$V_j \in L(K_j)$ such that V_j is the extension of S_j and $V_j N_j = N_j V_j$. The same argument for φ_j and V_j in place of ψ_j and T_j shows that $R_j = \varphi_j(V_j)$ commutes with N_j. Since V_j and $\varphi_j(V_j)$ have the same invariant subspaces, R_j is an extension of T_j. Evidently, R_j is subnormal. Hence for every $j \geqslant 0$ we get a subnormal extension of T_j which commutes with N_j, and our proof is complete.

Proposition 2 and Theorem 5 yield the following

THEOREM 6. *Let $A \in L(H)$ be a subnormal operator and suppose that the operator $T \in L(H)$ commutes with A. Assume that:*

1. *$X = \sigma(T)$ has connected complement.*
2. *There is a normal extension B of T such that $\sigma(B) \subset \partial X$.*

Then there is a normal extension $R \in L(K)$ and a normal extension $N \in L(K)$ of T such that N commutes with R.

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Multipliers on Banach algebras

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Abstract. This paper is concerned with the study and application of (left, right, double) multipliers on Banach algebras. We consider mainly Banach algebras with bounded (left, right) approximate identities and Banach algebras which are dense $*$-subalgebras of dual B^*-algebras. More specifically, in this second group of Banach algebras we are primarily interested in multipliers on modular annihilator A^*-algebras.

Let A be a Banach algebra with a bounded right approximate identity. Let $M_r(A)$ be the algebra of all bounded linear right multipliers on A. It follows that $M_r(A)$ can be embedded into the second conjugate space A^{**} of A, when A^{**} is considered as a Banach algebra with an Arens product. By using this embedding of $M_r(A)$ into A^{**}, we obtain various properties of A, A^{**}, and $M_r(A)$. Similarly, if A has a bounded left approximate identity we can embed the algebra $M_l(A)$ of continuous linear left multipliers on A into A^{**}. We also consider $M_l(A)$ and $M_r(A)$ with respect to their weak operator topologies and study the groups of isometric and onto (left, right, double) multipliers under these topologies.

The last section of the paper is devoted to the study of multipliers on a modular annihilator A^*-algebra A. Here we show how (left, right, double) multipliers on A are related to (left, right, double) multipliers on the completion $\mathfrak{A}$ of A.

Introduction. Let A be a Banach algebra and let $M_l(A)$ (resp. $M_r(A)$) be the algebra of continuous linear left (resp. right) multipliers on A. Let $M(A)$ be the algebra of double multipliers (S, T) on A such that $S \in M_l(A)$ and $T \in M_r(A)$. It was shown by L. Maté [14] that if A has a bounded right approximate identity then $M_r(A)$ can be embedded anti-isomorphically in the second conjugate space A^{**} of A, when A^{**} is considered as a Banach algebra with Arens product $F * G$, $F, G \in A^{**}$. This embedding is given by the map $T \to T^{**}(E)$, where E is the right identity of $(A^{**}, *)$. In § 5 we gather together various results on the algebras of multipliers as well as A and A^{**} coming out of Maté's representation. For example, we show that the canonical image $\pi(A)$ is a right ideal of $(A^{**}, *)$ if and only if every $F \in A^{**}$ is of the form $F = T^{**}(E) + G$, where $T \in M_r(A)$ and $G \in A^{**}$ with the property that $\pi(A) * G = (0)$.

In § 6 we consider the algebras $M_l(A)$ and $M_r(A)$ with respect to their weak operator topologies. Let $\mathscr{S}(M_l(A))$ (resp. $\mathscr{S}(M_r(A))$) be the closed unit ball of $M_l(A)$ (resp. $M_r(A)$). We show that if A has a right approximate identity bounded by one, then $\pi(A)$ is a right ideal of $(A^{**}, *)$ if and only if $\mathscr{S}(M_r(A))$ is compact in the weak operator topology τ_r on $M_r(A)$. In particular, if A is a B^*-algebra then this is equivalent to saying that A is dual. Section 7 is devoted to the study of groups of isometric and onto (left, right, double) multipliers. For example, we show that if A has a right approximate identity bounded by one and $\mathscr{S}(M_r(A))$ is τ_r-compact, then the set $G_r(A)$ of isometric and onto elements of $M_r(A)$ is a τ_r-compact group. We then use the group $G_r(A)$ to obtain a characterization of duality in a B^*-algebra.

In § 8 we are concerned with multipliers on a modular annihilator A^*-algebra A. We show that every left (right) multiplier on A is a continuous linear operator. Let $\mathfrak{A}$ be the completion of A. If A has a bounded approximate identity then every left (right) multiplier T on A is of the form $T = T'|A$, where T' is a left (right) multiplier on $\mathfrak{A}$. If A is a modular annihilator A^*-algebra which is an ideal of its completion $\mathfrak{A}$ then, for every $(S, T)\in M(A)$, there exists $(S', T')\in M(\mathfrak{A})$ such that $S = S'|A$, $T = T'|A$.

2. Notation and terminology. All algebras and vector spaces are over the complex field C. Let A be a Banach algebra. A^* and A^{**} will denote the first and second conjugate spaces of A. $\mathscr{S}(A)$ will denote the closed unit ball of A and π will stand for the canonical map of A into A^{**}. For any set $S\subset A$, $l(S)$ (resp. $r(S)$) will denote the left (resp. right) annihilator of S in A. S_A will denote the socle of A. For any $a\in A$, L_a and R_a will denote respectively the left and right multiplication operators determined by a.

If for every maximal modular left ideal M and for every maximal modular right ideal N in A we have $r(M)\neq 0$ and $l(N)\neq (0)$, A is called *modular annihilator*. If for every closed left ideal J and for every closed right ideal R in A we have $l(r(J)) = J$ and $r(l(R)) = R$, A is called *dual*. A Banach algebra with dense socle is modular annihilator and a modular annihilator B^*-algebra is dual ([24], pp. 41–42).

Let A be an A^*-algebra. The auxiliary norm in A will be denoted by $|\cdot|$. If A is modular annihilator, then $|\cdot|$ is unique and the completion $\mathfrak{A}$ in this norm is a dual B^*-algebra ([21], p. 422); moreover, if A is a two-sided ideal of $\mathfrak{A}$ then there exists a constant $k>0$ such that $\|xy\|\leqslant k\|x\||y|$, for all $x\in A$, $y\in\mathfrak{A}$ ([15], p. 18, Lemma 4).

