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Weak compact generating in duality

by

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Abstract. If X, Y are Banach spaces generated by a weakly compact set (WCG) and $Y \subset X^*$ is norming on X, then a projectional resolution of identity on X is constructed such that all projections are $w(X, Y)$ continuous and dual projections form resolution of identity on Y. In this case there exists an equivalent $w(X, Y)$-lower semicontinuous locally uniformly rotund norm on X the dual of which is rotund on X^* and locally uniformly rotund on Y. Also the existence of Gâteaux smooth partitions of unity on X is proved. Some results of [14] are generalized, namely it is shown that any WCG space has a large quotient with projectional basis and that if X^*, X^{**} are WCG, then X and X^* contain large reflexive subspaces. If X is a subspace of WCG space, then some sufficient conditions for X to be WCG are given: X is Fréchet smooth or X^* is WCG.

I. Introduction. Among nonseparable Banach spaces there is a class of spaces which behave well with respect to many geometrical and topological properties. This is a class of weakly compactly generated Banach spaces, introduced by H. Corson and studied by D. Amir, J. Lindenstrauss and others. A Banach space X is *weakly compactly generated* (WCG) if there is a weakly compact set $K \subset X$ such that X is the closed linear hull of K. These spaces form a very suitable roof over separable and reflexive spaces and include for example $c_0(\Gamma)$, $L_1(\mu)$ for finite μ, $l_p(\Gamma)$ – direct sum of WCG spaces, $p > 1$, $C(K)$ spaces, where K is weakly compact subset of a Banach space, etc. They possess a nice projectional resolution of identity ([1]), Markuševič bases, admit some nice norms ([1], [23], [10], [11]), and behave well as to some convex extremal problems ([15], [4], [5]). On the other hand, it was recently shown by H. Rosenthal ([20]) that the WCG property is not hereditary and W. Johnson and J. Lindenstrauss ([14]) constructed a non-WCG Fréchet smooth space whose dual is WCG.

In this paper we study weak compact generating in duality. In Section III we prove some results on projectional resolution of identity compatible with duality for WCG spaces. Also we prove a representation theorem for bounded subsets (in the weak topology) of Fréchet smooth WCG spaces (Proposition 5).

The lemmas from Section III are used in Section IV to extend and strengthen the following renorming theorem of M.I. Kadec (cf. e.g. [16], Theorem 1.9): If X, $Y \subset X^*$ are separable spaces, Y norming on X, then there is an equivalent norm on X which is $w(X, Y)$-lower semicontinuous and locally uniformly rotund. We prove, for example, that this equivalent norm may be also Gâteaux differentiable in the case if X, Y are WCG. More exactly, we prove: If X, $Y \subset X^*$ are WCG spaces, Y norming on X, then there is an equivalent norm $\|\cdot\|$ on X with the properties (i)–(vi) stated in Theorem 1. This is applied to show the existence of Gâteaux smooth partitions of unity in WCG spaces X for which a norming WCG $Y \subset X^*$ exists.

In Section V we use the projectional resolutions of identity to prove that if X has Fréchet differentiable norm (or X^* is WCG) and if X is a subspace of a WCG space, then X is WCG. The same result was independently obtained by D. Friedland [9] and also by W. Johnson and J. Lindenstrauss [14].

In Section VI we show that if X^* and X^{**} are WCG, then any subspace of X and X^* contains a reflexive subspace of the same density, extending (necessarily by a different proof) the results of W. Johnson and H. Rosenthal [13]. In the proof we use a concept of projectional bases introduced by C. Bessaga ([6]) for which some needed facts (which may be of an independent interest) are proved in Section VI. Let us remark that from the results of C. Bessaga and the result mentioned above immediately follows that if X^* and X^{**} are WCG, then X, X^*, X^{**} are all homeomorphic to a Hilbert space.

Appendix contains an application of the above methods to the problem of the existence of special Markuševič bases and quasicomplements.

II. Notations and definitions. We will work in nonseparable real Banach spaces. By a *subspace* of a Banach space we mean a norm closed subspace. Unless stated otherwise, if $Y \subset X^*$, then $Y_\perp$ ($Y^\perp$) denotes the *polar* of Y to X (X^{**}). If $\langle X, Y\rangle$ is a dual pair of vector spaces, then $w(X, Y)$ is the *weak topology* on X given by the duality $\langle X, Y\rangle$. $w(X^*, X)$ (respectively $w(X, X^*)$) topology is denoted by w^*-(respectively w-) topology. If X is a Banach space (shortly, a B-space and $M \subset X$, Y a subspace of X^*, total on X, then $\operatorname{sp} M$ ($w(X, Y)\operatorname{sp} M$) denotes the linear ($w(X, Y)$ closed linear) *span* of M in X. Also we put $\overline{\operatorname{sp}}\, M = w(X, X^*)\operatorname{sp} M$. If X is a B-space and $Y \subset X^*$, $\delta > 0$, then Y is called δ-*norming* if

$$\delta \leqslant \inf_{\|x\|=1} \left(\sup\left(f(x);\ f \in Y,\ \|f\| < 1\right)\right).$$

If Y is δ-norming for some $\delta > 0$, then we say Y is *norming* of X. A B-space X is *locally uniformly rotund* (LUR) if whenever $x_n, x \in X$, $\|x_n\| = \|x\| = 1$, $\lim\|x_n + x\| = 2$, then $\lim\|x_n - x\| = 0$. X is *rotund* if whenever $x, y \in X$, $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$, then $x = y$. X is an f-*space* if it admits an equivalent norm which is Fréchet differentiable at all nonzero points.

A topological space T is called an *Eberlein compact*, if it is homeomorphic to a weakly compact subset of Banach space (in its w-topology), $\operatorname{dens} T$ denotes the *density* of T, i.e. the smallest cardinal number of a dense subset in T. The *restriction* of a map φ onto a subset $A \subset T$ is denoted by φ/A. A system $\{x_a\} \subset X$, $\{f_a\} \subset X^*$, $a \in A$, is a *Markuševič basis of* X if $f_a(x_\beta) = \delta_{a\beta}$ (the Kronecker delta), $\overline{\operatorname{sp}}\{x_a\} = X$ and $\overline{\operatorname{sp}}\{f_a\}$ (= the coefficient space of the M-basis) is total on X, which means that $\bigcap_a f_a^{-1}(0) = \{0\}$. A Markuševič basis $\{(x_a, f_a)\}$ is *shrinking* if $\overline{\operatorname{sp}}\{f_a\} = X^*$ (cf. [13], [16]). A B-space X is *somewhat reflexive* (cf. [16], [14]) if any subspace $Y \subset X$ contains a reflexive subspace $Z \subset Y$ with $\operatorname{dens} Z = \operatorname{dens} Y$.

III. Dual projections in WCG spaces.

LEMMA 1. *Let $(X, \|\cdot\|)$ be a normed linear space, Y, B closed subspaces of X, B finite dimensional. Let $|\cdot|$ be another norm on Y such that either $|y| \leqslant \|y\|$ for all $y \in Y$ or $|y| \geqslant \|y\|$ for all $y \in Y$.*

Then there is a $\|\cdot\|$-continuous projection P of X onto B such that $PY \subset Y$ and P is $|\cdot|$-continuous on Y.

Proof. Let $|y| \leqslant \|y\|$ for all $y \in Y$. Let $(a_i)_1^p$ be a basis of $B \cap Y$, completed by $a_{p+1}, \ldots, a_n$ to a basis of B. Let $i \in \{1, \ldots, p\}$. We prove that there is a $\|\cdot\|$-continuous projection P_i of X onto $\operatorname{sp}\{a_i\}$ such that $P_i(a_j) = 0$ for $j \neq i$, $P_i Y \subset Y$ and P_i/Y is $|\cdot|$-continuous. For it let f_1 be a linear functional defined on $\operatorname{sp}(a_j)_{j=1}^p$:

$$f_1\Big(\sum_{j=1}^{p} a_i a_i\Big) = a_i.$$

Let f_2 be a $|\cdot|$-continuous extension of f_1 to Y. Then f_2 is a fortiori $\|\cdot\|$-continuous on Y. Let $Z = \operatorname{sp}\big(Y, (a_j)_{j=p+1}^n\big)$. Then Z is $\|\cdot\|$-closed and $Z = Y \oplus \operatorname{sp}(a_j)_{j=p+1}^n$, topologically. Let Q be a $\|\cdot\|$-continuous projections of Z onto Y such that $Q^{-1}(0) = \operatorname{sp}(a_j)_{j=p+1}^n$. Then define the functional f_3 on Z: $f_3(z) = f_2(Pz)$. Furthermore, extend f_3 $\|\cdot\|$-continuously on X to f and define on X a projection $P_i x = f(x)\cdot a_i$. If $i \in \{p+1, \ldots, n\}$, then $a_i \notin \overline{\operatorname{sp}}\big(Y, \operatorname{sp}(a_j, j \neq i)\big)$ and thus there is a $\|\cdot\|$-continuous linear functional f on X such that $f/\overline{\operatorname{sp}}\big(Y, \operatorname{sp}(a_j, j \neq i)\big) = 0$ and $f(a_i) = 1$. Then define projection P_i on X: $P_i x = f(x) a_i$. Now take $P = P_1 + \ldots + P_n$ to have the desired projection. Similarly in case $|y| \geqslant \|y\|$ for $y \in Y$.

