[12] B. M. Gurevič, *Topological entropy of denumerable Markov chains*, Dokl. Akad. Nauk SSSR, 187 (1969), pp. 715–718. (Soviet Math. Dokl. 10 (1969), pp. 911–915)
[13] F. Hahn, Y. Katznelson, *On the entropy of uniquely ergodic transformations*, Trans. Amer. Math. Soc., 126 (1967), pp. 335–360.
[14] J. L. Kelley, *General topology*, Van Nostrand, Princeton 1955.
[15] M. Misiurewicz, *Diffeomorphism without any measure with maximal entropy*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 21 (1973), pp. 903–910.
[16] M. Smorodinsky, *Ergodic theory, entropy*, Lect. Notes in Math. 214, Springer, Berlin 1971.
[17] P. Walters, *A variational principle for the pressure of continuous transformations*, preprint.

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A reflexive Banach space which is not sufficiently Euclidean

by

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Abstract. An example is given of a reflexive Banach space with unconditional basis which is not sufficiently Euclidean.

I. Introduction. In [8], Stegall and Retherford ask whether every reflexive Banach space Y is sufficiently Euclidean; i.e., whether Y contains a sequence (E_n) of subspaces with $\sup d(E_n, l_2^n) < \infty$ for which there are projections P_n of Y onto E_n satisfying $\sup\|P_n\| < \infty$. ($d(E, F)$ is the Banach–Mazur distance coefficient $\inf\{\|T\|\,\|T^{-1}\|: T$ is an isomorphism from E onto $F\}$.) This problem has a negative answer. In fact, we construct a reflexive Banach space Y with unconditionally monotone basis for which $\|P\| \geqslant 2^{-8} d(W, l_2^n)^{-2} n^{1/2}$ for any projection P from Y onto an n-dimensional subspace W.

We use standard Banach space theory notation as may be found e.g. in [6]. We would like to thank Professor T. Figiel for simplifying the proof that the example constructed in Section II is reflexive.

II. The example. We work with the space X of sequences of scalars which have only finitely many non-zero coordinates. Given a set E of integers and $x \in X$, Ex is the sequence which agrees with x in coordinates in E and is zero in the other coordinates. A sequence $(E_i)_{i=1}^n$ of sets of positive integers is called *allowable* provided $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E_i \subseteq [n+1, \infty)$ for $1 \leqslant i \leqslant n$. We will construct a norm $\|\cdot\|$ on X for which the unit vectors (e_n) form an unconditionally monotone basis so that the completion of $(X, \|\cdot\|)$ is reflexive and X satisfies

$$(*) \qquad \|x\| = \max\left(\|x\|_{c_0}, \tfrac{1}{2}\sup\left\{\sum_{i=1}^{n}\|E_i x\|: (E_i)_{i=1}^n \text{ is allowable}\right\}\right).$$

All Euclidean subspaces of $Y = [e_{p(n)}]$ are badly complemented if X satisfies $(*)$ and $(p(n))$ grows fast enough. The proof of this assertion makes use of the following proposition, whose proof is omitted because it involves only a nonessential modification of the argument for Theorem A in [7] and a standard perturbation argument.

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PROPOSITION 1. *For each positive integer m there is a positive integer $N = N(m)$ so that if Y is a space with unconditionally monotone basis and Z is a subspace of Y with $\dim Z \leqslant m$, then there are vectors $(y_i)_{i=1}^N$ in Y with pairwise disjoint and finite supports relative to the basis for Y and an automorphism T on Y which satisfies $\|T\|\,\|T^{-1}\| \leqslant 2$, $TZ \subseteq [y_i]_{i=1}^N$.*

Choose $1 \leqslant p(1) < p(2) < \ldots$, so that $p(m) \geqslant N(m)$ for each m, and set $Y = [e_{p(i)}]$. Suppose that W is an n-dimensional subspace of Y, where $n = 2m$ or $n = 2m-1$. Then obviously W contains a subspace Z of dimension m with $Z \subseteq [e_{p(i)}]_{i=m+1}^\infty$, so that each vector in Z has support contained in $[N(m)+1, \infty)$. By Proposition 1 there are unit vectors $(y_i)_{i=1}^{N(m)}$ in $[e_{p(i)}]_{i=m+1}^\infty$ which are disjointly supported relative to $(e_{p(i)})_{i=m+1}^\infty$ and there is an automorphism T on $[e_{p(i)}]_{i=m+1}^\infty$ with $\|T\|\,\|T^{-1}\| \leqslant 2$ and $TZ \subseteq [y_i]_{i=1}^{N(m)}$. Since $\operatorname{supp} y_i \subset [N(m)+1, \infty)$ for $1 \leqslant i \leqslant m$, we have from $(*)$ that $\frac{1}{2}\sum_{i=1}^m |a_i| \leqslant \left\|\sum_{i=1}^m a_i y_i\right\|$ for each choice $(a_i)_{i=1}^m$ of scalars. Now a result of Grothendieck's [3] yields that if U is an m-dimensional subspace of l_1^N and P is a projection of l_1^N onto U, then $\|P\| \geqslant \frac{1}{2} d(U, l_2^m)^{-1} m^{1/2}$. Hence if Q is a projection of $[y_i]_{i=1}^{N(m)}$ onto TZ, then $\|Q\| \geqslant 4^{-2}\frac{1}{2} d(TZ, l_2^m)^{-1} m^{1/2}$, whence if P is a projection of Y onto Z, then $\|P\| \geqslant 2^{-7} d(Z, l_2^m)^{-1} m^{1/2}$. Since there is a projection from W onto Z of norm at most $d(W, l_2^n)$, any projection from Y onto W must have norm at least