A Banach algebra A is said to be *left* (resp. *right*) *faithful* if $l(A) = (0)$ (resp. $r(A) = (0)$). A is faithful if $l(A) = r(A) = (0)$. If A contains a left (right) approximate identity, it is right (left) faithfull.

If T is a mapping of a set X into a set Y and S is a subset of X, $T|S$ will denote the *restriction* of T to S. If X is a normed vector space and $f\in X^*$, we shall occasionally denote the value $f(x)$ by (x, f), $x\in X$.

3. Multipliers. Let A be a Banach algebra. A mapping T on A into itself is called a *left* (resp. *right*) *multiplier* if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$), for all $x, y\in A$. If A has a bounded left (right) approximate identity then every left (right) multiplier on A is a continuous linear operator ([12], p. 640, Corollary). Let $M_l(A)$ (resp. $M_r(A)$) denote the set of all continuous linear left (resp. right) multipliers on A. $M_l(A)$ and $M_r(A)$ are Banach algebras with identity under the usual algebraic operations for operators and the norm given by the operator bound. Clearly, $L_a\in M_l(A)$ and $R_a\in M_r(A)$, for every $a\in A$. If A is left (resp. right) faithful, we may thus identify A as a subalgebra of $M_l(A)$ (resp. $M_r(A)$) by means of the mapping $a\to L_a$ (resp. $a\to R_a$). If A is commutative then $M_l(A) = M_r(A)$. We observe that if T is a left (right) multiplier on A for which T^{-1} exists, then T^{-1} is a left (right) multiplier. For suppose T is a left multiplier; then

$$\begin{aligned} T^{-1}(x)y &= T^{-1}T\big(T^{-1}(x)y\big)\\ &= T^{-1}\big((TT^{-1})(x)y\big)\\ &= T^{-1}(xy) \end{aligned}$$

for all $x, y\in A$. (See [3], p. 207.)

An ordered pair (S, T) of mappings from A into itself is called a *double multiplier* if $xS(y) = T(x)y$ for all $x, y\in A$. Let $M(A)$ denote the set of double multipliers (S, T) on A such that $S\in M_l(A)$ and $T\in M_r(A)$. Clearly, $(L_a, R_a)\in M(A)$ for every $a\in A$.

LEMMA 3.1. *If A is a faithful Banach algebra, then every double multiplier on A belongs to $M(A)$.*

Proof. Suppose that A is faithful and that (S, T) is a double multiplier on A. We show that $S\in M_l(A)$. A similar argument will show that $T\in M_r(A)$.

Let $x, y, z\in A$ and $\alpha, \beta\in C$. We have $zS(x)y = T(z)xy = zS(xy)$. By the right faithfulness of A, $S(xy) = S(x)y$ for all $x, y\in A$. Moreover, $zS(\alpha x+\beta y) = T(z)(\alpha x+\beta y) = \alpha T(z)x+\beta T(z)y = z\big(\alpha S(x)+\beta S(y)\big)$. Thus S is linear. Now let $\{x_n\}$ be a sequence in A such that $\|x_n - x\|\to 0$ and $\|S(x_n)-y\|\to 0$. Then

$$\begin{aligned} \|zS(x)-zy\| &\leqslant \|zS(x)-zS(x_n)\|+\|zS(x_n)-zy\|\\ &\leqslant \|T(z)\|\,\|x-x_n\|+\|z\|\,\|S(x_n)-y\|. \end{aligned}$$

Since the last term of this inequality tends to zero, $zS(x) = zy$ for all $z\in A$. Thus, by the right faithfulness of A, $S(x) = y$ and S has a closed graph. Hence, by the closed-graph theorem, S is continuous and consequently $S\in M_l(A)$. (See [5], p. 80, and [11], p. 301.) This completes the proof.

$M(A)$ is a Banach algebra under the norm $\|(S,T)\| = \max(\|S\|, \|T\|)$ and the algebraic operations given by $(S_1, T_1)+(S_2, T_2) = (S_1+S_2, T_1+T_2)$, $(S_1, T_1)(S_2, T_2) = (S_1S_2, T_2T_1)$ and $a(S, T) = (aS, aT)$, a a scalar. If A is a Banach*-algebra with a continuous involution, then $(S, T)\to(T^{\#}, S^{\#})$ is a continuous involution in $M(A)$, where $S^{\#}(x) = S(x^*)^*$ and $T^{\#}(x) = T(x^*)^*$, $x\in A$ ([11], p. 303). If A is a B^*-algebra then $M(A)$ is also a B^*-algebra ([5], p. 81, Theorem 2.11).

4. Arens products. Let A be a Banach algebra. Arens [1] has defined two products on A^{**} which make A^{**} into a Banach algebra. These are given as follows: Let $x, y\in A$, $f\in A^*$ and $F, G\in A^{**}$. Define $f*x\in A^*$ by $(f*x)y = f(xy)$. Define $F*f\in A^*$ by $(F*f)x = F(f*x)$. Define $F*G\in A^{**}$ by $(F*G)f = F(G*f)$. Define $x*'f\in A^*$ by $(x*'f)y = f(yx)$. Define $f*'F\in A^*$ by $(f*'F)x = F(x*'f)$. Define $F*'G\in A^{**}$ by $(F*'G)f = G(f*'F)$. The canonical map π is an isomorphism of A into the algebras $(A^{**}, *)$ and $(A^{**}, *')$. (See also [6].)

A Banach algebra A is called *Arens regular* if the two Arens products agree on A^{**}. Every B^*-algebra is Arens regular ([6], p. 869, theorem 7.1). From [6], p. 855, Lemma 3.3, and [13], p. 11, Proposition 1.6, it follows that $(A^{**}, *)$ has a right identity E if and only if A has a bounded right approximate identity. Similarly, $(A^{**}, *')$ has a left identity E' if and only if A has a bounded left approximate identity. We observe that if A has a right approximate identity bounded by one then $\|E\| = 1$. In particular, if A is a B^*-algebra then $\|E\| = 1$.

N.B. From now on, by a *(left, right) approximate identity* $\{u_a\}$ we shall mean a bounded (left, right) approximate identity with $\|u_a\|\leqslant 1$ for all a.

We remark that if A is a Banach algebra, then every continuous linear left (resp. right) multiplier on $\pi(A)$ is of the form $T^{**}|\pi(A)$ for some $T\in M_l(A)$ (resp. $T\in M_r(A)$). Moreover, $T^{**}(\pi(x)) = \pi(T(x))$ for all $x\in A$.

