Making use of the result of Grothendieck mentioned above that for (DF)-spaces the "problème des topologies" is settled in the affirmative, we get as an immediate consequence of Theorem 3.3 and Lemma 4.7 the following:

THEOREM 4.8. *Let E and F be locally convex spaces of type (DF) and let $\mathscr{A}$ be a Hom-stable ideal which is equivalent to its injective hull. Then $E \otimes_\pi F$ is of type co-$\mathscr{A}$ if and only if E and F are of type co-$\mathscr{A}$.*

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Two weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform

by

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Abstract. Necessary conditions are obtained on non-negative functions $U(x)$ and $V(x)$ so that $\int_{-\infty}^{\infty} |Tf(x)|^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx$, where $1 \leqslant p < \infty$, $Tf(x)$ denotes either the Hardy–Littlewood maximal function or the Hilbert transform of f and C is a constant independent of f. In the case $p = 1$, the necessary condition is also shown to be sufficient; in case $p > 1$ the necessary conditions are shown to be sufficient if various additional restrictions are placed on $U(x)$ and $V(x)$ or on $f(x)$.

1. Introduction. The first norm inequality of the form

$$\int_{-\infty}^{\infty} [f^*(x)]^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx, \tag{1.1}$$

where

$$f^*(x) = \sup_{y \neq x} \frac{1}{y-x} \int_x^y |f(t)|\,dt$$

is the Hardy–Littlewood maximal function of f and $1 < p < \infty$, was proved in [2] with $U(x) = V(x) = 1$. The first norm inequalities of the form

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p U(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx, \tag{1.2}$$

where

$$\tilde{f}(x) = \lim_{\varepsilon \to 0+} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y}\,dy$$

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is the Hilbert transform of f and $1 < p < \infty$, were proved in [9] for $U(x) = V(x) = 1$ and in [3] for $U(x) = V(x) = |x|^a$, $-1 < a < p-1$. Many results have been proved for both (1.1) and (1.2) with $U(x) = V(x)$: a necessary and sufficient condition for (1.1) with $U(x) = V(x)$ and $1 < p < \infty$ appears in [8]; a necessary and sufficient condition for (1.2) with $U(x) = V(x)$ and $p = 2$ was obtained in [4]; a necessary and sufficient condition for (1.2) with $U(x) = V(x)$ and $1 < p < \infty$ appears in [5]. Simlifications of the proofs in [5] and [8] can be found in [1].

Interest in inequalities (1.1) and (1.2) stems from the fact that they imply inequalities of the form

$$\int |S_n(f, x)|^p U(x)\,dx \leqslant C \int |f(x)|^p V(x)\,dx, \tag{1.3}$$

where $S_n(f, x)$ denotes the nth partial sum of an orthogonal expansion of f. Inequalities of the form (1.3) imply mean convergence results for orthogonal expansions. Inequality (1.1) is often useful in estimating error terms and in proving mean summability and almost everywhere convergence of summability methods for various orthogonal expansions.

Inequalities (1.1) and (1.2) are of interest with $U(x) \neq V(x)$ for several reasons. In the first place, for some expansions such as Laguerre and Hermite expansions for $p \leqslant 4/3$ or $p \geqslant 4$, inequalities of the form (1.3) with $U(x) = V(x)$ are impossible; see [6]. Another interesting feature of the general problem of determining pairs of non-negative functions $U(x)$ and $V(x)$ for which (1.1) and (1.2) are true is that the results are very different from the results for $U(x) = V(x)$ and, evidently, much more difficult. If $U(x) = V(x)$, then the condition

$$\left(\frac{1}{|I|}\int_I U(x)\,dx\right)\left(\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\,dx\right)^{p-1} \leqslant C, \tag{1.4}$$

where I is an arbitrary interval, $1 < p < \infty$ and C is independent of I is necessary and sufficient for (1.1) and (1.2). If $U(x) = V(x)$, condition (1.4) is also necessary and sufficient for the two weak type inequalities

$$\int_{f^*(x) > a} U(x)\,dx < Ca^{-p} \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx \tag{1.5}$$

and

$$\int_{|\tilde{f}(x)| > a} U(x)\,dx < Ca^{-p} \int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx \tag{1.6}$$

for $a > 0$. These results are contained in [1], [5] and [8].

