f\colon A\to R$ be an arbitrary mapping. There exists a $B\subset\bar{A}$ which is a G_δ-set in $\bar{A}$ and contains A and an extension $f'\colon B\to R$ of f (see [10], Theorem 4.3.20). From our assumptions and (*) we see that the set $\bar{A}$ is G_δ-open; hence B is G_δ-open in X, and by our initial observation there exists an extension $f''\colon B\cup\mathbf{C}(A)\to R$ of f'. The restriction $f''|\mathbf{C}(A)$ is the required extension of f. If X is realcompact, the equality $\upsilon A = \mathbf{C}(A)$ follows by the observation that $\mathbf{C}(A)$ is also realcompact (see [20], Theorem 2).

For a product $X = \mathop{\boldsymbol{P}}_{s\in S} X_s$ with X_s second-countable, a similar result was proved (by other means) by N. Noble [23] (see also [6], Theorem 5). From Proposition 3 follows, in particular, the theorem of M. Ulmer [25] about the C-embedding of the Σ-product in the product satisfying (7).

Lemma 3. *Let X satisfy* (*). *Then*

(i) *if $E\subset X$ and $X\setminus E$ is realcompact, then $E\subset\mathbf{C}(\mathbf{I}(E))$,*

(ii) *if both E and $X\setminus E$ are realcompact then both are G_δ-open.*

Proof. (i) Let K be an arbitrary functionally closed subset of X such that $K\cap E\neq\emptyset$. If we had $\mathbf{I}(K\cap E)=\emptyset$, then the set $K\cap(X\setminus E)$ would be G_δ-dense in the G_δ-open set K; hence by Proposition 3 it would be C-embedded in K (as a dense subspace). But this is impossible, as $K\cap(X\setminus E)$ is realcompact and different from K; hence $\mathbf{I}(K\cap E)\neq\emptyset$. Since K is arbitrary, we obtain (i).

(ii) By (i) we have $E\subset\mathbf{C}(\mathbf{I}(E))\subset\overline{\mathbf{I}(E)}$; hence by (*) E is dense in the G_δ-open set $\overline{\mathbf{I}(E)}$. By Proposition 3 the space E is C-embedded in its G_δ-closure $\mathbf{C}(E)$, and since it is realcompact, $E=\mathbf{C}(E)$. Thus $X\setminus E$ is G_δ-open, and, by symmetry, this concludes the proof.

From Lemma 3(ii) and from (8) we immediately obtain

Corollary 4. *If the product X satisfying (7) is decomposed into the union of two disjoint realcompact subspaces, then both of them are unions of $\aleph_0$-cubes.*

Remark 5. *A realcompact space X satisfies* (*) *if and only if the interior of each realcompact subspace E of X is G_δ-closed.*

Proof. Assume (*). If $E\subset X$ is realcompact, by Lemma 3(i) we have $X\setminus E\subset\overline{\mathbf{I}(X\setminus E)}$ and by (*) the set $\overline{X\setminus E}$ is G_δ-open. Thus $\operatorname{Int}E = X\setminus\overline{(X\setminus E)}$ is G_δ-closed. Suppose now that $\mathbf{C}(\operatorname{Int}E) = \operatorname{Int}E$ for an arbitrary realcompact subspace E of X. Let $A\subset X$ and $\mathbf{I}(A) = A$. The space $X\setminus A$ is realcompact ([20], Theorem 2); hence $\operatorname{Int}(X\setminus A)$ is G_δ-closed. Thus $\bar{A} = X\setminus\operatorname{Int}(X\setminus A)$ is G_δ-open.

Lemma 4. *Let the product X satisfy (7) and let the complement $X\setminus E$ of $E\subset X$ be realcompact. Then the closure $\bar{E}$ is equal to the sequential closure of E (i.e., each $x\in\bar{E}$ is the limit of some sequence of points of E).*

Proof. By (8) and Lemma 3(i) we have $\bar{E} = \overline{\mathbf{I}(E)}$. Take an $x\in\bar{E}$ and let $\Sigma=\Sigma(x)$ be the Σ-product with the base point x. By Lemma 2 we have $x\in\overline{\mathbf{I}(E)\cap\Sigma}$, and since Σ is a Fréchet space (see [22], Theorem 2.1, or [10], Exercise 3.10.D), we can choose a sequence $\{x_n\}\subset\mathbf{I}(E)\cap\Sigma$ converging to x.

