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A criterion for compositions of (p, q)-absolutely summing operators to be compact

by

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Abstract. If $(S_j)_{j=1,2,\ldots,M}$ are $(p_j, 2)$-absolutely summing operators, and $\sum_{j=1}^{M} p_j^{-1} > 1/2$, then the composition $S_M S_{M-1} \ldots S_1$ is compact.

Results. Let $\infty > p \geqslant q \geqslant 1$ and let X, Y be normed linear spaces. Recall that a bounded linear operator S: $X \to Y$ is said to be *(p, q)-absolutely summing* if there exists a positive constant C such that for all finite sequences $x_1, x_2, \ldots, x_n$ in X $(n = 1, 2, \ldots)$

$$\Big(\sum_{j=1}^{n} \|Sx_j\|^p\Big)^{1/p} \leqslant C \sup_{x^* \in X^*, \|x^*\| \leqslant 1} \Big(\sum_{j=1}^{n} |x^*(x_j)|^q\Big)^{1/q}.$$

The greatest lower bound of the constants C satisfying the above inequality is denoted by $\pi_{p,q}(S)$.

The main result of the present paper is:

THEOREM 1. *Let M be a positive integer, let X_k be Banach spaces $(k = 0, 1, \ldots, M)$, and let S_k: $X_{k-1} \to X_k$ be $(p_k, 2)$-absolutely summing operators $(2 \leqslant p_k < \infty$ for $k = 1, 2, \ldots, M)$. Then the condition*

$$\sum_{k=1}^{M} p_k^{-1} > 2^{-1}$$

implies the compactness of the composition $S_M S_{M-1} \cdot \ldots S_1$.

Combining Theorem 1 with the well-known fact (cf. Kwapień [4]) that if T: $X \to Y$ is a (p, q)-absolutely summing operator, then T is also $(\tilde{p}, \tilde{q})$-absolutely summing for every pair $(\tilde{p}, \tilde{q})$ such that $p^{-1} - q^{-1} = \tilde{p}^{-1} - \tilde{q}^{-1}$ and $\tilde{p} > p$, we get

COROLLARY 1. *Let M be a positive integer, let X_k be Banach spaces $(k = 0, 1, \ldots, M)$, let S_k: $X_{k-1} \to X_k$ be (p_k, q_k)-absolutely summing operators, and let $0 \leqslant q_k^{-1} - p_k^{-1} < 2^{-1}$, $1 \leqslant q_k \leqslant 2$ for $k = 1, \ldots, M$. Then the condition*

$$\sum_{k=1}^{M} 2^{-1} + p_k^{-1} - q_k^{-1} > 2^{-1}$$

implies the compactness of the composition $S_M S_{M-1} \ldots S_1$. *In particular, the composition of* $(p_k, 1)$*-absolutely summing operators with* $1 \leqslant p_k < 2$ *for* $k = 1, 2, \ldots, M$ *is compact whenever*

$$\sum_{k=1}^{M} p_k^{-1} > \frac{M+1}{2}.$$

An immediate consequence of Corollary 1 is the following generalization of the Dvoretzky–Rogers theorem:

COROLLARY 2. *Let* X *be a Fréchet space such that for some pair* (p, q) *with* $0 \leqslant q^{-1} - p^{-1} < 2^{-1}$ *for every continuous pseudonorm* $|\cdot|$ *on* X *there exists a continuous pseudonorm* $\|\cdot\|$ *on* X *such that the injection* $(X, \|\cdot\|) \to (X, |\cdot|)$ *is* (p, q)*-absolutely summing. Then* X *is a Schwartz space.*

Here by $(X, |\cdot|)$ (resp. $(X, \|\cdot\|)$) we denote the normed linear space which is a quotient of X by the $\{x \in X: |x| = 0\}$ (resp. $\{x \in X: \|x\| = 0\}$) with the norm induced by $|\cdot|$ (resp. by $\|\cdot\|$). For the definition of a Schwartz space cf. [3].