The following lemma is a slight generalization of Lemma 2 in [10], namely we are given another norm $|\cdot|_4$ on the subspace Y.

LEMMA 2. *Let $(X, |\cdot|_2)$ be a normed linear space with another norm $|\cdot|_1 \leqslant |\cdot|_2$ on X. Let $N \subset X$ and $Y \subset X$ be linear subspaces of X not necessarily closed. Further let $|\cdot|_3$, resp. $|\cdot|_4$ be norms of N, resp. Y. Let $|\cdot|_3 \geqslant |\cdot|_1$ on N and let on Y one of the following conditions hold:*

(i) $|y|_4 \geqslant |y|_1$ *for* $y \in Y$, *or*
(ii) $|y|_4 \leqslant |y|_1$ *for* $y \in Y$ *and* Y *is closed in* $(X, |\cdot|_1)$.

Then, given a finite-dimensional subspace $B \subset N$, n *elements* $f_1, \ldots, f_n \in (X, |\cdot|_2)^*$ *and integer* n, *there is an* $\aleph_0$-*dimensional subspace* $C \subset X$ *containing* B, *such that, for every* $\varepsilon > 0$ *and every subspace* $Z \subset X$, $Z \supset B$, $\dim Z/B = n$, *there is a linear operator*

$$T: Z \to C$$

with

$$T(Z \cap Y) \subset Y, \quad T(Z \cap N) \subset N, \quad |T|_i \leqslant 1+\varepsilon,\ i = 1, 2,$$

$$|T/Z \cap N|_3 \leqslant 1+\varepsilon, \quad |T/Z \cap Y|_4 \leqslant 1+\varepsilon, \quad Tb = b \text{ for } b \in B,$$

and

$$|f_k(z) - f_k(Tz)| \leqslant \varepsilon |z|_2 \quad \text{for} \quad z \in Z \text{ and } k = 1, 2, \ldots, m.$$

Proof. It goes similarly as the proof of Lemma 2 of [10], using in (ii) Lemma 1 of this paper instead of Lemma 1 of [10].

PROPOSITION 1. *Let* $(X, \|\cdot\|)$ *be a WCG B-space,* $Y \subset X$ *a closed subspace of* X, $|\cdot|$ *another norm on* Y, *equivalent to the norm* $\|\cdot\|$. *Let* $|\cdot|$ *be Fréchet differentiable at all nonzero points of* Y. *Denote by* μ *the first ordinal of cardinality* $\operatorname{dens} X$. *Then there is a long sequence* $\{P_\alpha;\ 0 \leqslant \alpha \leqslant \mu\}$ *of linear projections of* X *such that*

(i) $P_0 = 0$, $P_\mu =$ *identity on* X, $P_\alpha Y \subset Y$, $\|P_\alpha\| = |P_\alpha/Y| = 1$ *for* $\alpha > 0$;
(ii) $P_\alpha P_\beta = P_\beta P_\alpha$ *if* $\alpha < \beta$;
(iii) *for every* $x \in X$ *is* $\alpha \to P_\alpha x$ *a norm continuous function on ordinals*;
(iv) $\operatorname{dens} P_\alpha X \leqslant \overline{\overline{\alpha}}$, $\operatorname{dens}(P_\alpha/Y)^* Y^* \leqslant \overline{\overline{\alpha}}$ *for* $\alpha \geqslant \omega$;
(v) *for every* $y^* \in Y^*$, $\alpha \to (P_\alpha/Y)^* y^*$ *is a norm continuous function on ordinals.*

Proof. We may suppose that $|y| \geqslant \|y\|$ for $y \in Y$. Let K denote an absolutely convex weakly compact set generating X and contained in the $\|\cdot\|$-unit ball of X. We put in Lemma 2

(ii) $|x|_1 = |x|_2 = \|x\|$ for $x \in X$, $|x|_3 = \inf\{a > 0;\ x \in aK\}$ for $x \in \operatorname{sp} K = N$ and $|y|_4 = |y|$ for $y \in Y$.

Then we use the methods of [1] to derive the existence of projections (i), (ii), (iii), $\operatorname{dens} P_\alpha X \leqslant \overline{\overline{\alpha}}$ for $\alpha \geqslant \omega$.

Now, since $|P_\alpha/Y| \leqslant 1$ and $|\cdot|$ is Fréchet smooth, (v) follows from the results of [11], Lemma 3. Also, $\operatorname{dens}(P_\alpha/Y)^* Y^* = \operatorname{dens}(P_\alpha Y)^* = \operatorname{dens} P_\alpha Y$ since the norm $|\cdot|$ is Fréchet differentiable (cf. e.g. [15], the end of Section 5).

In the sequel, we will use the following lemmas. The first is a variant of Proposition 2.2 of [15].

LEMMA 3. *Let* X, Y *be WCG spaces,* $Y \subset X^*$. *Let* $P: X \to X$ *be a bounded linear projection of* X *such that* $P^* Y \subset Y$. *Then* $\operatorname{dens} P^* Y \leqslant \operatorname{dens} PX$ *and if moreover* Y *is total on* X, *then* $\operatorname{dens} PX = \operatorname{dens} P^* Y$.

Proof. Let K, K_1 be absolutely convex weakly compact sets generating X, Y, respectively. Let $M \subset PX$ be a dense subset of PX. Then, as J. Lindenstrauss in Proposition 2.2 of [15], using the restrictions of M to $P^* K_1$ and the Stone–Weierstrass theorem, we derive that $\operatorname{dens} C(P^* K_1) \leqslant \operatorname{card} M$ (where $P^* K_1$ is in w-topology) and thus there is a set $M_1 \subset P^* K_1$ such that M_1 is w-dense in $P^* K_1$ and $\operatorname{card} M = \operatorname{card} M_1$. Thus $\operatorname{dens} P^* Y \leqslant \operatorname{dens} PX$. The reversed inequality is proved similarly.

LEMMA 4. *Let* $X, Z \subset X^*$ *be B-spaces and let* $Z = \overline{\operatorname{sp}} K$, *where* K *is an absolutely convex weakly compact set in* Z. *Let* P *be a norm bounded linear operator of* X *into* X, *such that* $PK_0 \subset K_0$ (*where* K_0 *is the polar of* K *in* X). *Then* P *is* $w(X, Z) - w(X, Z)$-*continuous.*

Proof. For $P^*: X^* \to X^*$ we have $P^* K = P^*(K_0)^0 \subset (K_0)^0 = K$ and thus from boundedness of P, also $P^* Z \subset Z$.

PROPOSITION 2. *Let* Y *be a closed subspace of a WCG B-space* $(X, |\cdot|)$ *and let* $Z \subset Y^*$ *be a WCG subspace total on* Y. *Denote by* μ *the first ordinal of cardinality* $\operatorname{dens} X$. *Then there is a long sequence* $\{P_\alpha;\ 0 \leqslant \alpha \leqslant \mu\}$ *of linear projections of* X *such that:*

(i) $P_0 = 0$, $P_\mu =$ *identity,* $|P_\alpha| = 1$ *for* $\alpha > 0$, $P_\alpha Y \subset Y$;
(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ *if* $\alpha < \beta$;
(iii) *for every* $x \in X$, *the function* $\alpha \to P_\alpha x$ *is norm continuous on ordinals* $\langle 0, \mu \rangle$;
(iv) $(P_\alpha/Y)^* z$ *is norm continuous on ordinals for every* $z \in Z$;
(v) $\operatorname{dens} P_\alpha X \leqslant \overline{\overline{\alpha}}$ *and* $\operatorname{dens}(P_\alpha/Y)^* Z \leqslant \overline{\overline{\alpha}}$ *for* $\alpha \geqslant \omega$;
(vi) P_α/Y *are* $w(Y, Z) - w(Y, Z)$-*continuous.*

Proof. Denote by K, K_1 the absolutely convex weakly compact sets which generate X, Z, respectively and are contained in the unit balls. We put in Lemma 2(i): $|\cdot|_1 = |\cdot|_2 = |\cdot|$ on X, $|x|_3 = \inf\{a > 0;\ x \in aK\}$ for $x \in \operatorname{sp} K = N$ and $|y|_4 = \sup\{y(k_1);\ k_1 \in K_1\}$ for $y \in Y$. Then we use the methods of [1] and Lemma 4 to derive the existence of a long sequence $\{P_\alpha;\ 0 \leqslant \alpha \leqslant \mu\}$ of projections satisfying (i)–(iii), the first part of (v) and (vi). We prove that then also (iv) holds. For it let α be a limiting ordinal and $z \in K_1$. Then

$$\lim_{\beta \nearrow \alpha} P_\beta^* z = P_\alpha z$$

in the $w(Z, X)$ topology, as it is easily seen from (iii) and (i). Since $P_\xi z \in K_1$ for $\xi \leqslant \mu$, and on K_1 the weak and $w(Z, X)$ topologies coincide, we have

$$\lim_{\beta \nearrow \alpha} P_\beta^* z = P_\alpha^* z$$

in the weak topology of Z. From this, the boundedness of P_α's and from the argument of [1] (see Lemma 2 of [11]), (iv) follows. Furthermore Lemma 3 is used to derive (v).