If the requirement that $U(x) = V(x)$ is dropped, then (1.4) is still a necessary condition for (1.1), (1.2), (1.5) and (1.6); the proof for (1.1) and (1.5) is in [8], and the proof for (1.2) and (1.6) is a corollary of Theorem 3 of this paper. Condition (1.4) is also sufficient for (1.5) as shown in [8]. Condition (1.4) is not sufficient for (1.1), (1.2) or (1.6). For $p = 2$, a simple example of a pair that satisfies (1.4) but not (1.1) consists of the functions $U(x) = -x \log x$ on $(0, \frac{1}{2}]$ and 0 elsewhere and $V(x) = x(\log x)^2$ on $(0, \frac{1}{2}]$ and ∞ elsewhere. The function $f(x) = 1/(x\log^2 x)$ on $(0, \frac{1}{2}]$ and 0 elsewhere makes the right-hand side of (1.1) finite and the left-hand side infinite for $p = 2$. For $p = 2$, a simple example of a pair that satisfies (1.4) but not (1.2) or (1.6) consists of the functions $U(x) = x^{-1}|\log x|^{-5/2}$ on $(0, \frac{1}{2}]$ and 0 elsewhere and $V(x) = x^{-1}|\log x|^{-3/2}$ on $(0, \frac{1}{2}]$ and ∞ elsewhere. The function $f(x) = 1$ on $(0, \frac{1}{2}]$ and 0 elsewhere clearly violates (1.2) and (1.6). Also interesting in this case is that by Theorem 7 this pair does satisfy (1.1). Examples for other values of p, $1 < p < \infty$, can also be obtained easily.

Typical of the results contained in this paper is the fact that (1.1) for $1 < p < \infty$ implies the existence of a constant B such that

$$\left(\int_{-\infty}^{\infty} \frac{|I|^{p-1} U(x)}{(|I| + |x - x_I|)^p}\,dx\right)\left(\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\,dx\right)^{p-1} \leqslant B, \tag{1.7}$$

where I is any interval and x_I is the center of I. This and related results for $p = 1$ and the Hilbert transform are given in Section 2. In Sections 3 and 4, various theorems are proved in which (1.7) and some additional conditions on either U and V or on $f(x)$ are assumed in order to prove (1.1). The theorems are stated for functions with support on $[0, \infty]$; this can be easily modified so that the support is any semi-infinite or finite interval, and by putting two or more such integrals together, results for functions supported on $(-\infty, \infty)$ can be obtained. The additional conditions used to prove (1.1) are not necessary conditions, and we conjecture that (1.7) implies (1.1) without additional assumptions.

Section 5 contains a similar proof that the necessary conditions derived in Section 2 for (1.2) are also sufficient with some additional assumptions. Finally, in Section 6, the necessary condition derived in Section 2 for U and V which satisfy (1.1) or (1.2) with $p = 1$ is shown to be sufficient without additional assumptions.

Two theorems from [7] will be needed frequently in Sections 3, 4 and 5 and are quoted here for reference in a modified form. They are the following.

THEOREM A. *If $1 \leqslant p < \infty$, there is a finite C, independent of f, such that*

$$\int_0^\infty \left|\int_0^x f(t)\,dt\right|^p U(x)\,dx \leqslant C \int_0^\infty |f(x)|^p V(x)\,dx$$

if and only if there is a finite B, independent of r, such that for $r > 0$,

$$\Big[\int_r^\infty U(x)\,dx\Big]\Big[\int_0^r [V(x)]^{-1/(p-1)}dx\Big]^{p-1} \leqslant B.$$

THEOREM B. *If $1 \leqslant p < \infty$, there is a finite C, independent of f, such that*

$$\int_0^\infty \Big|\int_x^\infty f(t)\,dt\Big|^p U(x)\,dx \leqslant C\int_0^\infty |f(x)|^p V(x)\,dx$$

if and only if there is a finite B, independent of r, such that for $r > 0$,

$$\Big[\int_0^r U(x)\,dx\Big]\Big[\int_r^\infty [V(x)]^{-1(p-1)}dx\Big]^{p-1} \leqslant B.$$

Throughout this paper the convention $0\cdot\infty = 0$ is used, $|I|$ denotes the length of the interval I, and C is used to denote constants not necessarily the same at each occurrence.

2. Necessity results. This section consists of the proofs of the following four theorems.

THEOREM 1. *Assume that $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$ and that there is a constant C such that*

$$\int_{-\infty}^{\infty} [f^*(x)]^p U(x)\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx. \tag{2.1}$$

Then there is a constant B such that for every interval I

$$\Big[\int_{-\infty}^{\infty}\frac{|I|^{p-1}U(x)\,dx}{(|I|+|x-x_I|)^p}\Big]\Big[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}dx\Big]^{p-1} \leqslant B, \tag{2.2}$$

where $|I|$ denotes the length of I and x_I is the center of I.

