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Further results on integral representations

by

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Abstract. This paper is a continuation of my joint research with David Pollard ([1]). It is my object to establish integral representation theorems for a positive linear functional defined on a cone of functions in cases when this cone is not necessarily closed under differences.

We follow [1] and use definitions, notations and conventions from [1] without much further explanation.

LEMMA 1. *Let $\mathscr{K}$ be a $(\emptyset, \bigcup f, \bigcap f)$-paving on X and μ a $\mathscr{K}$-regular finitely additive measure.*

(i) *If (f_a) is a net on $[0, \infty]^X$ such that $f_a \to 0$, uniformly on $\mathscr{K}$-sets, and $\mu_*(\sup f_a) < \infty$, then $\mu_*(f_a) \to 0$.*

(ii) *If $\mu_*(f) < \infty$, then $\mu_*(f \wedge n^{-1}) \to 0$ as $n \to \infty$.*

(iii) *If $\mu_*(f) < \infty$, then, to every $\varepsilon > 0$, there exists $K \in \mathscr{K}$ such that $\mu_*(f \cdot 1_{\complement K}) \leqslant \varepsilon$; K can be chosen such that $f \gg 0$ on K (for some $\delta > 0$, $f(x) \geqslant \delta$ for all $x \in K$).*

(iv) *For any $f \in [0, \infty]^X$, $\mu_*(f) = \sup_n \mu_*(f_\wedge n)$.*

As usual, we only consider non-negative functions.

Proof. (iv) is trivial. Formally, (ii) and (iii) are easy consequences of (i). However, to prove (i) we start by proving (iii). To do this, assume that $\mu_*(f) < \infty$. Choose $k = \sum_1^n a_i 1_{K_i}$ such that all the a_i are positive, $k \leqslant f$ and $\mu_*(f) \leqslant \mu(k) + \varepsilon$. Put $K = \bigcup_1^n K_i$. Then $f \gg 0$ on K and

$$\mu_*(f \cdot 1_{\complement K}) + \mu_*(k) = \mu_*(f \cdot 1_{\complement K} + k) \leqslant \mu_*(f)$$

from which the desired result follows.

To prove (i), put $f = \sup f_a$. Then $\mu_*(f) < \infty$ by assumption. Choose K such that $\mu_*(f \cdot 1_{\complement K}) \leqslant \varepsilon$. Choose $\delta > 0$ such that $\delta \cdot \mu(K) \leqslant \varepsilon$. By assump-

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tion, there exists a_0 such that $f_a \leqslant \delta$ on K for all $a \geqslant a_0$. For $a \geqslant a_0$ we then have

$$\mu_*(f_a) \leqslant \mu_*(\delta\cdot 1_K + f\cdot 1_{CK}) = \mu_*(\delta\cdot 1_K) + \mu_*(f\cdot 1_{CK}) \leqslant 2\varepsilon. \quad \blacksquare$$

Remark. If μ is supported by a $\mathscr{K}$-set, i.e. if $\mu(CK) = 0$ for some K, then we need not assume that $\mu_*(\sup f_a) < \infty$ in (i), and in (iii) we can of course find K with $\mu_*(f_a\cdot 1_{CK}) = 0$ (but not necessarily such that $f \geqslant 0$ on K).

Now let $(\mathscr{C}, T, \mathscr{K})$ be given and assume that the following conditions, taken from [1], are satisfied:

A1′: $\mathscr{C}$ is a $(0, \vee f, \wedge f)$ convex cone in $[0, \infty)^X$ satisfying Stone's condition; (this is A1 of [1] with Stone's condition added),

A2: T: $\mathscr{C} \to [0, \infty)$ is a monotone positive linear functional;

A4: $\mathscr{K}$ is a $(\emptyset, \bigcup f, \bigcap f)$-paving on X;

A6: For $K_1 \cap K_2 = \emptyset$, $\varepsilon > 0$ there exist $h_1 \geqslant 1_{K_1}$, $h_2 \geqslant 1_{K_2}$ such that $T(h_1 \wedge h_2) < \varepsilon$.