PROPOSITION 3. *Let* $(X, |\cdot|)$ *be a WCG B-space*, $Y \subset X^*$ *a closed subspace of* $(X, |\cdot|)$ *and let one of the following two conditions hold:*

(a) Y *is WCG, or*

(b) X^* *is WCG.*

Let $\|\cdot\|$ *be an equivalent norm on* Y *and denote by* μ *the first ordinal of cardinality* $\operatorname{dens} Y$. *Then there is a long sequence* $\{P_\alpha; 0 \leqslant \alpha \leqslant \mu\}$ *of linear projections of* X *such that*

(i) $P_0 = 0, P_\mu =$ *identity on* Y, $P_\alpha^* Y \subset Y$, $|P_\alpha| = \|P_\alpha^*/Y\| = 1$ *if* $\alpha > 0$;

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ *if* $\alpha < \beta$;

(iii) $\operatorname{dens}_\alpha PX \leqslant \overline{\alpha}$, $\operatorname{dens} P_\alpha^* Y \leqslant \overline{\overline{\alpha}}$ *for* $\alpha \geqslant \omega$;

(iv) *for every* $x \in X$ *the function* $\alpha \to P_\alpha x$ *is norm continuous on ordinals;*

(v) *for every* $y \in Y$ *the function* $\alpha \to P_\alpha^* y$ *is norm continuous on ordinals.*

Proof of (a). This requires a different approach than that of the statements above. We may suppose that $\|y\| > |y|$ for $y \in Y$. Let K, K_1 be absolutely convex weakly compact sets generating X, Y, respectively. We put in Lemma 2(i) $X = X^*$, $|\cdot|_2 = |\cdot|$ on X^*, $|f|_1 = \sup\{f(k); k \in K\}$ for $f \in X^*$, $Y = Y$, $N = \operatorname{sp} K_1$, $|f|_3 = \inf\{\alpha > 0;\ f \in \alpha K_1\}$ on N, $|f|_4 = \|f\|$ on Y. We then work on X^* in the w^*-topology (details for this approach are in [12]). The limiting points of operators T_z and projections appearing in Amir–Lindenstrauss construction exist in the w^* sense by the w^*-compactness of the $|\cdot|$-unit ball of X^*. The fact that the w^*-cluster point Q of the sequence $\{T_n\}$ from the proof of Lemma 4 of [1] is a projection follows from the $w^* - w^*$ continuity of Q (see Lemma 4). As in Lemma 6 of [1] we choose a dense subset $\{y_\alpha; \alpha < \mu\}$ of N. Using the fact that all operators appearing in the construction are bounded and preserve K_1 and thus N and Y (Lemma 4), we see that the cluster points may be taken with respect to the w^*-topology on X and simultaneously with respect to the weak topology on Y. We use also Lemma 3 and the fact that if X is WCG, then $w^* \operatorname{dens} P^* X = \operatorname{dens} PX$ (Proposition 2.2 of [15]). From these remarks, and on Amir–Lindenstrauss construction (see [1], [10], [11]) the proof of part (a) follows.

Proof of (b). Let K, K_1 be absolutely convex weakly compact sets generating X, X^*, respectively. We use again Lemma 2(i), putting: $(X, |\cdot|_2) = (X^*, |\cdot|)$, $|f|_1 = \sup\{f(k); k \in K\}$ for $f \in X^*$, $Y = \operatorname{sp} K_1$, $N = Y$, $|f|_3 = \|f\|$ for $f \in Y$ and $|f|_4 = \inf\{\alpha > 0; f \in \alpha K_1\}$ for $f \in \operatorname{sp} K_1$. We then work on X^* and the cluster points of projections and linear operators in Amir–Lindenstrauss construction ([1]) are in weak topology on X^* because they preserve K_1. Further we proceed similarly as in (a). As in Lemma 6 of [1] we choose a dense subset $\{y_\alpha; \alpha < \mu\}$ of $Y = N$.

At the end of this section we show two propositions on M-bases:

PROPOSITION 4. *If* X *is a WCG space and* $Y \subset X^*$ *a WCG total subspace of* X^*, *then there is an M-basis* $\{(x_\alpha, f_\alpha); \alpha \in I\}$ *of* X, *such that*

$$\overline{\operatorname{sp}}\{f_\alpha; \alpha \in I\} = Y.$$

Proof. By transfinite induction on $\operatorname{dens} X = \operatorname{dens} Y$ (Lemma 3). If $\operatorname{dens} X = \operatorname{dens} Y = \aleph_0$, then we have Theorem III.1 of [12]. If $\operatorname{dens} X = \aleph$ and μ is the first ordinal of cardinality $\aleph$, then by Proposition 2 (or by Proposition 3) there is a long sequence of projections $\{P_\alpha; 0 \leqslant \alpha \leqslant \mu\}$ on X, $|P_\alpha| = 1$, $\operatorname{dens} P_\alpha X < \aleph$, $P_\alpha^* Y \subset Y$ and $\alpha \to P_\alpha^* y$ norm continuous for all $y \in Y$. By the induction hypothesis there is an M-basis $\{x_i^\alpha, i \in I_\alpha\}$ of $(P_{\alpha+1} - P_\alpha) X$ such that

$$\overline{\operatorname{sp}}\{f_i^\alpha;\ i \in I_\alpha\} = (P_{\alpha+1}^* - P_\alpha^*)\, Y \quad \text{for} \quad 0 \leqslant \alpha < \mu.$$

Evidently $\{(x_\alpha^i, f_\alpha^i);\ i \in I_\alpha,\ 0 \leqslant \alpha < \mu\}$ is an M-basis of X such that

$$\overline{\operatorname{sp}}\{f_i^\alpha; i \in I_\alpha,\ 0 \leqslant \alpha < \mu\} = Y.$$

PROPOSITION 5. *Let* X *be a WCG f-space. Then every bounded subset* $B \subset X$ *is isomorphic to a subspace of* $c_0(\Gamma)$ *(for certain* Γ*) by an affine homeomorphism with respect to weak topologies.*

Proof. By Theorem 1 of [11], there is a shrinking M-basis $\{(x_i, f_i); i \in \Gamma\}$ of X. Suppose that $|f_i| \leqslant 1$ for all $i \in \Gamma$. Then for every $x \in X$ is $T(x) = \{f_i(x)\} \in c_0(\Gamma)$ and thus T is continuous linear imbedding of X into $c_0(\Gamma)$. Now let $\{x_\alpha\} \subset B$ be a net and $x \in B$. Then we have

$$f(x_\alpha) \to f(x) \quad \text{for every} \quad f \in X^* \Leftrightarrow f_i(x_\alpha) \underset{\alpha}{\to} f_i(x) \quad \text{for every} \quad i \in \Gamma$$

because $\{x_\alpha, x\} \in B$ is bounded and $\overline{\operatorname{sp}}\{f_i\} = X^*$. But

$$f_i(x_\alpha) = (Tx_\alpha)_i \underset{\alpha}{\to} f_i(x) = (Tx)_i \Leftrightarrow Tx_\alpha \to Tx$$

in the weak topology on $c_0(\Gamma)$ because $\{Tx_\alpha, Tx\}$ is bounded in $c_0(\Gamma)$.

IV. A renorming theorem. The main result of this section is:

THEOREM 1. *If* X, $Y \subset X^*$ *are WCG Banach spaces,* Y *norming on* X, *then there is an equivalent norm* $\|\cdot\|$ *on* X *with the following six properties:*

(i) $\|\cdot\|$ *is* $w(X, Y)$ *lower semicontinuous;*

(ii) $\|\cdot\|$ *is locally uniformly rotund;*

(iii) *on the unit sphere* $\{x \in X; \|x\| = 1\}$ *the norm and the* $w(X, Y)$ *topologies coincide;*

(iv) *the dual norm* $\|\cdot\|^*$ *on* Y *is locally uniformly rotund;*

(v) *on the unit sphere* $\{y \in Y;\ \|y\|^* = 1\}$ *the norm and the* $w(Y, X)$ *topologies coincide;*

(vi) *the dual norm* $\|\cdot\|^*$ *on* X^* *is rotund.*

Before proving Theorem 1 we state some corollaries of it.

COROLLARY 1. *Under the assumptions of Theorem 1, X has a Gâteaux smooth partitions of unity (subordinated to any open covering) of X.*

Proof. By the results of H. Toruńczyk ([22], Theorem 1) it suffices to prove the following lemma.