THEOREM 2. *Assume that $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$, and that there is a constant C such that*

$$\int_{-\infty}^{\infty} |\tilde f(x)|^p U(x)\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx. \tag{2.3}$$

Then there is a constant B such that (2.2) holds for every interval I.

THEOREM 3. *Assume that $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$, and that there is a constant C such that for every $a > 0$*

$$\int_{|\tilde f(x)|>a} U(x)\,dx \leqslant Ca^{-p}\int_{-\infty}^{\infty} |f(x)|^p V(x)\,dx. \tag{2.4}$$

Then there is a constant B such that for every interval I

$$\Big[\frac{1}{|I|}\int_I U(x)\,dx\Big]\Big[\int_{-\infty}^{\infty}\frac{|I|^{p'-1}[V(x)]^{-1/(p-1)}}{(|I|+|x-x_I|)^{p'}}dx\Big]^{p-1} \leqslant B, \tag{2.5}$$

where $p' = p/(p-1)$.

THEOREM 4. *Assume that $U(x) \geqslant 0$, $V(x) \geqslant 0$ and either (2.1) or (2.3) holds with $p = 1$. Then there is a constant B such that for almost every x*

$$\int_{-\infty}^{\infty}\frac{U(y)}{|x-y|}dy \leqslant BV(x). \tag{2.6}$$

It should be noted that since (2.3) implies (2.4), then (2.3) implies (2.5). The fact that (2.3) implies (2.5) could also be obtained from Theorem 2 by a duality argument. This duality argument, however, is not particularly simpler than the proof of Theorem 3, and Theorem 3 is of interest since it suggests what the necessary and sufficient condition for the weak type inequality (2.4) is.

It should also be noted that a simple argument shows that (2.6) is equivalent to condition (2.2) with $p = 1$ with the usual interpretation of the L^∞ norm.

To prove Theorem 1, fix an I and let $Q = \int_I [V(x)]^{-1/(p-1)}dx$. If $Q = 0$, (2.2) follows for any B because of the convention $0\cdot\infty = 0$. If $Q = \infty$, $[V(x)]^{-1/p}$ is not in $L^{p'}$ on I, where $p' = p/(p-1)$. Then there is a function $g(x)$ that is in L^p on I such that $g(x)[V(x)]^{-1/p}$ is not integrable on I. Let $f(x) = g(x)[V(x)]^{-1/p}$ on I and 0 elsewhere. Then since the right-hand side of (2.1) is finite for this f and since $f^*(x) \equiv \infty$, it follows from (2.1) that $U(x) = 0$ almost everywhere. This implies (2.2). If $0 < Q < \infty$, let $f(x) = [V(x)]^{-1/(p-1)}$ on I and 0 elsewhere. Then $f^*(x) \geqslant (|I|+|x-x_I|)^{-1}Q$. Substituting this into the left-hand side of (2.1) and dividing by Q gives (2.2).

To prove Theorem 2, again fix an I and let $Q = \int_I [V(x)]^{-1/(p-1)}dx$. If $Q = 0$, (2.2) follows as before. If $Q = \infty$, let J be any subinterval of I for which $\int_J [V(x)]^{-1/(p-1)}dx = \infty$. Let $g(x)$ be a function that is in L^p and is 0 outside J, but such that $g(x)V(x)^{-1/p}$ is not integrable on J. Let $f(x) = g(x)[V(x)]^{-1/p}$ on J and 0 elsewhere. Then the right-hand side of (2.3) is finite for this f and $|\tilde f(x)| = \infty$ for x not in J. Therefore, $U(x) = 0$ almost everywhere outside J. Since repeated bisection of I produces J's that are arbitrarily short, $U(x) = 0$ almost everywhere in $(-\infty, \infty)$ and (2.2) follows.

If $0 < Q < \infty$, let $I = [a, a+h]$ and choose $r > 0$ so that

$$\int_a^{a+r} [V(x)]^{-1/(p-1)}\, dx = Q/2.$$

Now let $f(x) = [V(x)]^{-1/(p-1)}$ on $[a, a+r]$ and 0 elsewhere. Then for $x > a+r$, $|\tilde{f}(x)| \geqslant (|x-x_I|+|I|)^{-1} Q/2$. Using this fact in (2.3) and dividing by Q then shows that

$$\int_{a+r}^{\infty} \frac{U(x)\,dx}{(|I|+|x-x_I|)^p} Q^{p-1} 2^{-p} \leqslant C. \tag{2.7}$$

Similarly, by taking $f(x) = [V(x)]^{-1/(p-1)}$ on $[a+r, a+h]$, it follows that

$$\int_{-\infty}^{a+r} \frac{U(x)\,dx}{(|I|+|x-x_I|)^p} Q^{p-1} 2^{-p} \leqslant C. \tag{2.8}$$

Adding (2.5) and (2.6) proves (2.2).