Let us observe that Theorem 1 as well as Corollary 1 is the best possible in the following sense:

(*) Given a positive integer M and (p_k, q_k) such that $0 \leqslant q_k^{-1} - p_k^{-1} < 2^{-1}$, $1 \leqslant q_k \leqslant 2$ for $k = 1, 2, \ldots, M$ and $\sum_{k=1}^{M} 2^{-1} + p_k^{-1} - q_k^{-1} \leqslant 2^{-1}$ there exist Banach spaces $X_0, X_1, \ldots, X_M$ and (p_k, q_k)-absolutely summing operators $S_k: X_{k-1} \to X_k$ $(k = 1, 2, \ldots, M)$ such that the composition $S_M S_{M-1} \ldots S_1$ is not compact.

To see (*) observe first that, by the result of Kwapień [4] quoted above, it is enough to consider the case $q_k = 1$,

$$\sum_{k=1}^{M} p_k^{-1} \leqslant \tfrac{1}{2}(M+1).$$

The operators S_k can be constructed as follows. For $1 \leqslant a \leqslant b < \infty$, let $I_{a,b}$ denote the natural injection from l^a into l^b. G. Bennett [1] has shown that for $1 \leqslant a \leqslant b \leqslant 2$, $I_{a,b}$ is $(p, 1)$-absolutely summing, where $1/p = 1/a + 1/2 - 1/b$.

Let $p_1, \ldots, p_M \in [1, 2]$ such that $\sum_{k=1}^{M} p_k^{-1} \leqslant \frac{1}{2}(M+1)$ and define a sequence $a_0, a_1, \ldots, a_M$ by induction:

$$a_0 = 1; \quad 1/a_k = 1/a_{k-1} + 1/2 - 1/p_k, \quad k = 1, 2, \ldots, M.$$

It is easily seen that $a_k \leqslant a_{k+1}$, and $a_M \leqslant 2$.

Let us set $S_k = I_{a_{k-1}, a_k}$. Then S_k is $(p_k, 1)$-absolutely summing, and $S_M S_{M-1} \ldots S_1 = I_{a_0, a_M}$ is not compact.

Finally observe that the condition $0 \leqslant q_k^{-1} - p_k^{-1} < 2^{-1}$ in Corollary 1 is also necessary because the identity operator on l^2 is $(p, 1)$-absolutely summing for every $p \geqslant 2$.

Proof of Theorem 1. We shall employ the following notation. If (y_n) is a basis for a Banach space Y then

$$\operatorname{uc}(y_n) = \sup_m \sup_{|a_j| \leqslant 1} \sup_{\|\sum_{j=1}^{m} t_j x_j\| = 1} \Big\| \sum_{j=1}^{m} a_j t_j x_j \Big\|.$$

If X and Y are Banach spaces with bases (x_n) and (y_n), respectively, then the linear operator $T: X \to Y$ such that $Tx_n = y_n$ for all n is called the *natural injection from* X *into* Y *with respect to the bases* (x_n) *and* (y_n) or briefly a *natural injection*. To indicate that $T: X \to Y$ is a natural injection with respect to the bases (x_n) and (y_n) we shall write

$$T: \big(X, (x_n)\big) \to \big(Y, (y_n)\big).$$

If N is a fixed positive integer then l_N^p denotes the space of all scalar sequences $(t_1, t_2, \ldots, t_N)$ with the norm $\|(t_1, t_2, \ldots, t_N)\|_p = \big(\sum_{j=1}^{N} |t_j|^p\big)^{1/p}$ $(1 \leqslant p < \infty)$. The unit vector basis of l_N^p will be denoted by (e_n). By $I_{p,q}$ we shall denote the natural injection $\big(l_N^p, (e_n)\big) \to \big(l_N^q, (e_n)\big)$.