LEMMA 5. *Under the assumptions of Theorem 1, there is a homeomorphic imbedding u of X into $c_0(\Gamma)$ for some Γ such that $p_a \circ u$ is Gâteaux differentiable, where $p_a \in c_0(\Gamma)^*$ denotes the functional $(x_a) \to x_a$.*

Proof of Lemma 5. First, using Proposition 4, we see that there is an M-basis $\{x_a\} \subset X$, $\{f_a\} \subset Y$, $a \in \Gamma$, $\{f_a\}$ bounded, such that $\overline{\mathrm{sp}}\{f_a\} = Y$. Let $1 \notin \Gamma$ and define $u\colon X \to c_0(\Gamma \cup \{1\})$:

$$P_a \circ u(x) = \begin{cases} \|x\|^2 & \text{for} \quad a = 1, \\ f_a(x) & \text{for} \quad a \in \Gamma, \end{cases}$$

where $\|\cdot\|$ is a norm from Theorem 1.

Then to prove that u^{-1} is continuous, let

$$\lim_n |u(x_n) - u(x)| = 0.$$

Then

$$\lim_n \|x_n\| = \|x\| \quad \text{and} \quad \lim_n f_a(x_n - x) = 0.$$

Thus by the property (iii) of the norm $\|\cdot\|$ from Theorem 1,

$$\lim_n \|x_n - x\| = 0.$$

COROLLARY 2 ([10]). *If X, X^* are both WCG, then there is an equivalent norm on X which is LUR, the dual of which on X^* is LUR and bidual on X^{**} is rotund.*

Proof. Put in Theorem 1: $Y = X$, $X = X^*$.

Remark. Assumptions of Theorem 1 do not cover all spaces with nice renorming properties. For example, there is no WCG total subspace $Y \subset (c_0(\Gamma))^*$, Γ uncountable, since $l_1(\Gamma)$ has only separable WCG subspaces by the Phillips theorem (cf. e.g. [15], Section 2) and total Y cannot be w^* separable (Proposition 2.2 of [15]).

First we need the following two observations:

LEMMA 6 ([16]). *If X, $Y \subset X^*$ are Banach spaces, Y norming on X, then Y is 1-norming on $(X, |\cdot|)$, where $|\cdot|$ is an equivalent norm on X given by*

$$|x| = \sup\{f(x);\ f \in Y,\ \|f\| \leqslant 1\}.$$

Proof. Denote by K_1 the unit ball of X^* in its original norm. To see that Y is 1-norming on $(X, |\cdot|)$, it suffices to show that $((K_1 \cap Y)_0)^0 \cap Y$ is w^* dense in $((K_1 \cap Y)_0)^0$. But $((K_1 \cap Y)_0)^0 \cap Y \supset K_1 \cap Y$ and $K_1 \cap Y$ is w^* dense in $((K_1 \cap Y)_0)^0$ by the bipolar theorem.

LEMMA 7. *Let $(X, |\cdot|)$ be a Banach space, τ a linear Hausdorff topology on X such that $|\cdot|$ is τ-lower semicontinuous. If L is a finite-dimensional subspace of X, then*

$$\varrho(x, L) = \inf\{|x - l|;\ l \in L\}$$

is τ-lower semicontinuous on norm bounded sets.

Proof. We will show that $\liminf \varrho(x_a) \geqslant \varrho(x)$ if $\lim x_a = x$ in the τ-topology. First, from $\lim x_a = x$ in τ follows that there is a subnet $\{x_\nu\}$ of the net $\{x_a\}$ and a norm bounded net $\{y_\nu\} \subset L$ such that

$$\lim \varrho(x_\nu) = \liminf \varrho(x_a) = \lim |x_\nu - y_\nu|,$$

$$\lim |y_\nu - y| = 0 \quad \text{for some } y \in L.$$

Then

$$\liminf \varrho(x_a) = \lim |x_\nu - y_\nu| = \lim |x_\nu - y| \geqslant |x - y| \geqslant \varrho(x),$$

by the τ-lower semicontinuity of the norm $|\cdot|$.

Also, we need the following

LEMMA 8. *Let $Y \subset X^*$ be WCG B-spaces, Y total on X. Then there is a bounded linear one-to-one embedding of X into $c_0(\Gamma)$ for some Γ which is $w(X, Y) - w(c_0(\Gamma), l_1(\Gamma))$ continuous.*

Proof. We have $X \subset Y^*$ imbedded continuously in $w(X, Y) - w^*$ sense. By Proposition 2 of [1], there is $w^* - w$ continuous imbedding of Y^* into some $c_0(\Gamma)$.

LEMMA 9. *Under the assumptions of Theorem 1, there is an equivalent norm $\|\cdot\|_1$ on X with the properties* (i)–(iii) *of Theorem 1 and there is an equivalent norm $\|\cdot\|_2$ on X with the properties* (i), (iv), (v).

Proof. First we introduce an equivalent norm $|\cdot|$ defined in Lemma 6. Then in Proposition 2 we put $Y = X = (X, |\cdot|)$, $Z = Y$. Using this decomposition of X, Lemma 7 and the $w(X, Y)$ continuous imbedding of X into $c_0(\Gamma)$ from Lemma 8, we see that Troyanski's LUR-renorming construction on X ([23]) can be built up in the $w(X, Y)$ sense. The property (iii) follows easily from (i) and (ii): if $x_a, x \in X$, $\|x\|_1 = \|x_a\|_1 = 1$, $\lim x_a = x$ in the $w(X, Y)$ topology, then by (i),

$$2 \geqslant \limsup \|x_a + x\|_1 \geqslant \liminf \|x_a + x\|_1 \geqslant 2\|x\|_1 = 2.$$

Thus, by (ii), $\lim \|x_a - x\|_1 = 0$. For the second part of the statement we construct similarly a w^*-lower continuous LUR norm on Y. Then we use its dual norm on X.

LEMMA 10. *Under the assumptions of Theorem* 1, *there is an equivalent norm* $\|\cdot\|_3$ *on* X *with the properties* (i)–(v).

Proof. We use the following variant of an averaging procedure of E. Asplund ([2], [3]): Starting with $f_0 = \frac{1}{2}\|\cdot\|_1^2$, $g_0 = \frac{1}{2}\|\cdot\|_2^2$ and supposing that $g_0 \leqslant f_0 \leqslant (1+C)g_0$, we define

$$f_{n+1} = \tfrac{1}{2}(f_n+g_n), \qquad g_{n+1} = \left(\tfrac{1}{2}(f_n^*+g_n^*)\right)_*, \qquad n \geqslant 0,$$

where f_n^* denotes the dual function of f_n on Y in the sense of Fenchel and $\left(\frac{1}{2}(f_n^*+g_n^*)\right)_*$ means the dual function on X. Then, similarly as in [3], [2], we have

$$g_n \leqslant f_n, \quad g_n \leqslant g_{n+1}, \quad f_n \geqslant f_{n+1}, \quad n \geqslant 0, \quad g_{n+1} \leqslant f_{n+1} \leqslant (1+4^{-n}C)g_{n+1}.$$

From this follows ([3], [2]), that its common limit k is LUR, the dual of k on Y is also LUR. Furthermore, k is $w(X, Y)$-lower semicontinuous, since f_n, g_n are such and k is the supremum of g_n.

LEMMA 11. *If* X *is WCG and* $Y \subset X^*$ 1-*norming, then there is an equivalent norm* $\|\cdot\|_4$ *on* X *which is* $w(X, Y)$-*lower semicontinuous and whose dual norm on* X^* *is rotund.*

Proof. We will work on X with the norm $|\cdot|$ introduced in Lemma 6. Let T be a bounded linear one-to-one $w(X^*, X) - w(c_0(\Gamma), l_1(\Gamma))$ continuous mapping of X^* into $c_0(\Gamma)$ (see Lemma 8). Put for $x^* \in X^*$

$$p(x^*) = \tfrac{1}{2}|x|^2 + \tfrac{1}{2}|Tx|^2,$$

where $|Tx|$ means the Day's LUR norm on $c_0(\Gamma)$ (see e.g. [23]). Denote by K_1 the polar of the unit ball of Day's norm in $l_1(\Gamma)$ and by q the extended-valued Minkowski functional of the $w(X, Y)$ compact set $T^*K_1 \subset X$. Then for the Fenchel dual function of $\frac{1}{2}q^2$ in X^* we have

$$\begin{aligned}(\tfrac{1}{2}q^2)^*(x^*) &= \sup_{x\in X}(x^*, x) - \tfrac{1}{2}q^2(x) = \sup_{c>0}\Big(\sup_{q(x)=c}(x^*, x) - \tfrac{1}{2}q^2(x)\Big)\\ &= \sup_{c>0}\Big(\sup_{J\in cK_1}(x^*, T^*y) - \tfrac{1}{2}c^2\Big) = \sup_{c>0}(|Tx^*|c - \tfrac{1}{2}c^2) = \tfrac{1}{2}|T^*x^*|^2.\end{aligned}$$

Now if we put $r =$ inf-convolution of $\frac{1}{2}|\cdot|^2$, $\frac{1}{2}q^2$, then r is $w(X, Y)$-lower semicontinuous since $\frac{1}{2}q^2$ is inf-compact in $w(X, Y)$ (see [17], p. 22) and $\frac{1}{2}|\cdot|^2$ is $w(X, Y)$-lower semicontinuous ([17], p. 23). Furthermore, an easy calculation shows that $r^* = p$ on X^* (see [3], p. 22), so, $\sqrt{r}$ is the desired norm on X.