5. A^{**} and the algebras of multipliers on A.

LEMMA 5.1. *Let A be a Banach algebra with a right approximate identity. For each $T\in M_r(A)$, let $F^T\in A^{**}$ be given by $F^T = T^{**}(E)$, where E is a right identity of $(A^{**}, *)$. Then the following statements are true:*

(i) $(F^T*f)(x) = f(T(x))$ *for all* $x\in A$ *and* $f\in A^*$.

(ii) $T^{**}(\pi(x)) = \pi(x)*F^T$ *for all* $x\in A$.

(iii) $\pi(x)*F^T\in\pi(A)$ *for all* $x\in A$.

(iv) *The mapping $T\to F^T$ is an isometric anti-isomorphism of $M_r(A)$ into $(A^{**}, *)$.*

(v) $\|R_{F^T}|\pi(A)\| = \|F^T\|$ *for all* $T\in M_r(A)$.

Proof. (i) This follows from the proof of [14], p. 810, Theorem 1.

(ii) Using (i), we obtain

$$\begin{aligned} T^{**}(\pi(x))f &= \pi(x)(T^*(f)) = f(T(x)) = (F^T*f)x \\ &= \pi(x)(F^T*f) = (\pi(x)*F^T)f, \end{aligned}$$

for all $x\in A$ and $f\in A^*$. Hence

$$T^{**}(\pi(x)) = \pi(x)*F^T \quad (x\in A).$$

(iii) Since $T^{**}(\pi(x)) = \pi(T(x))$ and since $T(x)\in A$, for all $x\in A$, we obtain (iii).

(iv) From $F^T*f = T^*(f)$ we obtain $\|T^*(f)\|\leqslant\|F^T\|\|f\|$ so that $\|F^T\|\geqslant\|T^*\| = \|T\|$. On the other hand, $\|F^T\| = \|T^{**}(E)\|\leqslant\|T^{**}\|\|E\| = \|T^{**}\| = \|T\|$. Hence $\|F^T\| = \|T\|$. Thus, $T\to F^T$ is an isometry which is clearly linear. Now if $T_1, T_2\in M_r(A)$ then, by [14], p. 811, Theorem 2, we have

$$((F^{T_2}*F^{T_1})*f)x = (T_2^*T_1^*(f))(x) = f(T_1T_2(x)),$$

for all $x\in A$ and $f\in A^*$. Hence $T_1T_2\to F^{T_2}*F^{T_1}$. This proves (iv).

(v) We have

$$\begin{aligned} \|R_{F^T}|\pi(A)\| &= \sup_{\|x\|\leqslant 1}\|\pi(x)*F^T\| = \sup_{\|x\|\leqslant 1}\|T^{**}(\pi(x))\| \\ &= \sup_{\|x\|\leqslant 1}\|T(x)\| = \|T\| = \|F^T\|. \end{aligned}$$

COROLLARY 5.2. *Let A be a Banach algebra with a right approximate identity. If $x\in A$ such that $R_x\neq 0$, then*

$$\|R_x^{**}|\pi(A)\| = \|R_x\|.$$

Moreover, if $(A^{**}, *)$ is right faithful then $R_x^{**}(E) = \pi(x)$ for all $x\in A$.

DEFINITION. Let

$$N_A = \{G\in A^{**}\colon \pi(x)*G = 0 \text{ for all } x\in A\}.$$

Remark 1. We observe that if A has a left approximate identity $\{u_a\}$, then $N_A = (0)$. In fact, since

$$(f, \pi(u_a)*G) = (f, \pi(u_a)*'G)\to(f, E'*'G) \quad \text{for all } f\in A^*,$$

we have $G = E'*'G = 0$ for all $G\in N_A$. Hence $N_A = (0)$.

LEMMA 5.3. *Let A be a Banach algebra with a right approximate identity. Let $F\in A^{**}$. Then the following statements are equivalent:*

(i) $\pi(x)*F\in\pi(A)$ *for all* $x\in A$.

(ii) *There exist $T\in M_r(A)$ and $G\in N_A$ such that $F = T^{**}(E)+G$.*

(iii) *There exists $T\in M_r(A)$ such that*

$$(F*f)x = f(T(x)) \quad (x\in A, f\in A^*).$$

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Since $T \to T^{**}|\pi(A)$ is an isometric isomorphism of $M_r(A)$ onto $M_r(\pi(A))$, there exists $T \in M_r(A)$ such that $T^{**}(\pi(x)) = \pi(x)*F$ for all $x \in A$. Let $F^T = T^{**}(E)$. By Lemma 5.1, $\pi(x)*F^T = \pi(x)*F$ for all $x \in A$. Therefore $\pi(x)*(F-F^T) = 0$ for all $x \in A$ and so $G = F-F^T \in N_A$. Hence $F = F^T+G$, $G \in N_A$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $F^T = T^{**}(E)$. From Lemma 5.1 and the fact that $G \in N_A$ we get

$$(F^T*f)x = f(T(x)) = T^{**}(\pi(x))f \quad (x \in A,\ f \in A^*)$$

and

$$T^{**}(\pi(x)) = \pi(x)*F^T = \pi(x)*F \quad (x \in A).$$

Since $(F*f)x = (\pi(x)*F)f$, we have $(F*f)x = f(T(x))$ for all $x \in A, f \in A^*$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $F^T = T^{**}(E)$. Then, by Lemma 5.1,

$$(F*f)x = f(T(x)) = (F^T*f)x \quad (x \in A,\ f \in A^*),$$

whence

$$\pi(x)*F = \pi(x)*F^T \in \pi(A) \quad (x \in A).$$

This completes the proof.

