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The simultaneous existence of linear functionals

by

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Abstract. This paper contains a result on the simultaneous existence of many linear functionals that generalizes results of Kwapień and Le Presté, and applications to the theory of tensor product of (more than two) Banach spaces.

0. Introduction. In Section 1 of this paper we shall prove an abstract result on the existence of a family of linear functionals and apply our result to the existence of measures on compact spaces, obtaining generalizations of results of Kwapień and La Presté.

In Section 2 we give an application of the results of Section 1 to the theory of tensor products of a finite number ($\geqslant 2$) of Banach spaces.

In an earlier version of this paper a somewhat complicated proof of Theorem 1 was given. The proof appearing here was found independently by the author and the referee and was suggested to the author by [1], Proposition 2.

1. The simultaneous existence of many linear functionals. We suppose throughout that $n > 1$ and the index k will always run from 1 to n.

If $\lambda_1, \ldots, \lambda_n$ are given, we shall write $\bigtimes_k \lambda_k$ for the n-tuple $(\lambda_1, \ldots, \lambda_n)$.

For each k, let $r_k > 0$ and $\sum_k r_k = s \leqslant 1$. If $\lambda = \bigtimes_k \lambda_k \in (R^+)^n$ let

$$\varphi(\lambda) = \prod_k \lambda_k^{r_k}.$$

For each k, let C_k be a real linear space and S_k be a sublinear functional on C_k. Let $I \neq \emptyset$ and, for each $i \in I$, $x_{ik} \in C_k$ and $a_k \geqslant 0$.

1. THEOREM. *Let* $t = \dfrac{1}{1-s}$. *Then* (1) $\Leftrightarrow$ (2).

For each k, there exists a linear functional L_k on C_k dominated by S_k such that, for all $i \in I$,

(1)
$$L_k(x_{ik}) \geqslant 0 \quad \textit{and} \quad a_i \leqslant \varphi\big(\bigtimes_k L_k(x_{ik})\big).$$

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If J is a finite subset of I and for all j (i.e. for all $j \in J$) and k, $\lambda_{jk} \geqslant 0$ and $\delta_j \geqslant 0$ with $(\sum_j \delta_j^t)^{1/t} = 1$ then for all k

(2)
$$S_k\big(\sum_j \lambda_{jk} x_{jk}\big) \geqslant 0 \quad \text{and} \quad \varphi\big(\bigtimes_k S_k\big(\sum_j \lambda_{jk} x_{jk}\big)\big) \geqslant \sum_j \varphi\big(\bigtimes_k \lambda_{jk}\big)\,\delta_j\, a_j.$$

Proof. ($\Rightarrow$) We suppose that (1) is true and J, λ_{jk} and δ_j are as in (2). Then

$$S_k\big(\sum_j \lambda_{jk} x_{jk}\big) \geqslant L_k\big(\sum_j \lambda_{jk} x_{jk}\big) = \sum_j \lambda_{jk} L_k(x_{jk}) \geqslant 0$$

and

$$\begin{aligned}\varphi\big(\bigtimes_k S_k\big(\sum_j \lambda_{jk} x_{jk}\big)\big)^{1/s} &\geqslant \varphi\big(\bigtimes_k L_k\big(\sum_j \lambda_{jk} x_{jk}\big)\big)^{1/s}\\ &= \varphi\big(\sum_j \bigtimes_k \lambda_{jk} L_k(x_{jk})\big)^{1/s}\end{aligned}$$

noting from Hölder's inequality that $\varphi^{1/s}$ is superadditive

$$\geqslant \sum_j \varphi\big(\bigtimes_k \lambda_{jk} L_k(x_{jk})\big)^{1/s}$$

since φ is multiplicative

$$= \sum_j \varphi\big(\bigtimes_k \lambda_{jk}\big)^{1/s} \varphi\big(\bigtimes_k L_k(x_{jk})\big)^{1/s}$$

from (1)

$$\begin{aligned}&\geqslant \sum_j \varphi\big(\bigtimes_k \lambda_{jk}\big)^{1/s} a_j^{1/s}\\ &\geqslant \big(\sum_j \varphi\big(\bigtimes_k \lambda_{jk}\big) a_j\, \delta_j\big)^{1/s}\end{aligned}$$

from Hölder's inequality again, noting that $1/s$ and t are conjugate indices. It follows that (2) is true.