PROPOSITION 1. *Let* m *be a non-negative integer. Let* $X_0, X_1, \ldots, X_{m+1}$ *be Banach spaces, let* $S_k: X_{k-1} \to X_k$ *be bounded linear operators* $(k = 1, 2, \ldots, m+1)$ *such that the composition* $S_{m+1} S_m \ldots S_1$ *is not compact. Then there exists* $\delta > 0$ *such that for every positive integer* N *there exists for* $k = 0, 1, \ldots, m+1$ *an* N*-dimensional subspace* Y_k *of* X_k *with a basis* $(y_n^{(k)})$ *such that*

(i) $\sup_n \|y_n^{(0)}\| \leqslant 4$, $\inf_n \|y_n^{(m+1)}\| \geqslant \delta$, $\operatorname{uc}(y_n^{(k)}) \leqslant 3$ $(k = 0, 1, \ldots, m+1)$.

(ii) *the natural injection* $T_k: \big(Y_{k-1}, (y_n^{(k-1)})\big) \to \big(Y_k, (y_n^{(k)})\big)$ *coincides with the restriction of* S_k *to* Y_{k-1} $(k = 1, 2, \ldots, m+1)$.

Proof. Since $S_{m+1} S_m \ldots S_1$ is not compact, there exist an $\eta > 0$ and a sequence $(z_r^{(0)})$ in the unit ball of X_0 such that if $z_r^{(k)} = S_k S_{k-1} \ldots S_1(z_r^{(0)})$ for $k = 1, 2, \ldots, m+1$ then $\|z_{r'}^{(k)} - z_{r''}^{(k)}\| \geqslant \eta$ whenever $r' \neq r''$.

Now we fix a (large) integer M and we apply $m+2$ times the Brunel–Sucheston result which is based on the Ramsey combinatorial theorem (cf. [2], [5]) to extract an increasing infinite sequence (n_r) of integers such that if $u_r^{(k)} = z_{n_r}^{(k)}$ for all r and for $k = 0, 1, \ldots, m+1$, then the sequences $u_r^{(k)}$ are "almost M-subsymmetric" precisely:

$$\tfrac{9}{10} \Big\| \sum_{j=1}^{M} t_j u_j^{(k)} \Big\| \leqslant \Big\| \sum_{j=1}^{M} t_j u_{r_j}^{(k)} \Big\| \leqslant \tfrac{11}{10} \Big\| \sum_{j=1}^{M} t_j u_j^{(k)} \Big\|$$

for all scalars $t_1, t_2, \ldots, t_M$ for all sequences of the indices $r_1 < r_2 < \ldots < r_M$ and for $k = 0, 1, \ldots, M$. Now if N is given and if we choose $M = M(N)$ sufficiently large then, by a result of Brunel and Sucheston [2], the differences

$$y_n^{(k)} = u_{2n-1}^{(k)} - u_{2n}^{(k)} \quad (n = 1, 2, \ldots, N)$$

form for $k = 1, 2, \ldots, m+1$ a basic sequence in X_k of length N with $\mathrm{uc}\big((y_n^{(k)})\big) \leqslant 3$. We define Y_k to be the N-dimensional space generated by the $y_n^{(k)}$'s. Clearly, the spaces Y_k with the bases $y_n^{(k)}$ have the desired properties for $\delta = \eta/2$.

Proposition 1 allows us to reduce the problem of compactness of a composition of $(p, 2)$-summing operators to the case of natural injections between spaces with unconditional bases. Our next aim is a further reduction to the case of natural injections between l^r-spaces. We begin with two lemmas which are similar to an argument of Tzafriri [5].

LEMMA 1. *Let $1 \leqslant r < 2 \leqslant p$. Let N be a positive integer, let Y be an N-dimensional Banach space with a basis (y_n), and let $T\colon (l_N^r, (e_n)) \to (Y, (y_n))$ be the natural injection. Then*

$$\sup_{n_1 < n_2 < \ldots < n_j} \|y_{n_1} + y_{n_2} + \ldots + y_{n_j}\| \leqslant A j^{1/r - 1/p}$$

for $j = 1, 2, \ldots, N$, where

$$A = \begin{cases} \pi_{p,2}(T)\,\mathrm{uc}(y_n) & \text{for complex Banach spaces,} \\ 2\pi_{p,2}(T)\,\mathrm{uc}(y_n) & \text{for real Banach spaces.} \end{cases}$$