Corollary 5.4. *Let A be a Banach algebra with a right approximate identity. If $F \in A^{**}$ is of the form $F = T^{**}(E)+G$, for some $T \in M_r(A)$ and $G \in N_A$, then $\|R_F|\pi(A)\| = \|T\|$.*

As a consequence of the above results, we have:

Theorem 5.5. *Let A be a Banach algebra with a right approximate identity. Then the following statements are equivalent*:

(i) *$\pi(A)$ is a right ideal of $(A^{**}, *)$.*

(ii) *Every $F \in A^{**}$ is of the form $F = T^{**}(E)+G$ for some $T \in M_r(A)$ and $G \in N_A$.*

(iii) *For every $F \in A^{**}$, there is $T \in M_r(A)$ such that*

$$(F*f)x = f(T(x)) \quad (x \in A,\ f \in A^*).$$

Remark 2. If a Banach algebra A has a left approximate identity, we can state the left-hand version of the results above. In this case we consider $M_l(A)$ and the algebra $(A^{**}, *')$ with a left identity E'. We have $f*'E' = f$ for all $f \in A^*$. If $S \in M_l(A)$ and $F^S = S^{**}(E')$ then $(f*'F^S)(x) = f(S(x))$ for all $x \in A$ and $f \in A^*$, and $S^{**}(\pi(x)) = F^S*'\pi(x)$ for all $x \in A$. The mapping $S \to F^S$ is an isometric isomorphism of $M_l(A)$ into $(A^{**}, *')$.

Theorem 5.6. *Let A be a Banach algebra with left and right approximate identities. Let $(S, T) \in M(A)$. Then the following statements are true*:

(i) $S^{**}(E') = T^{**}(E)$.

(ii) *Let $F = S^{**}(E')$. Then $S^{**}(\pi(x)) = F*\pi(x)$ and $T^{**}(\pi(x)) = \pi(x)*F$ for all $x \in A$.*

(iii) $\|S\| = \|T\|$.

(iv) *S is a regular element of $M_l(A)$ if and only if T is a regular elements of $M_r(A)$, and $(S^{-1}, T^{-1}) \in M(A)$.*

(v) *S is an onto isometry if and only if T is an onto isometry.*

Proof. (i) to (iii). Let $F = S^{**}(E')$. By the left-hand version of Lemma 5.1, $S^{**}(\pi(x)) = F*\pi(x)$ for all $x \in A$. Since

$$T^{**}(\pi(x))*\pi(y) = \pi(x)*S^{**}(\pi(y)) = (\pi(x)*F)*\pi(y),$$

for all $x, y \in A$, the faithfulness of A implies that

$$T^{**}(\pi(x)) = \pi(x)*F \quad (x \in A).$$

Therefore, by Lemma 5.3 and Remark 1, $F = T^{**}(E)$. That $\|S\| = \|T\|$ follows directly from the proof of [5], p. 81, Lemma 2.6.

(iv) Suppose that T is a regular element of $M_r(A)$, i.e. T^{-1} exists and $T^{-1} \in M_r(A)$. Let $F = T^{**}(E)$. By (ii), $F*\pi(A) \subset \pi(A)$. Since $\pi(A)*F = \pi(A)$ and F has inverse $F^{-1} = (T^{-1})^{**}(E)$, it follows that $F*\pi(A) = \pi(A)$. Therefore $F^{-1}*\pi(A) = \pi(A)$. By the left-hand version of Lemma 5.3, there exists $S_1 \in M_l(A)$ such that $S_1^{**}(\pi(x)) = F^{-1}*\pi(x)$ for all $x \in A$. It is now easy to see that $S_1 = S^{-1}$. Since $\pi(A)*F^{-1} = \pi(A)$, we have $(S^{-1}, T^{-1}) \in M(A)$. Similarly, we can show that if S is regular then so is T.

(v) Suppose that S is an onto isometry. Since A has a right approximate identity and S is onto, we have

$$\begin{aligned}\|T(x)\| &= \sup\{\|T(x)y\|:\ y \in A,\ \|y\| \leqslant 1\}\\ &= \sup\{\|xS(y)\|:\ y \in A,\ \|y\| \leqslant 1\}\\ &= \sup\{\|xS(S^{-1}(z))\|:\ z \in A,\ \|z\| \leqslant 1\}\\ &= \sup\{\|xz\|:\ z \in A,\ \|z\| \leqslant 1\} = \|x\|,\end{aligned}$$

for all $x \in A$. Thus T is an isometry and, by (iv), it is also onto. Using a left approximate identity, we can similarly show that if T is an onto isometry, so is S.

Corollary 5.7. *Let A be a Banach algebra with left and right approximate identities. Then a pair (S, T), $S \in M_l(A)$, $T \in M_r(A)$, belongs to $M(A)$ if and only if $S^{**}(E') = T^{**}(E)$.*

Proof. This follows easily from Lemma 5.1, Remark 2 and Theorem 5.6.

Corollary 5.8. *Let A be a Banach algebra with left and right approximate identities. Then $\pi(A)$ is a two-sided ideal of $(A^{**}, *)$ if and only* *if for every $F \in A^{**}$ there exists $(S, T) \in M(A)$ such that $F = T^{**}(E)$. In this case we have $\|L_F \| \pi(A)\| = \|R_F | \pi(A)\|$ for all $F \in A^{**}$.*

6. Weak operator topologies on $M_l(A)$ and $M_r(A)$. Let X be a normed vector space and let B be an algebra of bounded linear operators on X into X. The weak operator topology on B is the topology on B generated by the seminorms $T \to |(T(x), f)|$, $x \in X$, $f \in X^*$. Under this topology B is a locally convex topological vector space in which multiplication is separately continuous. For any Banach algebra A, we shall denote the weak operator topology on $M_l(A)$ by τ_l and on $M_r(A)$ by τ_r.

THEOREM 6.1. *Let A be a Banach algebra with a right approximate identity. Then the following statements are equivalent:*

(i) *$\pi(A)$ is a right ideal of $(A^{**}, *)$.*

(ii) *The closed unit ball of $M_r(A)$ is τ_r-compact.*

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds and let $\{G_a\}$ be a τ_r-open cover of $\mathscr{S}(M_r(A))$. We may clearly assume that each G_a is of the following form: there exist $T_a \in \mathscr{S}(M_r(A))$, $\varepsilon > 0$, $x_1, \ldots, x_m \in A$ and $f_1, \ldots, f_n \in A^*$ such that

$$G_a = \{T \in M_r(A) \colon |((T - T_a)x_i, f_j)| < \varepsilon \text{ for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}.$$

For each a, let $F_a = T_a^{**}(E)$ and let

$$N_a = \{F \in A^{**} \colon |(f_j, \pi(x_i) * (F - F_a))| < \varepsilon \text{ for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}.$$

Each N_a is a w^*-neighborhood of F_a ([9], p. 116).