We shall see later on, how A3 and A5 of [1] are replaced by weaker assumptions.

At present, we consider some other conditions on $(\mathscr{C}, T, \mathscr{K})$, analogous to (i) and (iii) of Lemma 1. Conditions resembling (i) are kinds of continuity conditions and will be denoted by the letter C, conditions resembling (iii) are of the "exhaustion" type considered in [1] and will be denoted by the letter E. Furthermore, we introduce conditions, roughly speaking to the effect that $\mathscr{C}$ be closed under $\setminus$ relative to T; these conditions are of the "tightness" type considered in [2] and will be denoted by the letter t. Our conditions are as follows:

C: $\sup h_a \leqslant h$, $h \in \mathscr{C}$, $h_a \to 0$ uniformly on $\mathscr{K}$-sets implies $Th_a \to 0$;

C*: $\sup h_a \leqslant h$, $h \in \mathscr{C}$, $h_a \to 0$ uniformly on all $K \in \mathscr{K}$ for which $h \geqslant 0$ on K implies $Th_a \to 0$;

E: for all $h \in \mathscr{C}$, $\varepsilon > 0$, there exists $K \in \mathscr{K}$ such that $Th' \leqslant \varepsilon$ for any $h' \leqslant h$ with $h' = 0$ on K;

E*: for all $h \in \mathscr{C}$, $\varepsilon > 0$, there exists $K \in \mathscr{K}$ with $h \geqslant 0$ on K such that $Th' \leqslant \varepsilon$ for any $h' \leqslant h$ with $h' = 0$ on K;

t: for $h_1 \leqslant h_2$, $\varepsilon > 0$ there exists $h \leqslant h_2 - h_1$ such that $Th_1 + Th \geqslant Th_2 - \varepsilon$;

t*: for $h_1 \leqslant h_2$, $\varepsilon > 0$ and $K \in \mathscr{K}$, there exists h with $h \leqslant h_2$, $h \leqslant h_2 - h_1$ on K and such that $Th_1 + Th \geqslant Th_2 - \varepsilon$.

Note that if C holds, then

$$\inf_n T(h \wedge n^{-1}) = 0 \quad \text{for all } h \in \mathscr{C}. \tag{1}$$

LEMMA 2. *Assume that* t* *holds. Let* $h \in \mathscr{C}$, *let* $\varepsilon > 0$, *and suppose that*

$$Th' \leqslant \varepsilon \quad \text{for all } h' \leqslant h \text{ with } h' = 0 \text{ on } K.$$

Then

$$Th' \leqslant Th_0 + \varepsilon \quad \text{for all } h', h_0 \text{ with } h' \leqslant h \text{ and } h' \leqslant h_0 \text{ on } K.$$

Proof. Assume $h' \leqslant h_0$, $h' \leqslant h_0$ on K. We may assume that $h_0 \leqslant h'$. By t* we can find, given $\delta > 0$, $h^* \leqslant h'$ with $h^* \leqslant h' - h_0$ on K such that $Th^* + Th_0 \geqslant Th' - \delta$. Since $h^* = 0$ on K, we have $Th^* \leqslant \varepsilon$. $Th' \leqslant Th_0 + \varepsilon + \delta$ follows. δ being arbitrary, the result follows. ∎

LEMMA 3. *If* t* *holds and if* $tr(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$, *then* C, C*, E *and* E* *are all equivalent.*

Proof. Clearly, C* ⇒ C. Employing the condition $tr(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$, it is easy to see that C ⇒ C*.

Clearly, E* ⇒ E. Now assume that E holds. To establish E*, let h, ε be given. Choose K as specified by E. There exists $h_K \geqslant 1_K$. Choose $\varrho > 0$ such that $\varrho\cdot T(h_K) \leqslant \varepsilon$. Put $K_\varrho = K \cap \{h \geqslant \varrho\}$. Then $K_\varrho \in \mathscr{K}$. If $h' \leqslant h$, $h' = 0$ on K_ϱ, then $h' \leqslant \varrho h_K$ on K. By Lemma 2 we conclude that $Th' \leqslant T(\varrho h_K) + \varepsilon \leqslant 2\varepsilon$. Thus E* holds.