Proof. 1. *The case of complex spaces.* Fix $j \leqslant N$ and $n_1 < n_2 < \ldots < n_N$. Let $(a_{\nu,\mu})_{1 \leqslant \nu, \mu \leqslant j}$ be the unitary matrix defined by

$$a_{\nu,\mu} = j^{-1/2} e^{2\pi i \nu\mu/j} \quad \text{for} \quad 1 \leqslant \nu, \mu \leqslant j. \tag{1}$$

Let us set

$$z_\nu = \sum_{\mu=1}^{j} a_{\nu,\mu} e_{n_\mu} \quad \text{for} \quad 1 \leqslant \nu \leqslant j.$$

The unitarity of the matrix $(a_{\nu,\mu})$ and the inequality $\|x\|_r \leqslant j^{1/r-1/2}\|x\|_2$ for all $x \in l_j^r$ $(1 \leqslant r \leqslant 2)$ yield

$$\Big\|\sum_{\nu=1}^{j} t_\nu z_\nu\Big\|_r \leqslant j^{1/r-1/2}\Big\|\sum_{\nu=1}^{j} t_\nu z_\nu\Big\|_2 = j^{1/r-1/2}\Big(\sum_{\nu=1}^{j} |t_\nu|^2\Big)^{1/2}$$

for all scalars $t_1, t_2, \ldots, t_j$. Thus, by definition of $\pi_{p,2}(T)$,

$$\Big(\sum_{\nu=1}^{j} \|T(z_\nu)\|^p\Big)^{1/p} \leqslant \pi_{p,2}(T) j^{1/r-1/2}.$$

Since $|a_{\nu,\mu}| = j^{-1/2}$ for all $\nu, \mu = 1, 2, \ldots, j$, we have for $\nu = 1, 2, \ldots j$

$$\Big\|\sum_{\mu=1}^{j} y_{n_\mu}\Big\| \leqslant \mathrm{uc}(y_n) j^{1/2} \|T(z_\nu)\|.$$

Thus

$$\Big\|\sum_{\mu=1}^{j} y_{n_\mu}\Big\| \leqslant \mathrm{uc}(y_n) j^{1/2-1/p}\Big(\sum_{\nu=1}^{j}\|T(z_\nu)\|^p\Big)^{1/p} \leqslant \mathrm{uc}(y_n)\pi_{p,2}(T) j^{1/r-1/p}.$$

2. *The case of real spaces.* Let us set

$$u_\nu = \sum_{\mu=1}^{j} \mathrm{Re}(a_{\nu,\mu}) e_{n_\mu}, \quad v_\nu = \sum_{\mu=1}^{j} \mathrm{Im}(a_{\nu,\mu}) e_{n_\mu} \quad \text{for} \quad 1 \leqslant \nu \leqslant j,$$

where $a_{\nu,\mu}$ are defined by (1). Then for reals $s_1, s_2, \ldots, s_j$ we have

$$\Big\|\sum_{\nu=1}^{j} s_\nu u_\nu\Big\|_2 = \Big\|\sum_{\mu=1}^{j} \mathrm{Re}\Big(\sum_{\nu=1}^{j} s_\nu a_{\nu,\mu}\Big) e_{n_\mu}\Big\|_2 \leqslant \Big\|\sum_{\mu=1}^{j}\sum_{\nu=1}^{j} s_\nu a_{\nu\mu} e_{n_\mu}\Big\|_2$$
$$= \Big\|\sum_{\nu=1}^{j} s_\nu z_\nu\Big\|_2 = \Big(\sum_{\nu=1}^{j} s_\nu^2\Big)^{1/2}.$$

Hence, for all real scalars $s_1, s_2, \ldots, s_j$,

$$\Big\|\sum_{\nu=1}^{j} s_\nu u_\nu\Big\|_r \leqslant j^{1/r-1/2}\Big\|\sum_{\nu=1}^{j} s_\nu u_\nu\Big\|_2 \leqslant j^{1/r-1/2}\Big(\sum_{\nu=1}^{j} s_\nu^2\Big)^{1/2}$$

and similarly,

$$\Big\|\sum_{\nu=1}^{j} s_\nu v_\nu\Big\|_r \leqslant j^{1/r-1/2}\Big(\sum_{\nu=1}^{j} s_\nu^2\Big)^{1/2}.$$