Proof of Theorem 1. We use again an averaging procedure of E. Asplund ([3], [2]) introduced in the proof of Lemma 11, i.e. in the duality $\langle X, Y\rangle$, for the norms $\|\cdot\|_3$, $\|\cdot\|_4$ on X. The fact that the dual function of k on X from the proof of Lemma 10 is rotund follows from the inequalities

$$f_{n+1}^* \leqslant g_{n+1}^* \leqslant (1+4^{-n}C)f_{n+1}^*$$

(where the stars mean the dual functions on X^*) and from the fact that $g_n^* = 2^{-n}|T(x^*)|^2 + h_n$ on X, where T is a map from Lemma 11 and h_n is a $w(X^*, X)$-lower semicontinuous convex function on X^*. To sketch the proof of this here, assume it holds for numbers $\leqslant n$. Then

$$\begin{aligned}g_{n+1}^*(x) &= \Big(\big((\tfrac{1}{2}f_n^* + \tfrac{1}{2}g_n^*)(y)\big)_*\Big)^* = \big((\tfrac{1}{2}f_n^* + \tfrac{1}{2}h_n) + 2^{-n+1}|T(\cdot)|^2\big)\\ &= \Big(\text{inf-convolution}\big((\tfrac{1}{2}f_n^* + \tfrac{1}{2}h_n)_*, 2^{n-1}q^2\big)\Big)^* = h_{n+1} + 2^{-n-1}|T(x^*)|^2\end{aligned}$$

(we use the notations from the proof of Lemma 10).

V. On heredity of WCG property. First we state the following sufficient property for WCG.

LEMMA 12. *If a Banach space* X *has a shrinking Markuševič basis, then* X *is WCG.*

Proof. It is easy to see (cf. [23], [11]) that if $\{x_a\} \subset X$, $\{f_a\} \subset X^*$, $a \in A$ is a shrinking M-basis of X with $\{x_a\}$ bounded, then $\{x_a\} \cup \{0\}$ is a weakly compact set generating X.

Now we state the main result of this section.

THEOREM 2. *A Banach space* X *has a shrinking Markuševič basis if one of the following conditions holds:*

(i) X *has an equivalent Fréchet smooth norm and* X *is a subspace of a WCG* B-*space* Z;

(ii) X^* *is WCG and* X *is a subspace of a WCG* B-*space* Z.

Proof. Similarly as in [11] and using Proposition 1 in (i) and Proposition 2 in (ii).

COROLLARY. *A Banach space* X *is WCG if one of the following conditions holds:*

(i) X *is an* f-*space and* X *is a subspace of a WCG space*;

(ii) X *is an* f-*space and the unit ball of* X^* *with* w^* *topology is an Eberlein compact* (for the definition see Section II);

(iii) X^* *is WCG and the unit ball of* X *with* w^* *topology is an Eberlein compact.*

Proof. It follows from Theorem 1, Lemma 5 and the result of D. Amir and J. Lindenstrauss that if K is an Eberlein compact, then $C(K)$ is WCG ([1]).

VI. Long basic sequences and their application to the study of nonseparable Banach spaces. First we state some definitions.

A long sequence $\{S_a; 0 \leqslant a \leqslant \xi\}$ of linear projections of a Banach space X is called a *projectional basis of the type* ξ ([6]) if

(i) $S_0 = 0$, $S_\xi =$ identity on X, $\sup\{|S_a|; a \leqslant \xi\} < \infty$;

(ii) $S_\alpha S_\beta = S_\beta S_\alpha = S_\alpha$ if $\alpha < \beta$;

(iii) for every $x \in X$ is the function $\alpha \to S_\alpha x$ norm continuous on ordinals;

(iv) $\dim(S_{\alpha+1} - S_\alpha)X = 1$ for $\alpha < \xi$.

Then any system $\{(e_\alpha, f_\alpha); \alpha < \xi\}$ such that

$$e_\alpha \in X, \quad f_\alpha \in X^*, \quad f_\alpha(x) e_\alpha = (S_{\alpha+1} - S_\alpha)x$$

is said to be a *biorthogonal system* associated with $\{S_\alpha; \alpha \leqslant \xi\}$.

We call any system $\{e_\alpha; \alpha < \xi\}$ of the elements of X a *basis* of X *of the type* ξ if there is a projectional basis $\{S_\alpha; \alpha \leqslant \xi\}$ of X of the type ξ such that $e_\alpha \in (S_{\alpha+1} - S_\alpha)X$.

It is easy to see that a basis $\{e_\alpha; \alpha < \xi\}$ determines its projectional basis uniquely: if $x \in \operatorname{sp}\{e_\alpha\}$, $x = \sum_{i \in K} \lambda_i e_i$, K finite, then $S_\alpha x = \sum_{i < \alpha} \lambda_i e_i$.

A system $\{e_\alpha; \alpha < \xi\}$ is called a (*long*) *basic sequence* if it forms a basis for $\overline{\operatorname{sp}}\{e_\alpha; \alpha < \xi\}$ A system of projections $\{S_\alpha; \alpha \leqslant \xi\}$ of a w^* closed subspace Y of X is said to be a w^* *projectional basis* of Y if (i) and (ii) of the definition above hold for $\{S_\alpha\}$, S_α are $w^* - w^*$ continuous on Y and $S_\alpha y$ is a w^* continuous function on ordinals for every fixed $y \in Y$.

Dealing with these notions, we have two definitions of bounded completeness. First is a classical one analogous to that for Schauder bases:

A projectional basis $\{S_\alpha; \alpha \leqslant \xi\}$ of X is called S-*boundedly complete* if whenever $\gamma \leqslant \xi$ is a limiting ordinal and $\{y_\beta; \beta < \gamma\}$, where $y_\beta = S_\beta y_\alpha$ for $\gamma > \alpha > \beta$, is a norm bounded net, then $\lim_{\beta \to \gamma} y_\beta$ exists.

The second is that given for Markuševič bases in [13], [16]: A Markuševič basis $\{(e_\alpha, f_\alpha); \alpha \in I\}$ of a Banach space X is said to be M-*boundedly complete* if whenever $\{x_\nu\}$ is a bounded net in X such that $\lim_\nu f_\alpha(x_\nu)$ exists for each α, then there is an $x \in X$ such that $f_\alpha(x) = \lim_\nu f_\alpha(x_\nu)$ for each $\alpha \in I$.

A long basic sequence $\{e_\alpha; \alpha < \xi\}$ is called *shrinking* (boundedly complete) if $\{e_\alpha; \alpha < \xi\}$ is a shrinking (boundedly complete) basis of $\overline{\operatorname{sp}}\{e_\alpha; \alpha < \xi\}$.

Evidently M-basis (e_α, f_α) of X is M-boundedly complete if and only if every bounded subset of X is relatively compact in the $w(X, \operatorname{sp}\{f_\alpha\})$ topology. The "only if" part uses Tychonoff's theorem.

Remarks. 1. It is easy to see that the S-bounded completeness of a projectional basis $\{S_\alpha; \alpha \leqslant \xi\}$ implies that whenever β is a limiting ordinal, and $\{\beta_\nu; \nu \in I\}$ is a net of ordinals $\beta_\nu \leqslant \beta$ such that $\lim_\nu \beta_\nu = \beta$, and $\{y_\nu\}$ is a bounded net such that $y_\nu = S_{\beta_\nu} y_\mu$ for $\nu < \mu \in I$, then $\{y_\nu\}$ is a norm convergent net.

2. Similarly as in [13], if $T: X \to X/Y_\perp$, for a w^* closed subspace $Y \subset X^*$ is the quotient map, then $T^*: (X/Y_\perp)^* \to Y$ is a w^* isomorphism and norm isometry and $\{S_\alpha; \alpha \leqslant \xi\}: Y \to Y$ is a w^* projectional basis of Y iff $\{T^{*-1} S_\alpha T^*\}$ is a projectional basis of $X/Y_\perp$.

Furthermore, analogously to [13], we call $\{e_\alpha; \alpha < \xi\}$, $e_\alpha \in X^*$ a *long* w^* *basic sequence* if there is a w^* projectional basis $\{S_\alpha; \alpha \leqslant \xi\}$ of $Y = w^*\operatorname{sp}\{e_\alpha; \alpha < \xi\}$ such that $e_\alpha \in (S_{\alpha+1} - S_\alpha)X$.