Now assume that E holds and let us establish C. To do this, let (h_a), h be given with $h_a \leqslant h$ for all a, $h_a \to 0$ uniformly on $\mathscr{K}$-sets. To $\varepsilon > 0$ we choose K such that $Th' \leqslant \varepsilon$ whenever $h' \leqslant h$, $h' = 0$ on K. There exists $h_K \geqslant 1_K$. Choose $\varrho > 0$ such that $\varrho T(h_K) \leqslant \varepsilon$. Eventually, we have $h_a \leqslant \varrho h_K$ on K and hence, by Lemma 2, $Th_a \leqslant T(\varrho h_K) + \varepsilon \leqslant 2\varepsilon$. This proves C.

Lastly, assume that C holds and let us prove E. To a given h consider the set D consisting of all pairs (h', K) with $h' \leqslant h$, $h' = 0$ on K. Direct D by $(h_2', K_2) \geqslant (h_1', K_1) \Leftrightarrow K_2 \supseteq K_1$. For $a = (h', K) \in D$ define $h_a = h'$. Then (h_a) is a net with $h_a \leqslant h$ for all a and $h_a \to 0$, uniformly on $\mathscr{K}$-sets. Thus $Th_a \to 0$ and we conclude that E holds. ∎

As in [1], we consider μ_* defined on 2^X by

$$\mu K = \inf\{Th\colon h \geqslant 1_K\}; \quad K \in \mathscr{K}, \tag{2}$$

$$\mu_* A = \sup\{\mu K\colon K \subseteq A, K \in \mathscr{K}\}; \quad A \in 2^X. \tag{3}$$

LEMMA 4. *Assume that* $tr(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$, *and that* t* *and* E *hold. Then the restriction of* μ_* *to* $\mathscr{A}(\mathscr{K})$ *is a* $\mathscr{K}$*-regular finitely additive measure dominated by* T.

Proof. Going through the proof of Theorem 1 of [1], one finds that the only non-obvious fact that needs proof is, that for $K_1 \subseteq K_2$,

$$\mu K_1 + \mu_*(K_2 \setminus K_1) \geqslant \mu K_2 \tag{4}$$

holds. To establish (4), let $K_1 \subseteq K_2$ and $\varepsilon > 0$ be given. Choose $h_1 \geqslant 1_{K_1}$, $h_2 \geqslant 1_{K_2}$ such that

$$Th_1 \leqslant \mu K_1 + \varepsilon, \quad Th_2 \leqslant \mu K_2 + \varepsilon.$$

We may assume that $h_1 \leqslant h_2 \leqslant 1$.

Since $h_1 \leqslant h_2$ and since $h_2 - h_1 = 0$ on K_1, we may, according to t*, choose $h \leqslant h_2$ such that $h = 0$ on K_1 and such that

$$Th_1 + Th \geqslant Th_2 - \varepsilon.$$

By E*, we can find K such that $h \geqslant 0$ on K and such that

(5) $$Th' \leqslant \varepsilon \quad \text{for all } h' \leqslant h \text{ with } h' = 0 \text{ on } K.$$

Put $K_0 = K \cap K_2$. Then $K_0 \subseteq K_2 \setminus K_1$.