Thus, by definition of $\pi_{p,2}(T)$,

$$\sum_{\nu=1}^{j} \|T(u_\nu)\|^p \leqslant \big(\pi_{p,2}(T) j^{1/r-1/2}\big)^p \quad \text{and} \quad \sum_{\nu=1}^{j} \|T(v_\nu)\|^p \leqslant \big(\pi_{p,2}(T) j^{1/r-1/2}\big)^p.$$

Since $|\mathrm{Re}(a_{\nu,\mu})| + |\mathrm{Im}(a_{\nu,\mu})| \geqslant |a_{\nu,\mu}| = j^{-1/2}$ for all $\nu, \mu = 1, 2, \ldots, j$,

$$\Big\|\sum_{\mu=1}^{j} y_{n_\mu}\Big\| \leqslant j^{1/2}\mathrm{uc}(y_n)\big(\|T(u_\nu)\| + \|T(v_\nu)\|\big)$$
$$\leqslant 2^{1-1/p} j^{1/2}\mathrm{uc}(y_n)\big(\|T(u_\nu)\|^p + \|T(v_\nu)\|^p\big)^{1/p}.$$

Hence:

$$\Big\|\sum_{\mu=1}^{j} y_{n_\mu}\Big\| \leqslant 2^{1-1/p} j^{1/2}\mathrm{uc}(y_n) j^{-1/p}\Big(\sum_{\nu=1}^{j}\big(\|T(u_\nu)\|^p + \|T(v_\nu)\|^p\big)\Big)^{1/p}$$
$$\leqslant 2^{1-1/p} j^{1/2-1/p}\mathrm{uc}(y_n)\cdot 2^{1/p}\pi_{p,2}(T) j^{1/r-1/2}$$
$$= 2 j^{1/r-1/p}\pi_{p,2}(T)\,\mathrm{uc}(y_n).$$

LEMMA 2. *Let Y be an N-dimensional Banach space* ($N = 1, 2, \ldots$) *with a basis* (y_n). *If for some* $\varrho \in (1, \infty)$ *there exists* $A > 0$ *such that*

$$\sup_{n_1<n_2<\ldots<n_j} \Big\| \sum_{\mu=1}^{j} y_{n_\mu} \Big\| \leqslant A j^{1/\varrho} \quad \textit{for} \quad j = 1, 2, \ldots, N,$$

then for every s *with* $1 \leqslant s < \varrho$ *the natural injection* $T\colon (l_N^1, (e_n)) \to (Y, (y_n))$ *admits the factorization*

$$(l_N^1, (e_n)) \xrightarrow{I_{1,s}} (l_N^s, (e_n)) \xrightarrow{V} (Y, (y_n)),$$

i.e.

$$T = VI_{1,s} \quad \textit{and} \quad \|V\| \leqslant \frac{2}{2^{1-s/\varrho}-1} A\,\mathrm{uc}(y_n).$$

Proof. Fix scalars $t_1, t_2, \ldots, t_N$ with $\sum_{n=1}^{N} |t_n|^s = 1$. It is enough to show that

$$\Big\| \sum_{n=1}^{N} t_n y_n \Big\| \leqslant \frac{2}{2^{1-s/\varrho}-1} A\,\mathrm{uc}(y_n).$$

Let us set

$$N_m = \{n\colon 2^{-m+1} \geqslant |t_n| > 2^{-m}\}; \qquad k_m = \text{the cardinality of } N_m.$$

Clearly, $k_m 2^{-ms} \leqslant \sum_{n \in N_m} |t_n|^s \leqslant 1$, hence $k_m \leqslant 2^{ms}$ for all m. Thus, for all m,

$$\Big\| \sum_{n\in N_m} t_n y_n \Big\| \leqslant 2^{-m+1} \Big\| \sum_{n\in N_m} y_n \Big\| \mathrm{uc}(y_n) \leqslant 2^{-m+1} \mathrm{uc}(y_n) A k_m^{1/\varrho} \leqslant 2^{-m+1} \mathrm{uc}(y_n) A 2^{ms/\varrho}.$$