Theorem 3. *Let $X=\mathop{\boldsymbol{P}}_{s\in S} X_s$ be a product of separable metrizable spaces. Then*

(i) *a sequentially open* (⁶) *subspace $U\subset X$ is realcompact if and only if it is functionally open,*

(ii) *a G_δ-set G in X is realcompact if and only if it depends on countably many coordinates.*

Proof. (i) If U is a sequentially open realcompact subspace of X, then by Lemma 4 the complement $X\setminus U$ is closed. By Corollary 4, $X\setminus U$ is a union of $\aleph_0$-cubes and thus it depends on countably many coordinates (by Theorem 1(i) or [7], Theorem 5) and is functionally closed.

(ii) Let G be a realcompact G_δ-set in X. Since $X\setminus G$ is G_δ-closed it is also realcompact ([20], Theorem 2) and by Corollary 4 it is a union of $\aleph_0$-cubes. It is sufficient now to use Theorem 2. Conversely, each subspace of X which depends on countably many coordinates is realcompact as a product of realcompact spaces.

Corollary 5. *For a G_δ-set G in $N^{\mathfrak{m}}$, where $\mathfrak{m}>\aleph_0$, the following are equivalent:*

(i) *G is homeomorphic with $N^{\mathfrak{m}}$,*

(ii) *G can be embedded in $N^{\mathfrak{m}}$ as a closed set,*

(iii) *G is realcompact.*

(⁶) I.e., such that each converging sequence of elements of $X\setminus U$ has the limit point in $X\setminus U$.

Proof. Implications (i)⇒(ii)⇒(iii) are obvious. If G is realcompact, then by Theorem 3(ii) we have $G \underset{\text{top}}{=} G' \times N^{\mathfrak{m}}$ for some G_δ-set G' in $N^{\aleph_0}$. By the Mazurkiewicz Theorem (see [17], Theorem 3, § 36, II) we have $G' \times N^{\aleph_0} \underset{\text{top}}{=} N^{\aleph_0}$ and thus $G \underset{\text{top}}{=} G' \times N^{\aleph_0} \times N^{\mathfrak{m}} \underset{\text{top}}{=} N^{\aleph_0} \times N^{\mathfrak{m}} \underset{\text{top}}{=} N^{\mathfrak{m}}$.

COROLLARY 6. *For a G_δ-set G in a dyadic space X the following are equivalent*:

(i) *G is realcompact*,

(ii) *G is Lindelöf.*

Proof. Let $f\colon D^{\mathfrak{m}} \to X$ map a Cantor cube $D^{\mathfrak{m}}$ onto X. Suppose (i). Then the set $f^{-1}(G)$ is realcompact ([10], Corollary 3.11.8) G_δ-set in $D^{\mathfrak{m}}$ and, by Theorem 3 (ii), depends on countably many coordinates. Hence $f^{-1}(G)$ is a Lindelöf space as the product of a metrizable separable space and a compact space, and its continuous image G is also a Lindelöf space. The inverse implication is a well-known fact (see [10]).

Note that Corollary 6 can also be formulated in the following manner: *each Čech-complete, realcompact, subdyadic space* (see [6]) *is a Lindelöf space.*

COROLLARY 7. *The interior* $\operatorname{Int} A$ *of a realcompact subspace A of a Cantor cube $D^{\mathfrak{m}}$ is a Lindelöf space.*

Proof. By Remark 5 the complement $D^{\mathfrak{m}} \setminus \operatorname{Int} A$ is G_δ-open; hence it is a G_δ-set in $D^{\mathfrak{m}}$ (see [7], Corollary 1), and thus $\operatorname{Int} A$ is an F_σ-set in the compact space $D^{\mathfrak{m}}$.

The example given by R. Engelking ([7], p. 302), i.e., two Cantor cubes $D^{\aleph_1}$ in contact at exactly one point, shows that Corollary 7 does not hold for all dyadic spaces (but, as is easy to verify, it does hold for all retracts of Cantor cubes).