We claim that the following is true

(6) $$Th' \leqslant 3\varepsilon \quad \text{for all } h' \leqslant h \text{ with } h' = 0 \text{ on } K_0.$$

To prove this, assume that $h' \leqslant h$, $h' = 0$ on K_0. Choose $a > 0$ such that $T(h' \wedge a) \leqslant \varepsilon$ (cf. (1)). Since K_2 and $K \cap \{h' \geqslant a\}$ are disjoint $\mathscr{K}$-sets, we can by A6 find

$$h^* \geqslant 1_{K_2}, \quad h^{**} \geqslant 1_{K \cap \{h' \geqslant a\}}$$

such that $T(h^* \wedge h^{**}) \leqslant \varepsilon$. We may assume that $h^* \leqslant h_2$. Note that $Th_2 \leqslant \mu K_2 + \varepsilon \leqslant Th^* + \varepsilon$. Now we have

$$\begin{aligned} Th_2 + T(h^{**} \wedge h_2) &\leqslant Th^* + T(h^{**} \wedge h_2) + \varepsilon \\ &= T\big(h^* \vee (h^{**} \wedge h_2)\big) + T\big(h^* \wedge (h^{**} \wedge h_2)\big) + \varepsilon \\ &\leqslant Th_2 + 2\varepsilon. \end{aligned}$$

Hence

$$T(h^{**} \wedge h_2) \leqslant \varepsilon.$$

Now note that

$$h' \leqslant h' \wedge a + h^{**} \wedge h_2 \quad \text{on } K.$$

Use this together with $h' \leqslant h$, (5) and Lemma 2 to conclude that

$$\begin{aligned} Th' &\leqslant T(h' \wedge a + h^{**} \wedge h_2) + \varepsilon \\ &= T(h' \wedge a) + T(h^{**} \wedge h_2) + \varepsilon \\ &\leqslant 3\varepsilon. \end{aligned}$$

This proves (6).

For any $h'' \geqslant 1_{K_0}$ we have $h'' \geqslant h$ on K_0 and employing (6) in connection with Lemma 2 we get $Th \leqslant T(h'') + 3\varepsilon$. Since this inequality

holds for any $h'' \geqslant 1_{K_0}$, we have $Th \leqslant \mu(K_0) + 3\varepsilon$, hence

$$\begin{aligned} \mu K_1 + \mu_*(K_2 \setminus K_1) &\geqslant Th_1 + Th - 4\varepsilon \\ &\geqslant Th_2 - 5\varepsilon \\ &\geqslant \mu K_2 - 5\varepsilon \end{aligned}$$

which proves (4). ■

For the main result, we also need the condition

(7) $$Th = \sup_n T(h \wedge n) \quad \text{for all } h \in \mathscr{C}.$$

THEOREM 1 (cf. Theorems 2, 3 of [1]). *Assume that* $(\mathscr{C}, T, \mathscr{K})$ *satisfies* A1′, A2, A4, A6, t* *and that* $tr(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$.

Then the following equivalences hold:

(i) *There exists a $\mathscr{K}$-regular finitely aditive measure representing T*
$\Leftrightarrow$ E, (7)
$\Leftrightarrow$ C, (7).

(ii) *If $\mathscr{K}$ is closed under* $(\bigcap c)$ *then:*
There exists a $\mathscr{K}$-regular σ-additive measure representing T
$\Leftrightarrow$ E, (7), *T is σ-smooth at $\emptyset$* w.r.t. $\mathscr{K}$.

(iii) *If $\mathscr{K}$ is closed under* $(\bigcap a)$ *then:*
There exists a $\mathscr{K}$-regular τ-additive measure representing T
$\Leftrightarrow$ E, (7), *T is τ-smooth at $\emptyset$* w.r.t. $\mathscr{K}$.

Proof. We need only consider case (i), since (ii) and (iii) can then be handled by the methods in [1].

That E and C are equivalent has already been noted, and that E and (7) must hold in case there exists a $\mathscr{K}$-regular finitely additive measure representing T follows from Lemma 1.

Now assume that E and (7), and hence also C are satisfied. Let μ be the $\mathscr{K}$-regular finitely additive measure from Lemma 4. We shall prove that μ represents T. According to Lemma 4, we need only prove $\mu_*(h) \geqslant Th$ for all $h \in \mathscr{C}$. By (7) it is enough to consider bounded h's. Therefore, let h be a fixed bounded function in $\mathscr{C}$ (for the following it is in fact enough to assume that h is bounded on every $\mathscr{K}$-set).