($\Leftarrow$) We now suppose that (2) is true and write C for the linear space $C_1 \times \ldots \times C_n$ and S for the sublinear functional defined on C by

$$S\big(\bigtimes_k x_k\big) = \sum_k S_k(x_k) \qquad (x_k \in C_k).$$

We observe that $\sum_k (r_k/s) = 1$ hence, from the inequality of the arithmetic and geometric mean, for all $\beta \geqslant 0$,

(3)
$$\beta = \inf\big\{\sum_k (r_k/s)\lambda_k\colon\ \lambda_k \geqslant 0 \text{ and } \varphi\big(\bigtimes_k \lambda_k\big)^{1/s} = \beta\big\}.$$

Let J, λ_{jk} be as in (2) and $\beta_j \geqslant 0$ $(j \in J)$. Then

$$\sum_j \beta_j \varphi\big(\bigtimes_k \lambda_{jk}\big)^{1/s} a_j^{1/s} = \sum_j \varphi\big(\bigtimes_k \beta_j \lambda_{jk}\big)^{1/s} a_i^{1/s}.$$

From (2), by taking the inf over $\{\delta_j\}$,

$$\leqslant \varphi\big(\bigtimes_k S_k\big(\sum_j \beta_j \lambda_{jk} x_{jk}\big)\big)^{1/s}$$

from (3)

$$\begin{aligned}&\leqslant \sum_k (r_k/s) S_k\big(\sum_j \beta_j \lambda_{jk} x_{jk}\big)\\ &= S\big(\sum_j \beta_j \bigtimes_k (r_k/s)\lambda_{jk} x_{jk}\big).\end{aligned}$$

If λ_{jk} are constrained so that, for all k, $\varphi\big(\bigtimes_k \lambda_{jk}\big) = 1$, then

$$\sum_j \beta_j a_j^{1/s} \leqslant S\big(\sum_j \beta_j \bigtimes_k (r_k/s)\lambda_{jk} x_{jk}\big).$$

From the Mazur–Orlicz theorem, there exists a linear functional L on C, dominated by S, such that, for all $i \in I$,

$$\text{if} \quad \varphi\big(\bigtimes_k \lambda_k\big) = 1 \quad \text{then} \quad a_i^{1/s} \leqslant L\big(\bigtimes_k (r_k/s)\lambda_k x_{ik}\big).$$

For each k, there exists a linear functional L_k on C_k, dominated by S_k, such that

$$L\big(\bigtimes_k x_k\big) = \sum_k L_k(x_k) \qquad (x_k \in C_k).$$

Thus if $\varphi\big(\bigtimes_k \lambda_k\big) = 1$ then, for all $i \in I$,

$$a_i^{1/s} \leqslant \sum_k (r_k/s)\lambda_k L_k(x_{ik}).$$

Taking the inf over $\{\lambda_k\}$ and using (3), it follows that $L_k(x_{ik}) \geqslant 0$ and

$$a_i^{1/s} \leqslant \varphi\big(\bigtimes_k L_k(x_{ik})\big)^{1/s}.$$

It is immediate from this that (1) is satisfied.

2. Corollary. *Let $K_1, \ldots, K_n$ be compact Hausdorff spaces, $I \neq \emptyset$ and for each i and k, $a_i \in R^+$ and $f_{ik} \in C(K_k)$ with $f_{ik} \geqslant 0$. Let $p_k > 1$ and $t > 1$ and $\sum_k (1/p_k) + 1/t = 1$. Then*

for all k there exists a probability measure μ_k on K_k such that, for all $i \in I$, $a_i \leqslant \prod_k \mu_k(f_{ik}^{p_k})^{1/p_k}$

$\Leftrightarrow$

whenever J is a nonempty finite subset of I and, for all j and k, $\lambda_{jk} \geqslant 0$ and $\delta_j \geqslant 0$ with $(\sum_j \delta_j^t)^{1/t} = 1$ then

$$\sum_j \delta_j a_j \prod_k \lambda_{jk} \leqslant \prod_k \sup\nolimits_{a \in K_k} \big(\sum_j (\lambda_{kj} f_{kj})^{p_k}(a)\big)^{1/p_k}.$$

Proof. Immediate from Theorem 1 with $C_k = C(K_k)$, $r_k = 1/p_k$, $x_{ik} = f_{ik}^{p_k}$ and, for $f \in C_k$,

$$S_k(f) = \sup f(K_k).$$

2. An application to the theory of tensor products.

3. Notation. We suppose that, for each k, E_k and G_k are Banach spaces (all over $\mathscr{K}$) and r_k, p_k and t are as in Corollary 2. We write K_k for the unit ball of E_k', $E = \otimes_k E_k$ and $G = \otimes_k G_k$.

If $u \in G$ we write

$$g(u) = \inf\big\{\sum_i \prod_k \|y_{ik}\|\colon\ m \geqslant 1,\ y_{ik} \in G_k \text{ and } u = \sum_i \otimes_k y_{ik}\big\},$$

where i runs from 1 to m.

If $v\in E$ we write

$$\sigma(v) = \inf\{(\sum_i|\delta_i|^t)^{1/t}\prod_k \sup_{a\in K_k}(\sum_i|\langle x_{ik}, a\rangle|^{p_k})^{r_k}: \; m\geqslant 1,\ x_{ik}\in E_k \text{ and } v = \sum_i \delta_i \otimes_k x_{ik}\};$$

g and σ are norms on G and E, respectively. In fact, they are *cross-norms* in the sense that $g(\otimes_k y_k) = \prod_k \|y_k\|$ and $\sigma(\otimes_k x_k) = \prod_k \|x_k\|$.