We show that $\bigcup_a N_a \supseteq \mathscr{S}(A^{**})$. Let $F \in \mathscr{S}(A^{**})$. By Theorem 5.5, $F = T^{**}(E) + G$, where $T \in M_r(A)$ and $G \in N_A$, and, by Corollary 5.4, $\|R_F | \pi(A)\| = \|T\|$; so that $\|T\| \leqslant 1$. Since $\bigcup_a G_a \supseteq \mathscr{S}(M_r(A))$, $T \in G_a$, for some a, and therefore $|((T - T_a)x_i, f_j)| < \varepsilon$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. But

$$(f_j, \pi(x_i) * (F - F_a)) = ((T - T_a)x_i, f_j) \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n).$$

Hence $F \in N_a$ and consequently $\mathscr{S}(A^{**}) \subseteq \bigcup_a N_a$. Now, by the w^*-compactness of $\mathscr{S}(A^{**})$, there exists a finite subfamily, say $N_{a}, \ldots, N_{a_p}$, which covers $\mathscr{S}(A^{**})$. Hence $\mathscr{S}(M_r(A)) \subseteq \bigcup_{i=1}^{p} G_{a_i}$ and therefore $\mathscr{S}(M_r(A))$ is τ_r-compact.

(ii) $\Rightarrow$ (i). Suppose that $\mathscr{S}(M_r(A))$ is τ_r-compact. Let $F \in A^{**}$ such that $A^{**} * F \neq (0)$, $\|F\| \leqslant 1$. Let $\{x_a\}$ be a net in $\mathscr{S}(A)$ such that $\pi(x_a)$ w^*-converges to F. Let $R_a = R_{x_a}$ for all a. Then $\{R_a\} \subset \mathscr{S}(M_r(A))$ and therefore, by the τ_r-compactness of $\mathscr{S}(M_r(A))$, there is a subset, say $\{R_{a'}\}$, which τ_r-converges to an element $T \in \mathscr{S}(M_r(A))$. Let $F^T = T^{**}(E)$. We have

$$(f, \pi(x) * R_{a'}^{**}(E)) \to (f, \pi(x) * F^T) \qquad (x \in A,\ f \in A^*).$$

But $\pi(x) * R_{a'}^{**}(E) = \pi(x) * \pi(x_{a'})$ for all $x \in A$, and

$$(f, \pi(x) * \pi(x_a)) \to (f, \pi(x) * F) \qquad (x \in A,\ f \in A^*).$$

Hence

$$(f, \pi(x) * R_{a'}^{**}(E)) \to (f, \pi(x) * F) \qquad (x \in A,\ f \in A^*).$$

Therefore

$$\pi(x) * F = \pi(x) * F^T \qquad (x \in A).$$

Since $\pi(A) * F^T \subset \pi(A)$, it follows that $\pi(A) * F \subset \pi(A)$ and consequently $\pi(A)$ is a right ideal of $(A^{**}, *)$. This completes the proof.

As a consequence of Theorem 6.1 we have the following characterization of duality in a B^*-algebra.

THEOREM 6.2. *Let A be a B^*-algebra. Then the following statements are equivalent:*

(i) *A is dual.*

(ii) *$\mathscr{S}(M_r(A))$ is τ_r-compact.*

(iii) *$\mathscr{S}(M_l(A))$ is τ_l-compact.*

(iv) *Every $F \in A^{**}$ is of the form $F = T^{**}(E)$ for $T \in M_r(A)$.*

(v) *There exists an isometric isomorphism of $M(A)$ onto A^{**} which maps D_A onto $\pi(A)$, where $D_A = \{(L_a, R_a) \colon a \in A\}$.*

Proof. We observe that the map $S \to S^{\#}$ takes $M_l(A)$ onto $M_r(A)$ and the map $T \to T^{\#}$ takes $M_r(A)$ onto $M_l(A)$ with $\|S^{\#}\| = \|S\|$ and $\|T^{\#}\| = \|T\|$. Using the fact that A is dual if and only if $\pi(A)$ is a two-sided ideal of $(A^{**}, *)$ ([20], p. 533, Theorem 5.1), the equivalence of statements (i)–(v) follows at once. This completes the proof.

The next theorem gives us an example of a Banach algebra with an approximate identity which is Arens regular and which is not a B^*-algebra.

THEOREM 6.3. *Let X be a uniformly convex Banach space with an unconditionally monotone basis, and let A be the uniform closure of the algebra F of all linear operators of finite rank on X into itself. Then A has an approximate identity and $\mathscr{S}(M_l(A))$ and $\mathscr{S}(M_r(A))$ are compact in their respective weak operator topologies.*

Proof. By [23], p. 109, Theorem 1, X is reflexive and, by [23], p. 213, Theorem 7, the basis is shrinking. Since X has the property (F_0), [4], p. 165, Lemma 12 implies that A is the algebra of all compact linear operators on X. (See also the proof [10], p. 553, Proposition 2.4.) Since X has a basis, every $f \in A^*$ can be represented by a matrix [10]. Clearly, a matrix with a finite number of non-zero entries is an operator of finite rank. Now, by [10], p. 555, Proposition 3.1, $B(X) = A^{**}$ (in the sense that there exists a linear isometric map from $B(X)$ onto A^{**} such that each $a \in A$ is taken onto its usual image under the canonical map $\pi \colon A \to A^{**}$)

if and only if the set of matrices with finite number of non-zero entries in A^* is a dense linear subspace of A^*. Since X is uniformly convex, [10], p. 557, Theorem 3.2 gives us $B(X) = A^{**}$. Moreover, since A is the algebra of compact linear operators on X, by [10], p. 560, Theorem 4.1, A is Arens regular and, by [10], p. 561, Theorem 4.2, the Arens products coincide with operator multiplication in $B(X)$. Thus, in particular, A has an approximate identity. Since A is a two-sided ideal of $B(X)$, $\pi(A)$ is a two-sided ideal of $(A^{**}, *)$. Application of Theorem 6.1 and its left-hand version completes the proof.