To compare the two definitions of bounded completeness we state the following

Proposition 6. *If $\{S_\alpha; \alpha \leqslant \xi\}$ is a projectional basis of a Banach space X, and $\{x_\alpha, f_\alpha\}$ is its biorthogonal system, then*

(i) *If $\{x_\alpha, f_\alpha\}$ is M-boundedly complete, then $\{S_\alpha; \alpha \leqslant \xi\}$ is S-boundedly complete*;

(ii) *If $Y = \overline{\operatorname{sp}}\{f_\alpha\} \subset X^*$ is norming on X and $\{S_\alpha; \alpha \leqslant \xi\}$ is S-boundedly complete, then (x_α, f_α) is M-boundedly complete*;

(iii) *If $\{S_\alpha; \alpha \leqslant \xi\}$ is a w^* basis of a w^* closed subspace $Y \subset X^*$, and $\{f_\alpha\} \subset X$, then $\overline{\operatorname{sp}}\{f_\alpha\}$ is norming on Y.*

Proof. (i) Let $\gamma \leqslant \xi$ be a limiting ordinal and $\{y_\alpha; \alpha \leqslant \gamma\}$ be a bounded net such that $y_\alpha = S_\alpha y_\beta$ for $\alpha \leqslant \beta$. We are to prove that $\{y_\alpha\}$ is a norm convergent net. For it observe that if $\delta \geqslant \gamma$, then $f_\delta(y_\alpha) = f_\delta(S_\alpha y_\alpha) = 0$. If $\delta < \gamma$, then $f_\delta(y_\alpha)$ is equal to some number λ_δ for $\alpha > \delta$, so, $\lim_{\alpha \to \gamma} f_\delta(y_\alpha)$ exists for any $\delta < \xi$. Thus, by the M-bounded completeness of $\{x_\alpha; f_\alpha\}$, there is a $y \in X$ such that $f_\delta(y) = 0$ for $\delta \geqslant \gamma$ and $f_\delta(y) = \lambda_\delta$ for $\delta < \gamma$.

We have for every fixed $\alpha \leqslant \gamma$:

$$f_\delta(S_\alpha y) = f_\delta(y) = \lambda_\delta = f_\delta(y_\alpha) \quad \text{for} \quad \delta < \alpha$$

and

$$f_\delta(S_\alpha y) = 0 = f_\delta(y_\alpha) \quad \text{for} \quad \delta \geqslant \alpha.$$

Thus $S_\alpha y = y_\alpha$ by the totality of $\{f_\alpha\}$ on X. Furthermore, since for $\delta \geqslant \gamma$, $S_\delta y = S_\gamma y$ by the transfinite induction, we have that

$$\lim_{\alpha \to \gamma} S_\alpha y = S_\gamma y = \lim_{\delta \to \xi} S_\delta y = y.$$

Thus $\lim_{\alpha \to \gamma} y_\alpha = y$.

(ii) Suppose that the unit ball of X is $w(X, Y)$ closed (Lemma 6), where $Y = \overline{\operatorname{sp}}\{f_\alpha\}$. Let $\{x_i; i \in I\}$ be a bounded net in X with $\lim_i f_\alpha(x_i) = \lambda_\alpha$ for $\alpha < \xi$. We are to prove that there is an $x \in X$ such that $f_\alpha(x) = \lambda_\alpha$ for $\alpha < \xi$. First we show by induction on α that $\lim_i S_\alpha x_i$ equals to some y_α for any $\alpha \leqslant \xi$ in the $w(X, Y)$ sense, and $f_\gamma(y_\alpha) = 0$ if $\gamma \geqslant \alpha$ and $f_\gamma(y_\alpha) = \lambda_\gamma$ if $\gamma < \alpha$. This is true for $\alpha = 0, 1$ and also supposing it for $\beta - 1$, it holds for β if β is not limiting. Suppose it holds for $\alpha < \beta$, β limiting.

Then the set $\{S_\alpha(x_i);\ \alpha<\beta,\ i\in I\}$ is bounded and since the unit ball of X is $w(X, Y)$ closed, $\{y_\alpha;\ \alpha<\beta\}$ is a bounded net. Now, if $\beta>\delta\geqslant\alpha$, then

$$f_\gamma(S_\alpha y_\delta) = 0 = f_\gamma(y_\alpha) \quad \text{for} \quad \gamma\geqslant\alpha$$

and

$$f_\gamma(S_\alpha y_\delta) = f_\gamma(y_\delta) = \lambda_\delta \quad \text{for} \quad \gamma<\alpha\leqslant\delta.$$

Thus, by the totality of $\{f_\gamma\}$ on X,

$$S_\alpha y_\delta = y_\alpha \quad \text{for} \quad \alpha\leqslant\delta<\beta.$$

Therefore by the B-bounded completeness of $\{S_\alpha;\ \alpha\leqslant\xi\}$, $\{y_\alpha;\ \alpha<\beta\}$ is norm convergent to some y_β. Now we observe that if $\delta\geqslant\beta$, then

$$f_\delta(y_\beta) = \lim_{\alpha\to\beta} f_\delta(y_\alpha) = 0$$

and if $\delta<\beta$, then

$$f_\delta(y_\beta) = \lim_{\alpha\to\beta} f_\delta(y_\alpha) = \lambda_\delta.$$

Moreover, if $\delta\geqslant\beta$, then

$$f_\delta(S_\beta x_i) = 0$$

and if $\delta<\beta$, then

$$\lim_i f_\delta(S_\beta x_i) = \lim_i f_\delta(x_i) = \lambda_\delta.$$

Thus $\lim_i f_\delta(S_\beta x_i) = f_\delta(y_\beta)$ for all δ, showing that $\lim S_\beta x_i = y_\beta$ in the $w(X, Y)$ sense. Hence the inductive proof is finished. If we take now $\alpha=\xi$ and put $x=y_\xi$, then we obtain

$$\lim_i S_\xi x_i = \lim_i x_i = y_\xi = x$$

in the $w(X, Y)$ sense and $f_\alpha(x) = \lambda_\alpha$ for $\alpha<\xi$. The proof is thus finished.

(iii) From the definition of the w^* basis of Y easily follows that if $T: X\to X/Y$ is the canonical mapping, then $\{Tf_\alpha, e_\alpha\}$ is a basis of $X/Y_\perp$. Thus $\overline{\mathrm{sp}}\{Tf_\alpha\} = X/Y$ and thus $\overline{\mathrm{sp}}\{f_\alpha\}$ is norming on Y.

In the connection with Proposition 6 let us remark that W. Johnson proved in [12], p. 173, that a coefficient space of a boundedly complete Markuševič basis of X is norming on X.

If $\{S_n\}$ is a countable Schauder basis of X, then its biorthogonal coefficients $\{f_n\}$ are norming on X (cf. e.g. [16]). We show it here briefly for completeness:

It is known that X can be renormed so that $\{S_n\}$ is orthogonal ($|x| = \sup_n \|S_n x\|$). Let $x\in X$, $|x|=1$ and let $\varepsilon>0$. Choose $f\in X^*$ with $|f| = f(x) = 1$. Then

$$1 = f(x) = \lim f(S_n x) = \lim (P_n^* f)(x).$$

Let n be such that $(P_n^*f)(x) > 1-\varepsilon$. Evidently

$$P_n^* f\in \mathrm{sp}\{f_i\} \quad \text{and} \quad |P_n^* f|\leqslant |f| = 1.$$

Thus Proposition 6 says that in the separable case is a basis M-boundedly complete iff it is S-boundedly complete (c.f. [12], Theorem II.3.(1)).

We will need the following two observations.

Lemma 13. *Let $\{x_\alpha;\ \alpha<\xi\}$ be a linearly independent long sequence. Define on $\mathrm{sp}\{x_\alpha\}$ the projections S_α,*

$$S_\alpha\Big(\sum \lambda_i x_i\Big) = \sum_{i<\alpha} \lambda_i x_i.$$

Then $\{x_\alpha;\ \alpha<\xi\}$ is basic iff

$$\sup\{|S_\alpha|;\ \alpha<\xi\}<\infty.$$

Proof. $S_\alpha y$ is obviously continuous on $\mathrm{sp}\{x_\alpha\}$ and thus continuous on $\overline{\mathrm{sp}}\{x_\alpha\}$ since $\sup\{|S_\alpha|;\ \alpha<\xi\}<\infty$.