4. Remark. g is the *projective tensor product* norm on G. It is well known that the g-continuous linear functionals ψ on G are in (1, 1)-correspondence with the bounded n-linear functionals χ on $G_1\times\ldots\times G_n$ by the rule $\chi(\bigtimes_k y_k) = \psi(\otimes_k y_k)$. Further

$$\|\psi\| = \sup\{|\chi(\bigtimes_k y_k)|: \|y_1\|, \ldots, \|y_n\|\leqslant 1\}.$$

5. Notation. If, for each k, $Q_k\in L(E_k, G_k)$ then there exists a linear map Q: $E\mapsto G$ such that, for all $x_k\in E_k$, $Q(\otimes_k x_k) = \otimes_k Q_k(x_k)$. We shall write $\otimes_k Q_k$ for this map Q.

In the theorem below we shall characterize the σ-continuous linear functionals on E.

6. Theorem. *Let $E_1, \ldots, E_n$ be given and let β be a linear functional on E. Then $\|\beta\|$ (= the σ-norm of β) $\leqslant 1$ $\Leftrightarrow$ for all k there exist G_k and $Q_k\in\prod_{p_k}(E_k, G_k)$ with $\pi_{p_k}(Q_k)\leqslant 1$ and there exists a linear functional ψ on G with $\|\psi\|$(= the g-norm of ψ) $\leqslant 1$ such that*

$$\beta = \psi\circ(\otimes_k Q_k). \tag{4}$$

Proof. ($\Leftarrow$) If $V = \sum_i \delta_i\otimes_k x_{ik}$ then, by (4) and linearity

$$\beta(V) = \psi(\sum_i \delta_i \otimes_k Q_k x_{jk}).$$

Since $\|\psi\|\leqslant 1$,

$$|\beta(V)|\leqslant g(\sum_i \delta_i\otimes_k Q_k x_{jk})\leqslant \sum_i \delta_i \prod_k \|Q_k x_{ik}\|$$

from Hölder's inequality

$$\leqslant (\sum_i |\delta_i|^t)^{1/t}\prod_k(\sum_i \|Q_k x_{ik}\|^{p_k})^{r_k}$$

since $\pi_{p_k}(Q_k)\leqslant 1$,

$$\leqslant (\sum_i|\delta_i|^t)^{1/t}\prod_k \sup_{a\in K_k}(\sum_i|\langle x_{ik}, a\rangle|^{p_k})^{r_k}.$$

It follows by taking the infimum over all representations of V that

$$|\beta(V)|\leqslant\sigma(V),$$

hence $\|\beta\|\leqslant 1$, as required.

($\Rightarrow$) Suppose $m\geqslant 1$ and, for all j and k ($1\leqslant j\leqslant m$, $1\leqslant k\leqslant n$), $x_{jk}\in E_k$ and $\delta_j\geqslant 0$ with $(\sum_j|\delta_j|^t)^{1/t}\leqslant 1$. Then, since $\|\beta\|\leqslant 1$,

$$\sum_j \delta_j\beta(\otimes_k x_{jk}) = \beta(\sum_j \delta_j\otimes_k x_{jk})\leqslant \sigma(\sum_j\delta_j\otimes_k x_{jk}) \leqslant \prod_k \sup_{a\in K_k}(\sum_j|\langle x_{jk}, a\rangle|^{p_k})^{r_k}.$$

It follows from Corollary 2 that for all k there exists a probability measure μ_k on K_k such that

$$|\beta(\otimes_k x_{jk})|\leqslant \prod_k \mu_k(|\langle x_k, \cdot\rangle|^{p_k})^{r_k} \quad \text{for all } x_k\in E_k.$$

Let H_k be the canonical image of E_k in $L_{p_k}(K_k, \mu_k)$ and $G_k = \overline{H_k}$; further let Q_k be the canonical map of E_k into G_k. Clearly, $Q_k\in\prod_{p_k}(E_k, G_k)$ and $\pi_{p_k}(Q_k)\leqslant 1$. Using the density of H_k in G_k, it is easily seen that there exists an n-linear functional χ on $G_1\times\ldots\times G_n$ such that, for all $y_k\in G_k$,

$$|\chi(\bigtimes_k y_k)|\leqslant \prod_k\|y_k\|.$$

The result follows with ψ defined as in Remark 4.

7. Corollary. *If $v\in E$ then*

$$\sigma(v) = \sup\{|\psi\circ(\otimes_k Q_k)(r)|: \psi \text{ and } Q_k \text{ as in Theorem 6}\}.$$

Proof. Immediate from Theorem 6 and the Hahn–Banach theorem.

References

[1] S. Kwapień, *On operators factorizable through L_p spaces*, Bull. Soc. Math. France 31/32 (1972), pp. 215–225.

[2] J.-T. La Presté, *Sur une généralization de la notion d'opérateurs nucléaires et sommants dans les espaces de Banach*, C. R. Acad. Sci. Paris 275 No. 1, pp.45–48.

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