Hence

$$\Big\| \sum_{n=1}^{N} t_n y_n \Big\| \leqslant \sum_m \Big\| \sum_{n\in N_m} t_n y_n \Big\| \leqslant 2A\,\mathrm{uc}(y_n) \sum_{m=1}^{\infty} 2^{(s/\varrho-1)m} \leqslant \frac{2}{2^{1-s/\varrho}-1} A\,\mathrm{uc}(y_n).$$

PROPOSITION 2. *Let N and $m+1$ be positive integers. Let* $2 \leqslant p_k < \infty$ *for* $k = 1, 2, \ldots, m$ *and let* $\sum_{k=1}^{m} p_k^{-1} \leqslant 2^{-1}$. *For* $k = 0, 1, \ldots, m$, *let* Y_k *be an N-dimensional Banach space with a basis* $(y_n^{(k)})$ *and let* $T_k\colon (Y_{k-1}, (y_n^{(k-1)})) \to (Y_k, (y_n^{(k)}))$ *denote the natural injection. Then, for every s such that* $s = 1$ *for* $m = 0$ *and* $1 \geqslant s^{-1} > 1 - \sum_{k=1}^{m} p_k^{-1}$ *for* $m \geqslant 1$, *the natural injection* $\mathscr{I}\colon (l_N^1, (e_n)) \to (Y_m, (y_n^{(m)}))$ *admits the factorization:*

$$(l_N^1, (e_n)) \xrightarrow{I_{1,s}} (l_N^s, (e_n)) \xrightarrow{V} (Y_m, (y_n^{(m)})),$$

i.e. $\mathscr{I} = VI_{1,s}$ *with*

$$\|V\| \leqslant C \prod_{k=1}^{m} \pi_{p_k,2}(T_k) [\mathrm{uc}(y_n^{(k)})]^2 \sup_n \|y_n^{(0)}\|, \tag{2}$$

where $C = C(s, p_1, p_2, \ldots, p_m)$ *is a universal constant depending on* $s, p_1, p_2, \ldots, p_m$ *only.*

We proceed by induction with respect to m. The case $m = 0$ is trivial. Assume that for some $m-1 \geqslant 0$ the assertion of Proposition 2 is true. For $k = 0, 1, \ldots, m$ let Y_k be N-dimensional Banach spaces with bases $(y_n^{(k)})$, respectively, and for $k = 1, 2, \ldots, m$ let p_k be real with $2 \leqslant p_k < \infty$ and $\sum_{k=1}^{m} p_k^{-1} \leqslant 2^{-1}$. Fix s with $1 \geqslant s^{-1} > 1 - \sum_{k=1}^{m} p_k^{-1}$ and consider separately two cases:

1) either $m-1 = 0$ or $m > 1$ and $1 \geqslant s^{-1} > 1 - \sum_{k=1}^{m-1} p_k^{-1}$. Then, by the inductive hypothesis, for the natural injection

$$V_1\colon (l_N^s, (e_n)) \to (Y_{m-1}, (y_n^{(m-1)}))$$

we have $\|V_1\| = \sup_n \|y_n^{(0)}\|$ for $m-1 = 0$, and for $m-1 > 0$

$$\|V_1\| \leqslant C(s, p_1, \ldots, p_{m-1}) \prod_{k=1}^{m-1} \pi_{p_k,2}(T_k) [\mathrm{uc}(y_n^{(k)})]^2 \sup_n \|y_n^{(0)}\|.$$

If $V\colon (l_N^s, (e_n)) \to (Y_m, (y_n^{(m)}))$ is the natural injection, then $V = T_m V_1$. Hence $\|V\| \leqslant \|V_1\| \|T_m\|$. Taking into account that $\|T_m\| \leqslant \pi_{p_m,2}(T_m)$ and $\mathrm{uc}(y_n^{(m)}) \geqslant 1$, we get (2).