$$2^{-7} d(W, l_2^n)^{-1} d(Z, l_2^m)^{-1} m^{1/2} \geqslant 2^{-8} d(W, l_2^n)^{-2} n^{1/2}.$$

To finish the proof we need show that the completion of $(X, \|\cdot\|)$ is reflexive when it satisfies $(*)$. By [4] it is enough to show that the completion of X does not contain an isomorphic copy of c_0 or l_1. The c_0 case is trivial; the l_1 case follows from:

LEMMA 1. *Suppose that $(y_i)_{i=0}^m$ are unit vectors in X with $\operatorname{supp} y_0 \subseteq [1, k)$, and $m > 17(18k)^k$. Assume that $k \max\operatorname{supp} y_i < \min\operatorname{supp} y_{i+1}$ for $0 \leqslant i < m$. Then*

$$\left\|y_0 + \frac{1}{m}\sum_{i=1}^m y_i\right\| \leqslant 2 - 1/17.$$

After proving Lemma 1, we complete the proof that the completion of X does not contain an isomorph of l_1 as follows: Assume the completion of X contains a copy of l_1. Then by a result of James [5], there is a sequence (x_n) of unit vectors in X which is $1+\frac{1}{34}$ equivalent to the unit vector basis of l_1. By using a standard gliding hump argument, we can assume that $\max\operatorname{supp} x_i < \min\operatorname{supp} x_{i+1}$ for $1 \leqslant i < \infty$. Letting $k = \max\operatorname{supp} x_1$, we may assume by passing to a subsequence of (x_i) that $k \max\operatorname{supp} x_i < \min\operatorname{supp} x_{i+1}$. Letting $y_0 = x_1$ and $y_i = x_{i+1}$ for $1 \leqslant i < \infty$ and choosing m so that $m > 17(18k)^k$ we have from Lemma 1 that

$$\left\|y_0 + \frac{1}{m}\sum_{i=1}^m y_i\right\| \leqslant 2 - \frac{1}{17},$$

which contradicts the $1+\frac{1}{34}$ equivalence of (x_i) to the unit vector basis of l_1.

We turn now to the proof of Lemma 1. Set $z = \frac{1}{m}\sum_{i=1}^m y_i$ and let $\delta = \frac{1}{17}$. Suppose that $2 - \delta < \|y_0 + z\|$. Since $\|y_0 + z\|_{c_0} \leqslant 1$, we have from $(*)$ that there is an allowable sequence $(E_i)_{i=1}^p$ for which

$$(1)\qquad 2 - \delta < \tfrac{1}{2}\sum_{j=1}^p \|E_j(y_0 + z)\|.$$

It must be that $p < k$, for otherwise $E_j y_0 = 0$ for all j and thus the right side of (1) would be no more than $\frac{1}{2}\sum_{j=1}^p \|E_j z\| \leqslant 1$. We have

$$2 - \delta < \tfrac{1}{2}\sum_{j=1}^p \|E_j y_0\| + \tfrac{1}{2}\sum_{j=1}^p \|E_j z\| \leqslant 1 + \tfrac{1}{2}\sum_{j=1}^p \|E_j z\|,$$

hence

$$(2)\qquad 1 - \delta < \tfrac{1}{2}\sum_{j=1}^p \|E_j z\|.$$

Let $\mathscr{A}$ be a collection of pairwise disjoint two element subsets of $\{y_1, \ldots, y_m\}$ maximal with respect to the property that $\{v, w\} \in \mathscr{A}$ implies

$$\max_{1 \leqslant j \leqslant p} \big|\|E_j v\| - \|E_j w\|\big| < \frac{\delta}{p}.$$

Let $D = \{1, \ldots, m\} \setminus \{i \colon (y_i, y_j) \in \mathscr{A}$ for some $j\}$. Then

$$(3)\qquad \frac{1}{m}\sum_{i \in D}\frac{1}{2}\sum_{j=1}^p \|E_j y_i\| \leqslant m^{-1}\operatorname{card} D \leqslant m^{-1}\left(\frac{p(1+\delta)}{\delta}\right)^p = m^{-1}(18p)^k < \delta.$$

Indeed, for $i \in D$ let C_i be the open l_∞^p-ball of radius $\frac{\delta}{2p}$ around $(\|E_j y_i\|)_{j=1}^p$ in $\boldsymbol{R}^p$. Then the C_i's are pairwise disjoint cubes in $\boldsymbol{R}^p$ contained in the cube $\left[-\frac{\delta}{2}, 1+\frac{\delta}{2}\right]^p$, hence

$$\operatorname{card} D\left(\frac{\delta}{p}\right)^p = \sum_{i \in D}\operatorname{volume} C_i \leqslant (1+\delta)^p.$$

We have from the condition on m, (3), (2), and the triangle inequality that

$$1-2\delta < \frac{1}{2m} \sum_{\{a,b\}\in\mathscr{A}} \sum_{j=1}^{p} \|E_j(a+b)\|,$$

hence there are $v = y_s$, $w = y_t$, $s < t$ with $\{v, w\} \in \mathscr{A}$ and

(4) $$4(1-2\delta) < \sum_{j=1}^{p} \|E_j(v+w)\|.$$

We will show that the right side of (4) is less than $3+9\delta$, which will show that $\delta > \frac{1}{17}$.