Below we always indicate by the letter k a $\mathscr{K}$-simple function. If $k = \sum_1^n a_i 1_{K_i}$ with all the a_i positive, we put $\operatorname{supp}(k) = \bigcup_1^n K_i$.

Direct the set

$$D = \{(h', h'', k):\ k \leqslant h' \leqslant h,\ h'' \leqslant h,\ h' + h'' \leqslant h \text{ on } \operatorname{supp}(k)\}$$

by defining

$$(h_2', h_2'', k_2) \geqslant (h_1', h_1'', k_1) \Leftrightarrow k_2 \geqslant k_1.$$

For $\alpha = (h', h'', k) \in D$ put $h_\alpha = h''$. Then $(h_\alpha)_{\alpha \in D}$ is a net on $\mathscr{C}$ with $h_\alpha \leqslant h$ for all α. To prove that $h_\alpha \to 0$, uniformly on $\mathscr{K}$-sets, let K and $\varepsilon > 0$ be given. Since h is bounded on K, we may as well assume that $h \leqslant 1$ on K. Choose n such that $n^{-1} \leqslant \varepsilon$. Put $K_\nu = K \cap \{h \geqslant \nu n^{-1}\}$; $\nu = 1, \ldots, n$. Since $tr(\mathscr{C}) \subseteq \mathscr{F}(\mathscr{K})$, these sets are all members of $\mathscr{K}$. Put $k = \sum_1^n n^{-1} 1_{K_\nu}$. Put $\alpha_0 = (h, 0, k)$. Then $\alpha_0 \in D$ and since $h - k \leqslant \varepsilon$ on K, we see that $h_\alpha \leqslant \varepsilon$ on K for all $\alpha \geqslant \alpha_0$. Thus $h_\alpha \to 0$, uniformly on $\mathscr{K}$-sets.

Given $\varepsilon > 0$, we can by C find $\alpha_0 = (h_0', h_0'', k_0)$ such that $Th_\alpha \leqslant \varepsilon$ for all $\alpha \geqslant \alpha_0$. Put $K_0 = \operatorname{supp}(k_0)$.

Consider a fixed $h' \geqslant k_0$, $h' \leqslant h$. By t* we can find h'' such that $h'' \leqslant h$, $h'' \leqslant h - h'$ on K_0 and such that $Th'' + Th' \geqslant Th - \varepsilon$. Then $(h', h'', k_0) \in D$ and $(h', h'', k_0) \geqslant \alpha_0$. Therefore $Th'' \leqslant \varepsilon$. It follows that $Th' \geqslant Th - 2\varepsilon$. Since this argument applies to any $h' \geqslant k_0$ with $h' \leqslant h$, we conclude that $\mu_*(k_0) \geqslant Th - 2\varepsilon$. $\mu_*(h) \geqslant Th - 2\varepsilon$ follows. ■

Remarks 1. In case we ask for the necessary and sufficient conditions that there exists a $\mathscr{K}$-regular finitely additive [or σ-additive or τ-additive] measure supported by a $\mathscr{K}$-set which represents T, then we need only replace E by the condition

E_0: there exists K such that $Th = 0$ for all $h = 0$ on K

and C by the condition

C_0: $h_\alpha \to 0$ uniformly on $\mathscr{K}$-sets implies $Th_\alpha \to 0$.

2. As pointed out in [1], Section 1 it is not difficult to generalize to the situation in which T is allowed to take the value $+\infty$. Also, it is possible to handle $\mathscr{C}$-functions which may take the value $+\infty$; this only requires a proper definition of $\setminus$, cf. Section 3 of [2].

3. As usual, the $\mathscr{K}$-regular representing measure is unique and is determined in all cases by (2).

4. For the sufficiency part in (ii), we need not assume that $\mathscr{K}$ be closed under $(\cap c)$; what we obtain then is a $\mathscr{K}_\sigma$-regular σ-additive measure representing T, where $\mathscr{K}_\sigma$ is the paving of countable intersections of sets in $\mathscr{K}$. A similar remark applies to (iii). To establish these results, proceed as explained in Section 4 of [1].