2) $m-1 > 0$ and $1 - \sum_{k=1}^{m-1} p_k^{-1} \geqslant s^{-1} > 1 - \sum_{k=1}^{m} p_k^{-1}$. Let us put

$$r = \max\Big\{1, \Big[2^{-1}\Big(s^{-1} + 1 - \sum_{k=1}^{m} p_k^{-1}\Big) + p_m^{-1}\Big]^{-1}\Big\}.$$

Clearly, $1 \geqslant r^{-1} > 1 - \sum_{k=1}^{m-1} p_k^{-1}$. Thus, by the inductive hypothesis for the natural injection $V_1\colon (l_N^r, (e_n)) \to (Y_{m-1}, (y_n^{(m-1)}))$, we have

$$\|V_1\| \leqslant C(s, p_1, \ldots, p_{m-1}) \prod_{k=1}^{m-1} \pi_{p_k,2}(T_k) [\mathrm{uc}(y_n^{(k)})]^2 \sup_n \|y_n^{(0)}\|. \tag{3}$$

Let $T = T_m V_1\colon (l_N^r, (e_n)) \to (Y_m, (y_n^{(m)}))$ be the natural injection and let $\varrho = (r^{-1} - p_m^{-1})^{-1}$. Clearly,

$$r^{-1} > 1 - \sum_{k=1}^{m-1} p_k^{-1} > 2^{-1}.$$

Hence $1 \leqslant r < 2 \leqslant p_m$. Thus, by Lemma 1, for $j = 1, 2, \ldots, N$ we have

$$\sup_{n_1<n_2<\ldots<n_j} \|y_{n_1}^{(m)} + y_{n_2}^{(m)} + \ldots + y_{n_j}^{(m)}\| \leqslant 2\pi_{p_m,2}(T)\,\mathrm{uc}(y_n^{(m)}) j^{1/\varrho}. \tag{4}$$

Next observe that $s < \varrho$. For, if $r > 1$ then

$$\varrho^{-1} = r^{-1} - p_m^{-1} = 2^{-1}\Big(s^{-1} + 1 - \sum_{k=1}^{m} p_k^{-1}\Big) < s^{-1},$$

and if $r = 1$ then $2^{-1}\left(s^{-1}+1-\sum_{k=1}^{m} p_k^{-1}\right)+p_m^{-1} \geqslant 1$ hence

$$s^{-1} > 2^{-1}\left(s^{-1}+1-\sum_{k=1}^{m} p_k^{-1}\right) \geqslant 1-p_m^{-1} = r^{-1}-p_m^{-1} = \varrho^{-1}.$$

Denote by $V\colon \big(l_N^s, (e_n)\big) \to \big(Y_m, (y_n^{(m)})\big)$ the natural injection. Since $s < \varrho$ and $\pi_{p_m,2}(T) \leqslant \pi_{p_m,2}(T_m)\|V_1\|$, Lemma 2 and (4) yield

$$\|V\| \leqslant \frac{4}{2^{1-s/\varrho}-1}[\mathrm{uc}(y_n)]^2 \pi_{p_m,2}(T) \tag{5}$$

$$\leqslant \frac{4}{2^{1-s/\varrho}-1}[\mathrm{uc}(y_n)]^2 \pi_{p_m,2}(T_m)\|V_1\|.$$

Combining (3) with (5) we get (2) with

$$C = C(s, p_1, \ldots, p_m) = \frac{4}{2^{1-s/r-s/p_m}-1} C(s, p_1, \ldots, p_{m-1}).$$

For the proof of Theorem 1 we shall also need the following simple:

LEMMA 3. *Let $1 < s < 2 \leqslant p$ and let $p^{-1}+2^{-1}-s^{-1} > 0$. Let N be a positive integer, let Y be an N-dimensional Banach space with a basis (y_n), and let $V\colon \big(l_N^s, (e_n)\big) \to \big(Y, (y_n)\big)$ be the natural injection. Then*

$$\pi_{p,2}(V) \geqslant \inf_n \|y_n\| N^{1/p+1/2-1/s}.$$

Proof. Using the Hölder inequality for the exponents

$$q = \frac{s}{2(s-1)} \quad \text{and} \quad q^* = \frac{s}{2-s},$$

we get

$$\sup_{\|x^*\|=1,\, x^* \in (l_N^s)^*} \left(\sum_{n=1}^{N} |x^*(e_n)|^2\right)^{1/2}$$