Let $A = \{j\colon \|E_j(v+w)\| = \max(\|E_j v\|, \|E_j w\|)\}$, and let us set $B = \{1, \ldots, p\}\setminus A$. Since $\{v, w\} \in \mathscr{A}$, we have for $j \in A$ that

$$\|E_j(v+w)\| < \delta/p + \min(\|E_j v\|, \|E_j w\|) = \delta/p + \|E_j v\| + \|E_j w\| - \|E_j(v+w)\|$$

hence using the triangle inequality and (*) we get

(5) $$\sum_{j\in A} \|E_j(v+w)\| < \sum_{j=1}^{p} \big(\delta/p + \|E_j v\| + \|E_j w\| - \|E_j(v+w)\|\big) \leqslant \delta + 2 + 2 - 4(1-2\delta) = 9\delta.$$

Let $j \in B$. Since

$$\|E_j(v+w)\|_{c_0} \leqslant \max(\|E_j v\|, \|E_j w\|) < \|E_j(v+w)\|,$$

there is an allowable sequence $(F_{j,i})_{i=1}^{k(j)}$ of subsets of E_j, for which

$$\|E_j(v+w)\| = \tfrac{1}{2} \sum_{i=1}^{k(j)} \|F_{j,i}(v+w)\|.$$

Clearly, $k(j) \leqslant \max\operatorname{supp} v$, since otherwise we would have $\|F_{j,i}(v+w)\| = \|F_{j,i} w\|$ so that by (*) $\|E_j(v+w)\| \leqslant \|E_j w\|$, contrary to the definition of B. Let $G_{j,i} = F_{j,i} \cap \operatorname{supp} w$. Since $\sum_{j=1}^{p} k(j) \leqslant p \max\operatorname{supp} v < \min\operatorname{supp} w$, $(G_{j,i})_{j\in B,\, i=1}^{k(j)}$ is an allowable system. Therefore

(6) $$\sum_{j\in B} \|E_j(v+w)\| \leqslant \sum_{j\in B} \tfrac{1}{2} \sum_{i=1}^{k(j)} \|F_{j,i} v\| + \tfrac{1}{2} \sum_{j\in B} \sum_{i=1}^{k(j)} \|F_{j,i} w\| \leqslant 2 + 1 = 3.$$

From (5) and (6) we get that the right-hand side of (4) is less than $3+9\delta$, which completes the proof.

Remark 1. Of course, it is trivial that if Z is an n-dimensional subspace of a Banach space W then there is a projection from W onto Z of norm at most $d(Z, l_2^n)n^{1/2}$. (It is in fact known that there is a projection from W onto Z of norm at most $n^{1/2}$.)

Remark 2. A trivial modification of the construction given in [2] yields the desired norm on X. Indeed, let $\|x\|_0 = \|x\|_{c_0}$, and set $\|x\|_{n+1} = \max\big(\|x\|_n, 1/2 \sup\{\sum_{i=1}^{k} \|E_i x\|_n\colon (E_i)_{i=1}^{k} \text{ is allowable}\}\big)$. $\|x\| = \lim_{n\to\infty} \|x\|_n$ is an unconditionally monotone norm on X which satisfies (*).

Remark 3. The construction given above can be easily modified to give for any $0 < c < (2/\pi)^{1/2}$ a reflexive Banach space Y with unconditionally monotone basis in which $\|P\| \geqslant cd(W, l_2^n)^{-2} n^{1/2}$ for any projection P of Y onto an n-dimensional subspace W.

References

[1] W. J. Davis and W. B. Johnson, *Compact, non-nuclear operators*, Studia Math. 51 (1974), pp. 81–85.

[2] T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no l_p*, Compositio Math. 28 (1974), pp. 37–49.

[3] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Matem. Sao Paulo 8 (1956), pp. 1–79.

[4] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. 52 (1950), pp. 518–527.

[5] — *Uniformly non-square Banach spaces*, ibid. 80 (1964), pp. 542–550.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture notes in Mathematics, Springer-Verlag 1973.

[7] A. Pełczyński and H. P. Rosenthal, *Localization techniques in L^p spaces*, Studia Math. 52 (1975), pp. 263–289.

[8] C. P. Stegall and J. R. Retherford, *Fully nuclear and completely nuclear operators with applications to $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces*, Trans. Amer. Math. Soc. 163 (1972), pp. 457–492.

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