Lemma 14. *Let $\{P_\alpha;\ \alpha\leqslant\xi\}$ be a long sequence of linear projections of a Banach space X such that*

$$\sup\{|P_\alpha|;\ \alpha\leqslant\xi\}<\infty$$

and

$$P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha \quad \text{if} \quad \alpha<\beta \quad \text{and} \quad P_\alpha\neq P_{\alpha+1}.$$

If $0\neq x_\alpha\in(P_{\alpha+1}-P_\alpha)X$, then $\{x_\alpha;\ \alpha<\xi\}$ is a long basic sequence where $\{P_\alpha/Z;\ \alpha\leqslant\xi\}$ is the associated projectional basis of $Z=\overline{\mathrm{sp}}\{x_\alpha;\ \alpha<\xi\}$. If moreover the function $\alpha\to P_\alpha x$ is norm continuous on ordinals and

$$0\neq y_\alpha\in(P_{\alpha+1}-P_\alpha)^* X^*,$$

then $\{y_\alpha;\ \alpha<\xi\}$ is a w^* long basic sequence with its biorthogonal functionals $f_\alpha\in(P_{\alpha+1}-P_\alpha)X$ and with the associated w^* projectional basis $\{P_\alpha^*/Y;\ \alpha\leqslant\xi\}$ of $Y = w^*\mathrm{sp}\{y_\alpha\}$. Furthermore

$$\mathrm{dens}\,\overline{\mathrm{sp}}\{x_\alpha\} = \mathrm{dens}\,\overline{\mathrm{sp}}\{y_\alpha\} = \overline{\overline{\xi}}.$$

Proof. Put $S_\alpha = P_\alpha/\overline{\mathrm{sp}}\{x_\alpha\}$. Then for $z=\sum\lambda_i x_i\in\mathrm{sp}\{x_\alpha\}$ we have that $S_\alpha z = \sum_{i<\alpha}\lambda_i x_i$ and thus $\{x_\alpha;\ \alpha<\xi\}$ is a long basic sequence, according to Lemma 13. If moreover the function $\alpha\to P_\alpha x$ is continuous and $y_\alpha\in(P_{\alpha+1}-P_\alpha)^* X^*$ we put $S_\alpha = P_\alpha^*/Y$. Then $S_\alpha y$ is a w^* continuous function on ordinals, $S_\alpha Y\subset Y$, $\sup\{|S_\alpha|;\ \alpha\leqslant\xi\}<\infty$. Thus $\{S_\alpha;\ \alpha\leqslant\xi\}$ is a w^* projectional basis of Y. It follows from $P_\alpha^* Y\subset Y$ that $P_\alpha Y_\perp\subset Y_\perp$. Consider the duality $\langle Z, Y\rangle$, where $Z = X/Y_\perp$ and denote by S_α^* the dual of S_α in Z (we use the w^* continuity of P_α^*). Then if $z_\alpha\in(S_{\alpha+1}^*-S_\alpha^*)Z$ is so that $z_\alpha(y_\alpha)=1$, then $\{(z_\alpha, y_\alpha)\}$ is a biorthogonal system and if $x_\alpha\in z_\alpha$, $x_\alpha\in X$, then $\{(P_{\alpha+1}-P_\alpha)x_\alpha, y_\alpha\}$, forms a biorthogonal system. The statement on the density of $\overline{\mathrm{sp}}\{x_\alpha\}$ follows from the fact that

$$\left|\frac{x_\alpha}{|x_\alpha|}-\frac{x_\beta}{|x_\beta|}\right|\geqslant\frac{1}{2}\left|(P_{\alpha+1}-P_\alpha)\left(\frac{x_\alpha}{|x_\alpha|}-\frac{x_\beta}{|x_\beta|}\right)\right|=\frac{1}{2}\left|\frac{x_\alpha}{|x_\alpha|}\right|=\frac{1}{2}$$

if $\alpha\neq\beta$.

Remark. A similar approach to that in Lemma 14 is contained in [19]. We will now use the following variant of the result of Johnson and Rosenthal ([13], Proposition II.1) which has also a similar proof by using Proposition 6 with the remark under it. We enclose it here for the completeness.

PROPOSITION 7. *Assume that* $\{y_\alpha;\alpha<\xi\}\subset X^*$ *and* $T: X\to X/\{y_\alpha\}_\perp$ *be the quotient map. Then*

(a) $\{y_\alpha;\alpha<\xi\}$ *is* w^* *basic iff* $X/\{y_\alpha\}_\perp$ *has a basis* $\{x_\alpha;\alpha<\xi\}$ *with associated biorthogonal functionals* $\{x_\alpha^*\}$ *such that* $T^*x_\alpha^*=y_\alpha$, $\alpha<\xi$. *Thus if* $\{y_\alpha;\alpha<\xi\}$ *is* w^* *basic, then* $\{y_\alpha;\alpha<\xi\}$ *is basic.*

(b) *The following are equivalent:*

(i) $\{y_\alpha;\alpha<\xi\}$ *is an* S $(=M$ *here)-boundedly complete long* w^* *basic sequence;*

(ii) $\{y_\alpha;\alpha<\xi\}$ *is* w^* *basic and* $\overline{\mathrm{sp}}\{y_\alpha\}=w^*\mathrm{sp}\{y_\alpha\}$;

(iii) $X/\{y_\alpha\}_\perp$ *has a shrinking basis* $\{x_\alpha;\alpha<\xi\}$ *with associated biorthogonal functionals* $\{x_\alpha^*\}$ *such that* $T^*x_\alpha^*=y_\alpha$.

(c) $\{y_\alpha,\alpha<\xi\}$ *is a shrinking* w^* *long basic sequence iff* $X/\{y_\alpha\}_\perp$ *has an* M*-boundedly complete basis* $\{x_\alpha;\alpha<\xi\}$ *with associated biorthogonal functionals* $\{x_\alpha^*\}$ *such that* $T^*x_\alpha^*=y_\alpha$.

Here and also in the sequel the term $\{y_\alpha\}$ is a shrinking w^* long basic sequence means that $\{y_\alpha\}$ is a w^* long basic sequence which is shrinking as a basis of $\overline{\mathrm{sp}}\{y_\alpha\}$ (see (a)). Similarly for the case of bounded completeness.

In the sequel we present some results which are nonseparable analogues to some results of [13] and have necessarily different proofs. In these results in [13] is often a typical assumption that some space, say Z, has separable dual Z^*. We consider two generalizations of this assumption in nonseparable case: Z^* is WCG or Z is f. They both coincide in the separable case. Thus our propositions have two alternatives.

PROPOSITION 8. *Let* X *be a WCG Banach space,* $Y\subset X^*$, μ *be the first ordinal of cardinality* $\operatorname{dens} Y$. *Assume that*

(i) X^* *is WCG or*

(ii) X *has an equivalent Fréchet smooth norm and* Y *is WCG.*

Then there is a boundedly complete w^* *basic long sequence* $\{y_\alpha;\alpha<\mu\}\subset Y$. $\{y_\alpha;\alpha<\mu\}$ *may be orthogonal if* Y *is nonseparable.*

Proof. If $\operatorname{dens} Y=\aleph_0$, then there is a proposition $P: X\to X$, $|P|=1$ such that PX is separable and $P^*X^*\supset Y$ ([1]). Now we may use Theorem III. 2 of [13] for PX, $Y\subset P^*X^*=(PX)^*$, since in both cases $\operatorname{dens}P^*X^*=\aleph_0$ ([15]). If $\operatorname{dens} Y>\aleph_0$, let $|\cdot|$ be the norm of X and $\|\cdot\|$ be an equivalent norm on X such that its dual on X^* is LUR (cf. [10], Proposition 9 for case (i) and [11], Theorem 1 for case (ii)). Then in both cases there is a long sequence of linear projections $\{P_\alpha;\omega\leqslant\alpha\leqslant\mu\}$ of X such that $|P_\alpha|=\|P_\alpha\|=1$ for $\alpha\geqslant\omega$, $P_\alpha^*Y\subset Y$, $P_\mu^*y=y$ for $y\in Y$, $P_\alpha^*\neq P_{\alpha+1}^*$ on Y and the functions $\alpha\to P_\alpha x$, $\alpha\to P_\alpha^*y$ are norm continuous for all $x\in X$, $y\in Y$ (by Proposition 3(a), (b)).

If we take $0\neq y_\alpha\in(P_{\alpha+1}^*-P_\alpha^*)Y$, we have a w^* basic long sequence (Lemma 14) which is orthogonal ($|P_\alpha|\leqslant 1$) and exactly as in the proof of Theorem III. 2 of [13] we show that $\overline{\mathrm{sp}}\{y_\alpha\}$ is w^* closed: if $y\in w^*\mathrm{sp}\{y_\alpha\}$, then $\lim\limits_{\alpha\to\mu}S_\alpha y=y$ in the w^* sense and since $\|y\|\geqslant\|S_\alpha y\|$, we have by the w^* lower semicontinuity of $\|\cdot\|$ on X^* that $\lim\|S_\alpha y\|=\|y\|$ and by the LUR of $\|\cdot\|$, $\lim\|S_\alpha y-y\|=0$. Now by induction, $S_\alpha y\in\overline{\mathrm{sp}}\{y_\alpha\}$ and the result follows by using Proposition 7(b).

Remark. Let us observe that from Proposition 8 follows (in its notation), that Y contains a w^* closed subspace of the same density.

PROPOSITION 9 (cf. Theorem IV. 1 of [13]). *If* X *is a WCG Banach space, then* X *has a quotient space of the density* $\operatorname{dens}X$ *with a projectional basis. If* $\operatorname{dens}X>\aleph_0$, *then this projectional basis may be orthogonal.*

Proof. If $\operatorname{dens}X>\aleph_0$, choose a long sequence of linear projections $\{P_\alpha;\alpha<\xi\}$ (ξ is the first ordinal of cardinality $\operatorname{dens}X$ constructed by D. Amir and Lindenstrauss in [1]). Then we use Lemma 14 and Proposition 7(a).

The following is a nonseparable version of Theorem III. 3 of [13].