$$= \sup_{\sum_{n=1}^{N} |a_n|^{s/(s-1)}=1} \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2}$$

$$\leqslant \sup_{\sum_{n=1}^{N} |a_n|^{s/(s-1)}=1} \left(\sum_{n=1}^{N} |a_n|^{s/(s-1)}\right)^{(s-1)/s} N^{(2-s)/2s} = N^{1/s-1/2}.$$

Hence

$$N^{1/p} \inf_n \|y_n\| \leqslant \left(\sum_{n=1}^{N} \|T(e_n)\|^p\right)^{1/p} \leqslant \pi_{p,2}(T) N^{1/s-1/2}.$$

Thus

$$\pi_{p,2}(T) \geqslant \inf_n \|y_n\| N^{1/p+1/2-1/s}.$$

Proof of Theorem 1. Let M be a positive integer and let X_k be Banach spaces ($k = 0, 1, \ldots, M$). Let $S_k\colon X_{k-1} \to X_k$ be $(p_k, 2)$-absolutely summing operators, let $2 \leqslant p_k < \infty$ and $\sum_{k=1}^{M} p_k^{-1} > 2^{-1}$ ($k = 1, 2, \ldots, M$). Choose the index $m < M$ so that

$$\sum_{k=1}^{m} p_k^{-1} \leqslant 2^{-1} < \sum_{k=1}^{m+1} p_k^{-1}$$

and pick s so that

$$1 \geqslant s^{-1} > 1-\sum_{k=1}^{m} p_k^{-1} \quad \text{and} \quad p_{m+1}^{-1}+2^{-1}-s^{-1} > 0.$$

We shall show that the composition $S_{m+1}S_m \ldots S_1$ is compact. Suppose not. Then, by Proposition 1, there exists $\delta > 0$ such that for a positive integer N with

$$N > \left(4\delta^{-1}C3^{2m}\prod_{k=1}^{m+1}\pi_{p_k,2}(T_k)\right)^{1/(p_{m+1}^{-1}+2^{-1}-s^{-1})},$$

where $C = C(s, p_1, \ldots, p_m)$ is the constant appearing in Proposition 2, there exist, for $k = 0, 1, \ldots, m+1$, N-dimensional subspaces Y_k of X_k with bases $(y_n^{(k)})$ satisfying the conditions (i) and (ii) of Proposition 1. Let $V\colon \big(l_N^s, (e_n)\big) \to \big(Y_m, (y_n^{(m)})\big)$ and $T\colon \big(l_N^s, (e_n)\big) \to \big(Y_{m+1}, (y_n^{(m+1)})\big)$ denote the natural injections. By Proposition 2 and by (i),

$$\|V\| \leqslant C\prod_{k=1}^{m}\pi_{p_k,2}(T_k)[\mathrm{uc}(y_n^{(k)})]^2 \sup_n \|y_n^{(0)}\|$$

$$\leqslant 4C\,3^{2m}\prod_{k=1}^{m}\pi_{p_k,2}(T_k),$$

while, by Lemma 3 with $p = p_{m+1}$, and by (i),

$$\pi_{p_{m+1},2}(T) \geqslant \inf_n \|y_n^{(m+1)}\| N^{p^{-1}+2^{-1}-s^{-1}} \geqslant \delta N^{p^{-1}+2^{-1}-s^{-1}}.$$

Since $T = T_{m+1}V$, we get

$$N^{p^{-1}+2^{-1}-s^{-1}} \leqslant \delta^{-1}\pi_{p_{m+1},2}(T_{m+1})\|V\| \leqslant 4\delta^{-1}C3^{2m}\prod_{k=1}^{m+1}\pi_{p_k,2}(T_k),$$

which contradicts the choice of N.

Thus the composition $S_{m+1}S_m \ldots S_1$ is compact and therefore the composition $S_M S_{M-1} \ldots S_1$ is compact.

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