COROLLARY 6.4. *Let X be a uniformly convex Banach space with an unconditionally monotone basis, and let $B(X)$ be the algebra of all bounded linear operators on X. Then the closed unit ball of $B(X)$ is compact in the weak operator topology on $B(X)$.*

Proof. Let A be as above, and let e be a minimal idempotent in A. Then $(Ae)^* = eA$ ([4], p. 161) and there exist $\eta \in X$, $\varphi \in X^*$, $\|\eta\| = \|\varphi\| = 1$, such that $e = \eta \otimes \varphi$, where $(\eta\otimes\varphi)(\xi) = (\xi, \varphi)\eta$, $\xi \in X$; moreover, $Ae = \{\xi\otimes\varphi\colon \xi \in X\}$ and $eA = \{\eta\otimes\psi\colon \psi\in X^*\}$ ([17], p. 67). Now for $a \in B(X)$, $\xi\in X$, $\psi \in X^*$, we have $\big(a(\xi), \psi\big)e = yax$, where $x = \xi\otimes\varphi$ and $y = \eta\otimes\psi$. By the proof above, every $T\in M_1(A)$ is of the form $T = L_a|A$, for some $a\in B(X)$, and $\|T\| = \|a\|$. Hence if $\{U_a\}$ is an open cover of $\mathscr{S}\big(B(X)\big)$ in the weak operator topology τ_X on $B(X)$, then the image of $\{U_a\}$ by the map $a \to L_a|A$ is an open cover of $\mathscr{S}\big(M_1(A)\big)$. Since $\mathscr{S}\big(M_1(A)\big)$ is τ_1-compact, it follows that $\mathscr{S}\big(B(X)\big)$ is τ_X-compact.

COROLLARY 6.5. *The closed unit ball of $B(H)$, for any Hilbert space H, is compact in the weak operator topology on $B(H)$.*

7. Groups of isometric multipliers.

LEMMA 7.1. *Let A be a Banach algebra and $G_r(A)$ be the subset of $M_r(A)$ consisting of all elements which are isometric and onto. If the closed unit ball of $M_r(A)$ is compact in the weak operator topology on $M_r(A)$, then $G_r(A)$ is a compact topological group.*

Proof. We follow [3], p. 207. Clearly, $G_r(A)$ is a group and, since multiplication in $M_r(A)$ is separately continuous in τ_r, $G_r(A)$ is a topological semigroup. Suppose that $\mathscr{S}\big(M_r(A)\big)$ is τ_r-compact and let T belong to the τ_r-closure of $G_r(A)$. Since $\mathscr{S}\big(M_r(A)\big)$ is τ_r-closed, $\|T\| \leqslant 1$. Let $\{T_a\}$ be a net in $G_r(A)$ which τ_r-converges to T. By the τ_r-compactness of $\mathscr{S}\big(M_r(A)\big)$, the net $\{T_a^{-1}\}$ contains a subnet $\{T_{a'}^{-1}\}$ which τ_r-converges to $S\in\mathscr{S}\big(M_r(A)\big)$. Clearly, $ST = TS = E$, where E is the identity of $M_r(A)$. Since $\|S\|\leqslant 1$ and $\|T\|\leqslant 1$, S and T are isometries and so $T\in G_r(A)$. Thus $G_r(A)$ is a τ_r-closed subset of $\mathscr{S}\big(M_r(A)\big)$ and therefore τ_r-compact. Application of [8], p. 124, Theorem 2 completes the proof.

THEOREM 7.2. *Let A be a Banach algebra with a right approximate identity. If $\pi(A)$ is a right ideal of $(A^{**}, *)$, then $G_r(A)$ is a τ_r-compact group.*

Proof. This follows from Theorem 6.1 and Lemma 7.1.

THEOREM 7.3. *Let A be a Banach algebra with left and right approximate identities. Let $G(A)$ be the subset of $M(A)$ consisting of elements (S, T) such that either S or T is an onto isometry. Then the following statements are true:*

(i) *$G(A)$ is a group and, for every $(S, T)\in G(A)$, both S and T are onto isometries.*

(ii) *If $\pi(A)$ is a two-sided ideal of $(A^{**}, *)$, then $G(A)$ is a compact group in the product topology $\tau_1\times\tau_r$.*

Proof. (i) This follows from Theorem 5.6.

(ii) This follows from Theorem 7.2 and its left-hand version, and [9], p. 116, Lemma 1.5.

As an application of the groups discussed above, we give the following characterization of duality for a B^*-algebra. We observe that if B is a commutative B^*-algebra, then $G_1(B) = G_r(B) = G(B)$. We recall that the algebra of multipliers on B is isometrically isomorphic to $C_b(\Omega_B)$, the algebra of all bounded continuous complex-valued functions on the carrier space Ω_B of B ([24]).

THEOREM 7.4. *Let A be a B^*-algebra. Then A is dual if and only if the following conditions hold:*

(i) *$G_r(A)$ is τ_r-compact.*

(ii) *$G(B)$ separates the points of the carrier space Ω_B, for every maximal commutative $*$-subalgebra B of A.*

Proof. Suppose that A is dual. Then, by Theorem 6.2 and Lemma 7.1, $G_r(A)$ is τ_r-compact and, by [16], p. 179, Theorem 1, Ω_B is discrete. Therefore, representing the elements of $G(B)$ as bounded continuous functions on Ω_B, it is easy to see that $G(B)$ separates the points of Ω_B, for every maximal commutative $*$-subalgebra B of A.

Conversely, suppose (i) and (ii) hold. Since B is convex and norm-closed, it is weakly closed. From this it follows that $G(B)$ is compact in the weak operator topology on $M(B)$. Since $G(B)$ separates the points of Ω_B by [3], p. 209, Theorem 5, Ω_B is discrete and so A is dual by [16], p. 179, Theorem 1.

As another application of the group $G_r(A)$, we have the following:

THEOREM 7.5. *Let G be a locally compact group and $L(G)$ the group algebra of G. Then G is compact if and only if $\pi\big(L(G)\big)$ is a right ideal of $(L^{**}(G), *)$, where $L^{**}(G)$ is the second conjugate space of $L(G)$.*

Proof. Let G have a right invariant Haar measure μ. If G is compact then $L(G)$ is a dual algebra ([30], p. 699, Theorem 15) and therefore $\pi(L(G))$ is a two-sided ideal of $(L^{**}(G), *)$ ([24], p. 82, Theorem 3.3) Conversely, suppose that $\pi(L(G))$ is a right ideal of $(L^{**}(G), *)$. Since $L(G)$ contains a bounded approximate identity, $G_r(L(G))$ is τ_r-compact. By [32], p. 254. Theorem 3, $T \in G_r(L(G))$ if and only if $T = \lambda R_g$, where λ is a scalar with $|\lambda| = 1$ and R_g is the right translation operator on $L(G)$ given by some $g \in G$. Thus G may be identified as a subset of $G_r(L(G))$. Let τ be the given topology on G. Let $\{g_a\}$ be a net in G which τ-converges to $g \in G$. Now, if $x \in L(G)$ then $R_s(x)$, as an element of $L(G)$, is a continuous function of $s \in G$ ([31], p. 118, 30c). Therefore,

$$\int f(h)x(hg_a^{-1})\mu(dh) \to \int f(h)x(hg^{-1})\mu(dh)$$

for all $x \in L(G)$ and $f \in L_\infty(G)$, where $L_\infty(G)$ is the set of all essentially bounded measurable complex-valued functions on G. That is, R_{g_a} τ_r-converges to R_g. Thus $g \to R_g$ is continuous and, consequently, the topology on G induced by τ_r is weaker than τ.