THEOREM 2. *Assume that* $(\mathscr{C}, T)$ *satisfies* A1′, A2 *and* t. *Put* $\mathscr{K} = tr(\mathscr{C})$ *and assume that to any pair* K_1, K_2 *of disjoint* $\mathscr{K}$*-sets there exist* $h_1, h_2 \in \mathscr{C}$ *with* $h_1 \geqslant 1_{K_1}$, $h_2 \geqslant 1_{K_2}$ *and* $h_1 \wedge h_2 = 0$.

Then a necessary and sufficient condition that T *can be represented by a* $\mathscr{K}$*-regular finitely additive measure is that* (1) *and* (7) *hold. When it exists, this measure is unique and determined by* (2).

We leave the simple proof to the reader, only noting that E is obtained from (1).

Of course, the separation property assumed in the theorem could be replaced by A6. The example below shows that some separation property is needed.

EXAMPLE 1. Let X be the set of integers, $\mathscr{C}$ the set of bounded non-negative functions on X for which both limits $h(\infty) = \lim_{n\to\infty} h(n)$ and $h(-\infty) = \lim_{n\to-\infty} h(n)$ exist and are equal, and define T by $Th = h(\infty)$. In this case, $tr(\mathscr{C}) = 2^X$. All properties considered, except the separation property A6 hold. μ determined by (2) is not additive. However, T can be represented by a $\mathscr{K}$-regular finitely additive measure, indeed, every finitely additive measure μ on 2^X with $\mu X = 1$ and vanishing on all finite sets represents T. Thus we have no uniqueness in this case. ■

THEOREM 3 (compare with Theorem 4 of [1] and Theorem 3.13 of [2]). *Assume that* $(\mathscr{C}, T)$ *satisfies* A1′, A2 *and* t, *and that* $h \setminus 1 \in \mathscr{C}$ *for all* $h \in \mathscr{C}$. *Denote by* $\mathscr{K}_\sigma [\mathscr{K}_\tau]$ *the paving on* X *of countable [arbitrary] intersections of sets in* $tr(\mathscr{C})$.

Then a necessary and sufficient condition that T *can be represented by a* $\mathscr{K}_\sigma$*-regular* σ*-additive measure [*$\mathscr{K}_\tau$*-regular* τ*-additive measure] is that* T *be* σ*-smooth [*τ*-smooth] at* 0.

Due to the condition $h \setminus 1 \in \mathscr{C}$ for all $h \in \mathscr{C}$, the proof can be carried out in the same way as the proof of Theorem 4 of [1].

When we add the assumption that $\mathscr{C}$ be closed under $(\wedge c)$, we end up with (in the $\mathscr{K}_\sigma$-case) the situation dealt with in Theorem 3.13 of [2], except for the fact that the condition $h \setminus 1 \in \mathscr{C}$ for all $h \in \mathscr{C}$ was not needed in that result. It should be noted that the general formula (with μ determined by (2))

$$\mu_*(f) = \sup_{k \leqslant f} \inf_{h \geqslant k} Th; \quad f \in [0, \infty)^X,$$

k's referring to $\mathscr{K}$-simple functions, then reduces to

$$\mu_*(f) = \sup_{h \leqslant f} Th; \quad f \in [0, \infty]^X$$

since $1_K \in \mathscr{C}$ for every $K \in tr\,\mathscr{C}$ $[nh \setminus (n-1) \downarrow \{h = 1\}$ for $h \leqslant 1]$.

Throughout the paper our main effort has been to avoid the condition that $\mathscr{C}$ be closed under $\setminus$. That this can be done is theoretically revealing, in particular, when comparing with the analogous situation for the construction of measures. The typical applications which are possible with the refined results but not with the earlier results are to situations when $\mathscr{C}$ is a class of upper semicontinuous functions.

References

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