PROPOSITION 10. *Let* X, Y *be WCG Banach spaces,* $Y\subset X^*$. *Assume that* Y *admits an equivalent Fréchet smooth norm. Denote by* ξ *the first ordinal of* $\operatorname{dens} Y$. *Then* Y *contains a shrinking* w^* *basic long sequence* $\{y_\alpha;\alpha<\xi\}$. *If* Y *is nonseparable, then* $\{y_\alpha;\alpha<\xi\}$ *may be chosen to be orthogonal. This* X *has a quotient space with an orthogonal* M*-boundedly complete basis and* Y *contains a subspace isomorphic to a second conjugate space.*

Proof. If $\operatorname{dens} Y=\aleph_0$, we proceed as in the first part of the proof of Proposition 8. If $\operatorname{dens} Y>\aleph_0$, let $|\cdot|$ be the norm on X and $\|\cdot\|$ be an equivalent Fréchet smooth norm on Y. By Proposition 3(a), there is a long sequence $\{P_\alpha;\omega\leqslant\alpha\leqslant\xi\}$ of linear projections of X such that $|P_\alpha|=\|P_\alpha^*/Y\|=1$ for $\alpha\geqslant\omega$, $P_\alpha^*Y\subset Y$, $P_{\alpha+1}^*\neq P_\alpha^*$ on Y, $\alpha\to P_\alpha x$, $\alpha\to P_\alpha^*y$ are continuous on ordinals for all $x\in X$, $y\in Y$. If we take $0\neq y_\alpha\in(P_{\alpha+1}^*-P_\alpha^*)Y$, then $\{y_\alpha;\alpha<\xi\}$ is a w^* basic sequence (Lemma 14), basic on $\overline{\mathrm{sp}}\{y_\alpha\}$ (Proposition 7(a)). From the orthogonality of $\{P_\alpha\}$ in the F-norm $\|\cdot\|$ it follows that $\{y_\alpha; \alpha < \xi\}$ is shrinking (cf.e.g. Lemma 3 of [11]). The rest of the proposition follows from Proposition 7(c) and [12], Theorem II.5.

Before proceeding we will need the following definition (cf. [16], [13]).

DEFINITION. A Banach space X is called *somewhat reflexive* if any closed subspace $Y \subset X$ contains a reflexive subspace of the same density.

PROPOSITION 11 (cf. [13], Theorem IV.2). *Let X, $Y \subset X^*$ be Banach spaces. Then Y is somewhat reflexive if*

(i) *X, Y are WCG which both admit equivalent Fréchet smooth norms, or*

(ii) *X, X^* are WCG and Y admits an equivalent Fréchet smooth norm.*

Proof. First, let us recall that if X, X^* are WCG, then X admits an equivalent Fréchet smooth norm ([10]). Now let $Z \subset Y$. If Z is separable, we proceed exactly as in the first part of the proof of Proposition 8 and use Theorem IV. 2 of [13]. Suppose $\operatorname{dens} Z > \aleph_0$. Z is WCG as a subspace of WCG space which admits an equivalent Fréchet smooth norm ([11] or Corollary (i) of Theorem 2). According to Proposition 8 there is boundedly complete orthogonal w^* basic long sequence $\{y_\alpha; \alpha < \xi\} \subset Z$ (ξ is the first ordinal of $\operatorname{dens} Z$) which is orthogonal also with respect to a Fréchet smooth norm on Y. Therefore as in the proof of Proposition 10, $\{y_\alpha; \alpha < \xi\}$ is a shrinking basic sequence.

THEOREM 3. *Let X, X^* be WCG Banach spaces and let X^* admit an equivalent Fréchet smooth norm. Then X, X^* are somewhat reflexive.*

Proof. X^* is somewhat reflexive by Proposition 11 (ii). Now let $Y \subset X$. Then $Y \subset X^{**}$ and Y, X^* are WCG Banach spaces which admit equivalent Fréchet smooth norms (for Y this follows from [10] and Corollary (i) of Theorem 2). Hence we may use Proposition 1(i).

COROLLARY. *If X^*, X^{**} are WCG, then X, X^* are somewhat reflexive.*

Proof. X is then WCG space by Theorem 4 of [11]. X admits an equivalent Fréchet smooth norm ([10]).

COROLLARY. *If X^*, X^{**} are WCG and* $\operatorname{dens} X = \aleph$, *then X, X^*, X^{**} are all homeomorphic to the Hilbert space $l_2(\aleph)$.*

Proof. Since X, X^*, X^{**} are WCG, $\operatorname{dens} X = \operatorname{dens} X^* = \operatorname{dens} X^{**}$ (Proposition 2.2 of [15]). Furthermore, X, X^*, X^{**} contain reflexive subspaces of density character $\aleph$ (X^{**} that from X). Further we use the following results of C. Bessaga and A. Pełczyński (cf. [6]): The first says that all reflexive Banach spaces X with $\operatorname{dens} X = \aleph$ are homeomorphic to $l_2(\aleph)$, and the second, the well-known Bessaga–Pełczyński lemma, says that if a Banach space X of density character $\aleph$ contains a subspace homeomorphic to $l_2(\aleph)$, then X is homeomorphic to $l_2(\aleph)$.

VII. Appendix. Here we apply the above results to some problems discussed in [10].

Remark. Similarly as in Proposition 3, using Lemma 3 of [11] we see that if X is a WCG Banach space which admits an equivalent Fréchet smooth norm $\|\cdot\|$ and $Y \subset X^*$ is a WCG Banach space, then there is a long sequence of linear projections $\{P_\alpha; 0 \leqslant \alpha \leqslant \xi\}$, where ξ is the first ordinal of $\operatorname{dens} X$ such that $\|P_\alpha\| = 1$ for $\alpha > 0$, $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\alpha < \beta$, $\operatorname{dens} P_\alpha X = \operatorname{dens}(P_\alpha X)^* \leqslant \overline{\overline{\alpha}}$ for $\alpha \leqslant \xi$, $P_\alpha^* Y \subset Y$, and the functions $\alpha \to P_\alpha x$, $\alpha \to P_\alpha^* x^*$ are norm continuous on ordinals. (The assumption on Y to be WCG is necessary — cf. [11].)

From this and the results of [10] follows that Propositions 5, 6, 7 of [10] remain valid if the assumptions X, X^* are WCG are replaced by X is WCG and admits an equivalent Fréchet smooth norm and $Y \subset X^*$ is a WCG Banach space.

Therefore if we say that a subspace $Z \subset X$ is a quasicomplement of a subspace Y if $\overline{Z \oplus Y} = X$ and $Z \cap Y = \{0\}$, Proposition 7 of [10] (where some results of [7] are used) gives:

THEOREM. *If X is a WCG Banach space which admits an equivalent Fréchet smooth norm and $Y \subset X^*$ is a WCG subspace of X,* then Y has a w^* closed quasicomplement.*

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Localisation des sommes de Riesz sur un groupe de Lie compact

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Résumé. Grâce à une étude précise des noyaux des sommes de Riesz, on obtient des résultats de localisation pour les développements de Peter–Weyl sur un groupe de Lie compact.

Ce travail est la suite de l'article [3], dont il reprend les notations: G est un groupe de Lie réel, compact, connexe et simplement connexe (donc semi-simple), de dimension n, et de rang l; si f est une fonction sommable sur G, on pose

$$S_R^\delta f = \sum_{\lambda\in A}\left(1-\frac{\langle\lambda+\beta,\lambda+\beta\rangle}{R^2}\right)_+^\delta d_\lambda\chi_\lambda * f,$$

où $\delta\geqslant 0$ et $R>0$ (sommes de Riesz d'indice δ). L'opérateur S_R^δ s'interprète comme une convolution avec une fonction centrale s_R^δ, dont un développement a été obtenu dans [3], lorsque $\delta>(l-1)/2$

$$s_R^\delta(\exp H) = C\frac{R^n}{D(\exp H)}\sum_{\zeta\in t_e}\Big(\prod_{\alpha\in R^+}(\alpha, H+\zeta)\Big)\mathscr{J}_{n/2+\delta}(R\,|H+\zeta|)\ (^1),$$

où $\mathscr{J}_\nu(\varrho) = \varrho^{-\nu}J_\nu(\varrho)$ et J_ν est la fonction de Bessel d'indice ν. On se propose ici d'améliorer les estimations obtenues précédemment et d'obtenir des résultats de localisation.

LEMME. *Soit $\delta>(l-1)/2$, et soit $\varepsilon>0$. Il existe une constante $C>0$, telle que*

$$x\in G \quad et \quad d(x,e)>\varepsilon \Rightarrow |s_R^\delta(x)|\leqslant CR^{\frac{n-1}{2}+\frac{n-l}{2}-\delta}.$$

Rappelons que Q est un domaine fondamental d'un tore maximal, centré à l'origine; et soit B_ε la boule ouverte de centre 0 et de rayon ε. Soit $H_0\in\bar{Q}\cap\complement B_\varepsilon$. Soit J l'ensemble des racines positives qui prennent en

(1) La série converge absolument pour tout H, et le membre de droite de l'égalite défini a priori pour H régulier se prolonge par continuité.