Let τ_s be the strong operator topology on $M_r(L(G))$. Let V be a compact neighborhood of the identity $e \in G$, and let x be the characteristic function of V. Then $x \in L(G)$ and $V = \{g\colon |x(g)-x(e)| < 1\}$. Let $W = \{T \in G_r(L(G))\colon \|T(x)-R_e(x)\| < 1\}$; W is a τ_s-neighborhood of the identity in $G_r(L(G))$. If $R_g \in W$ then $g \in V^{-1}$, so that, if we identify G as a subset of $G_r(L(G))$, we obtain $W \cap G \subset V^{-1}$. Since the τ-compact neighborhoods V of e form a basis for the τ-neighborhoods of e, the topology on G induced by τ_s is stronger than τ. As $\tau_r = \tau_s$ on $G_r(L(G))$, we have $\tau = \tau_r$ on G. Thus if $\{R_{g_a}\}$ is a net converging to $T \in G_r(L(G))$ then $T = R_g$, for some $g \in G$. That is, G is a closed subset of $G_r(L(G))$ and hence compact.

8. Multipliers on modular annihilator A^*-algebras.

LEMMA 8.1. *Let A be a modular annihilator A^*-algebra. Then every left (right) multiplier on A is linear.*

Proof. Let T be a left multiplier on A. Let $a, b \in S_A$ and λ, μ scalars. Since $a, b \in S_A$, a, b belong to a right ideal of finite order and therefore, by [2], p. 286, Lemma 2.3, there exists a self-adjoint idempotent $e \in A$ such that $a = ea$ and $b = eb$. Hence

$$\begin{aligned} T(\lambda a+\mu b) = T\big(e(\lambda a+\mu b)\big) &= T(e)(\lambda a+\mu b) \\ &= \lambda T(a)+\mu T(b), \end{aligned}$$

which shows that T is linear on S_A. Thus if $a, b \in A$, $\lambda, \mu \in C$ and e is a minimal idempotent in A, then

$$\big(T(\lambda a+\mu b)-\lambda T(a)-\mu T(b)\big)e = 0.$$

Since every $x \in S_A$ can be expressed in the form

$$x = e_1x_1+\ldots+e_nx_n,$$

where e_i is a minimal idempotent in A, $x_i \in A$, $i \leqslant i \leqslant n$, and since $l(S_A) = (0)$, it follows that $T(\lambda a+\mu b) = \lambda T(a)+\mu T(b)$, for all $a, b \in A$ and $\lambda, \mu \in C$. Thus T is linear. Similarly, if T is a right multiplier on A then T is linear. This completes the proof.

The following lemma is actually the "Main Boundedness Theorem" of Bade and Curtis ([26], p. 592, Theorem 2.1, [27], p. 285, Theorem 4.1) applied to multipliers.

LEMMA 8.2. *Let T be a linear left (right) multiplier on a Banach algebra A. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A such that $x_ny_m = 0$, $n \neq m$. Then there exists a constant $M > 0$ such that*

$$\|T(x_ny_n)\| \leqslant M\|x_n\|\,\|y_n\|, \qquad n \geqslant 1. \tag{1}$$

Proof. Let T be a linear right multiplier on A. Suppose that there does not exist $M > 0$ for which (1) is true. Then there exists a doubly indexed subsequence of distinct elements $\{u_{ij}\}$ of $\{x_n\}$ and a corresponding subsequence $\{v_{ij}\}$ of $\{y_n\}$ such that

$$\|T(u_{ij}v_{ij})\| > 4^{i+j}\|u_{ij}\|\,\|v_{ij}\| \quad \text{for all } i, j.$$

Let

$$h_i = \sum_{k=1}^{\infty}\frac{1}{2^k}\,\frac{v_{ik}}{\|v_{ik}\|} \in A, \qquad i \geqslant 1.$$

Then

$$u_{pq}h_i = 0 \quad \text{if} \quad p \neq i$$

and

$$u_{iq}h_i = \frac{1}{2^q}\cdot\frac{u_{iq}v_{iq}}{\|v_{iq}\|}.$$

We observe that $T(h_i) \neq 0$, for $i \geqslant 1$, since

$$u_{iq}T(h_i) = T(u_{iq}h_i) = \frac{1}{2^q}\cdot\frac{1}{\|v_{iq}\|}T(u_{iq}v_{iq})$$

and

$$\|T(u_{iq}v_{iq})\| > 4^{i+j}\|u_{iq}\|\,\|v_{iq}\| > 0.$$

Choose $\{j_i\}$ a sequence of distinct positive integers such that $2^{j_i} > \|T(h_i)\|$, $i \geqslant 1$. Let

$$h = \sum_{i=1}^{\infty}\frac{1}{2^i}\,\frac{u_{ij_i}}{\|u_{ij_i}\|} \in A.$$

Then

$$hh_k = \frac{1}{2^k}\frac{u_{kj_k}}{\|u_{kj_k}\|}\cdot\frac{1}{2^{j_k}}\frac{v_{kj_k}}{\|v_{kj_k}\|}$$

and

$$\|hT(h_k)\| = \|T(hh_k)\|$$
$$= \frac{1}{2^{k+j_k}}\frac{\|T(u_{kj_k}v_{kj_k})\|}{\|u_{kj_k}\|\,\|v_{kj_k}\|}$$
$$> \frac{1}{2^{k+j_k}}\cdot 4^{k+j_k} = 2^{k+j_k} > 2^k\|T(h_k)\|, \qquad k\geqslant 1.$$

Thus $\|h\| = \infty$ which is clearly a contradiction since $h\in A$ and $\|h\|<\infty$.

In a similar way we can show that (1) holds for a linear left multiplier.