Remark 6. (a) Theorem 3(ii) and Corollary 6 hold if realcompactness is replaced by Dieudonné-completeness (for the definition see [10], Problem 8.5.13). Indeed, in both cases the condition $c(G) \leqslant \aleph_0$ holds ([7], Theorem 3), and for such spaces these two concepts coincide (see [12], 15.21).

(b) Theorem 3, (ii) and Corollary 6 are not true for sets G which are unions of $\aleph_0$-cubes. One can in fact define (by using the Continuum Hypothesis) a subset G of $D^{\aleph_1}$ such that both G and $D^{\aleph_1} \setminus G$ are unions of $\aleph_0$-cubes, and yet G contains as a closed subset a space $D(\aleph_1)$.

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Added in proof.

1. A similar approach to the Efimov theorem (Remark 3) appeared recently in the book of A. V. Arhangel'skiĭ and V. I. Ponomarev, *Foundations of general topology in problems and exercises*, Moskva 1974 (Russian), Problem 391, § 5, II.

2. The positive answer to Question 1(a) (where X is even product of arbitrary metrizable spaces) follows from recent results of E. V. Ščepin, DAN SSSR 226 (1976), pp. 527–529, Theorem (Russian).

3. In Corollary 6 it is enough to assume that G is a Borel set in X with $X \setminus G$ realcompact, as was shown by the first of the authors in *A note on Borel sets in dyadic spaces*, Bull. Acad Polon. Sci. (1976) (to appear).

References

[1] K. Alster, *Subparacompactness in Cartesian products of generalized ordered spaces*, Fund. Math. 87 (1975), pp. 7–28.
[2] — R. Engelking, *Subparacompactness and product spaces*, Bull. Acad. Polon. Sci. 20 (1972), pp. 763–767.
[3] M. Bockstein, *Un théorème de séparabilité pour les produits topologiques*, Fund. Math. 35 (1948), pp. 242–246.
[4] W. W. Comfort and S. Negrepontis, *On families of large oscillation*, Fund. Math. 75 (1972), pp. 275–290.
[5] H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. 81 (1959), pp. 785–796.
[6] B. Efimov and R. Engelking, *Remarks on dyadic spaces*, II, Colloq. Math. 13 (1965), pp. 181–197.
[7] R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. 57 (1965), pp. 287–304.
[8] — *On functions defined on Cartesian products*, Fund. Math. 59 (1966), pp. 221–231.
[9] — *On the double circumference of Alexandroff*, Bull. Acad. Polon. Sci. 16 (1968), pp. 629–634.
[10] — *General Topology*, Warszawa (sous presse).
[11] — and M. Karłowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. 57 (1965), pp. 275–285.
[12] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Princeton 1960.
[13] J. R. Isbell, *Uniform spaces*, Providence 1964.
[14] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tracts 34, Amsterdam 1971.
[15] A. P. Kombarov, *The product of normal spaces. Uniformities on Σ-products* (in Russian), Dokl. Akad. Nauk SSSR 205 (1972), pp. 1033–1035.
[16] — and V. I. Malyhin, *On Σ-products* (in Russian), Dokl. Akad. Nauk SSSR 213 (1973), pp. 774–776.
[17] K. Kuratowski, *Topology*, vol. I, II, New York–London–Warszawa 1966/1968.
[18] A. Miščenko, *Theorems on Cartesian products of topological spaces* (in Russian), Fund. Math. 58 (1966), pp. 259–284.
[19] A. Mostowski, *Constructible sets with applications*, Amsterdam 1969.
[20] S. Mrówka, *Some properties of Q-spaces*, Bull. Acad. Polon. Sci. 5 (1957), pp. 947–950.
[21] J. Mycielski, *α-incompactness of N^α*, Bull. Acad. Polon. Sci. 12 (1964), pp. 437–438.
[22] N. Noble, *The continuity of functions on Cartesian products*, Trans. Amer. Math. Soc. 149 (1970), pp. 187–198.
[23] N. Noble, *C-embedded subsets of products*, Proc. Amer. Math. Soc. 31 (1972) pp. 613–614.
[24] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), pp. 977–982.
[25] M. Ulmer, *C-embedded Σ-spaces*, Pacific J. Math. 46 (1973), pp. 591–602.
[26] J. M. Worrel and Jr. H. H. Wicke, *Characterizations of developable topological spaces* Canad. J. Math. 17 (1965), pp. 820–830.

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