THEOREM 8.3. *Let A be a modular annihilator A^*-algebra. Then every left (right) multiplier on A is a continuous linear operator.*

Proof. Let T be a right multiplier on A. By Lemma 8.1, T is linear. We shall first show that T is continuous on every minimal right ideal of A. Let I be a minimal right ideal, $I = eA$, where e is a minimal idempotent. Let $J = \{x\in A\colon y\to T(xy)$ is continuous on $A\}$. Then one can verify that J is a two-sided ideal of A ([29], p. 153), and that an idempotent g is in J if and only if $T\,|\,gA$ is continuous. We can now follow the argument given in the proof of [28], p. 308, Theorem 2.1 to show that $T\,|\,eA$ is continuous. (We replace the homomorphism ν there by T.)

To show that T is continuous on all of A, it suffices to show that the graph of T is closed. Let $\{x_n\}$ be a sequence in A, $x, y\in A$ be such that $x_n\to x$ and $T(x_n)\to y$. Let e be a minimal idempotent in A. Then $ex_n\to ex$ and $T(ex_n) = eT(x_n)\to ey$. By the continuity of T on eA, we have $T(ex_n)\to T(ex) = eT(x)$. Hence $e\big(T(x)-y\big) = 0$ for all minimal idempotents e in A, so that $z\big(T(x)-y\big) = 0$ for all $z\in S_A$. Since $r(S_A) = (0)$, we obtain that $T(x) = y$. This shows that the graph of T is closed and hence that T is continuous. (See also [18], p. 145.)

Similarly we can show that every left multiplier on A is continuous.

Remark. We note that Lemma 8.1 and Theorem 8.3 hold for any semisimple modular annihilator Banach algebra A, since $l(S_A) = r(S_A) = (0)$.

THEOREM 8.4. *Let A be a modular annihilator A^*-algebra which is an ideal of its completion $\mathfrak{A}$. Then every double multiplier (S, T) on A has a unique extension to a double multiplier on $\mathfrak{A}$.*

Proof. By [21], p. 422, Lemma 3.1, $A\cap J\neq 0$, for every non-zero two-sided ideal J of A. Therefore, by the argument in the proof of [15], p. 31, Theorem 18, the norm

$$\|x\|_1 = \sup\{\|xy\|\colon y\in A,\ \|y\|\leqslant 1\} \qquad (x\in A)$$

is equivalent to the auxiliary norm $|\cdot|$ on A. By Lemma 3.1, every double multiplier (S, T) on A belongs to $M(A)$. Let $x\in\mathfrak{A}$ and let $\{x_n\}$ be a sequence in A such that $|x_n - x|\to 0$. We have

$$(x_n - x_m)S(y) = T(x_n - x_m)y,$$

for all $y\in A$ and all positive integers m, n. But

$$\begin{aligned}\|T(x_n)-T(x_m)\|_1 &= \sup\{\|(T(x_n)-T(x_m))y\|\colon y\in A,\ \|y\|\leqslant 1\}\\ &\leqslant \sup\{k\,|x_n-x_m|\,\|S(y)\|\colon y\in A,\ \|y\|\leqslant 1\}\\ &\leqslant k\,|x_n-x_m|\,\|S\|.\end{aligned}$$

Hence $\{T(x_n)\}$ is a Cauchy sequence in $\mathfrak{A}$. By the completeness of $\mathfrak{A}$, there exists $z\in\mathfrak{A}$ such that $|T(x_n)-z|\to 0$. Let $T'(x) = z$. It is easy to see that the value $T'(x)$ is independent of the sequence $\{x_n\}$. This gives us a unique extension T' of T to $\mathfrak{A}$. Similarly, we obtain a unique extension S' of S to $\mathfrak{A}$. It follows quite readily that $xS'(y) = T'(x)y$ for all $x, y\in\mathfrak{A}$. By Lemma 3.1, $(S', T')\in M(\mathfrak{A})$.

COROLLARY 8.5. *Let A be a modular annihilator A^*-algebra which is an ideal of its completion $\mathfrak{A}$. Then every $T\in M_l(A)\cap M_r(A)$ has a unique extension T' to $\mathfrak{A}$ and $T'\in M_l(\mathfrak{A})\cap M_r(\mathfrak{A})$.*

THEOREM 8.6. *Let A be a modular annihilator A^*-algebra with an approximate identity, and let $\mathfrak{A}$ be the completion of A. Then for every left (right) multiplier T on A there exists a left (right) multiplier T' on $\mathfrak{A}$ such that $T = T'\,|\,A$.*

Proof. Let $\pi_{\mathfrak{A}}$ denote the canonical map of $\mathfrak{A}$ into $\mathfrak{A}^{**}$. Since $|x| \leqslant \beta\|x\|$, $x\in A$ ([17], p. 187, Corollary (4.1.16)), for each $g\in\mathfrak{A}^*$, $g_A = g\,|\,A\in A^*$. For every $F\in A^{**}$, let $F'\in\mathfrak{A}^{**}$ be given by $F'(g) = F(g_A)$ for all $g\in\mathfrak{A}^*$. (See [24], p. 82.) Then $\big(F*\pi(x)\big)' = F'*\pi_{\mathfrak{A}}(x)$, for all $F\in A^{**}$, $x\in A$. Since A is dense in $\mathfrak{A}$ and $\mathfrak{A}$ is dual, we have

$$F'*\pi_{\mathfrak{A}}(x)\in\pi_{\mathfrak{A}}(\mathfrak{A}) \qquad (x\in\mathfrak{A},\ F\in A^{**}).$$

Now if $T\in M_l(A)$ and $F = T^{**}(E)$, then $T^{**}\big(\pi(x)\big) = F*\pi(x)$, for all $x\in A$. Hence if $T'\in M_l(\mathfrak{A})$ is given by $T'^{**}\big(\pi_{\mathfrak{A}}(x)\big) = F'*\pi_{\mathfrak{A}}(x)$ for all $x\in\mathfrak{A}$, then $T = T'\,|\,A$. Similarly we can show that if $T\in M_r(A)$ then $T = T'\,|\,A$, for some $T'\in M_r(\mathfrak{A})$. This completes the proof.

Let A and $\mathfrak{A}$ be as in Theorem 8.6. If $A \neq \mathfrak{A}$ then the mapping $T \to T'$ cannot be onto $M_r(\mathfrak{A})$. For otherwise, A would be a two-sided ideal of $\mathfrak{A}$ and consequently the norms $\|\cdot\|$ and $|\cdot|$ would be equivalent which would give $A = \mathfrak{A}$, a contradiction. (See the proof of Theorem 8.4.)

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