To prove Theorem 3, fix an $I = [a, a+h]$, let

$$K(x) = \frac{[V(x)]^{-1/(p-1)}}{(|I|+|x-x_I|)^{p'}} \quad \text{and} \quad Q = \int_{-\infty}^{\infty} K(x)\,dx.$$

If $Q = 0$, (2.4) follows from the convention $0 \cdot \infty = 0$. If $0 < Q < \infty$, choose r so that $\int_{-\infty}^{r} K(x)\,dx = Q/2$ and define $f(x) = [V(x)(|x-x_I|+|I|)]^{-1/(p-1)}$ on $[r, \infty)$ and 0 elsewhere. Now if x is in $I \cap (-\infty, r]$ and t is in $[r, \infty)$, then $0 \leqslant t-x \leqslant |t-x_I|+|x_I-x| < |t-x_I|+|I|$. Consequently, if x is in $I \cap (-\infty, r]$,

$$\tilde{f}(x) = \int_r^{\infty} \frac{f(t)}{x-t}\,dt > \int_r^{\infty} K(t)\,dt = Q/2.$$

Using this fact in (2.4) with $a = Q/2$ shows that

$$\int_{(-\infty,r]\cap I} U(x)\,dx \leqslant C(Q/2)^{1-p}. \tag{2.9}$$

A similar argument shows that

$$\int_{[r,\infty)\cap I} U(x)\,dx \leqslant C(Q/2)^{1-p}. \tag{2.10}$$

Adding (2.9) and (2.10) gives (2.5); this completes the proof if $0 < Q < \infty$.

If $Q = \infty$, let $V_n(x) = 1/n + V(x)$. Then (2.4) is true with the same constant if V is replaced by V_n. Since $\int_{-\infty}^{\infty} \frac{V_n(x)^{-1/(p-1)}}{(|I|+|x-x_I|)^{p'}}\,dx < \infty$, it follows from the last paragraph that (2.5) holds if V is replaced by V_n and the constant does not depend on n. Then letting $n \to \infty$ and applying the monotone convergence theorem shows that (2.5) is true and completes the proof of Theorem 3. This proof for the case $Q = \infty$ is due to D. Kurtz.

To prove Theorem 4, let I be an interval with center x_I, let $a = \operatorname{ess\,inf}_{y\in I} V(y)$ and, given $\varepsilon > 0$, let E be the subset of I where $V(x) \leqslant a+\varepsilon$. Let $f(x)$ equal $|E|^{-1}$ times the characteristic function of E. For y not in I, both $|\tilde{f}(y)|$ and $f^*(y)$ are bounded below by $(|x_I-y|+|I|)^{-1}$, so the hypothesis of Theorem 4 shows that

$$\int_{y\notin I} \frac{U(y)}{|I|+|x_I-y|}\,dy \leqslant C \int_{-\infty}^{\infty} f(y)V(y)\,dy \leqslant C(a+\varepsilon).$$

Since ε was arbitrary, it follows that

$$\int_{y\notin I} \frac{U(y)\,dy}{|x_I-y|+|I|} \leqslant C \operatorname*{ess\,inf}_{y\in I} V(y). \tag{2.11}$$

Now let $I = [x-h, x+h]$ and take the limit of both sides of (2.11) as $h \to 0^+$. The left-hand side converges to $\int_{-\infty}^{\infty} U(y)|x-y|^{-1}\,dy$ by the monotone convergence theorem. The proof of Theorem 4 can then be completed by showing that for almost every x

$$\lim_{h\to 0^+} [\operatorname*{ess\,inf}_{|x-y|\leqslant h} V(y)] \leqslant V(x). \tag{2.12}$$

If (2.12) failed on a set of positive measure, then there would be a set D of positive measure and rationals r and s such that $r < s$ and, for x in D, $V(x) < r$ and $\lim_{h\to 0^+} [\operatorname*{ess\,inf}_{|x-y|\leqslant h} V(y)] > s$. Let z be a point of density of D. Then every interval about z contains a set of positive measure where $V(x) < r$; therefore, for every $h > 0$, $\operatorname*{ess\,inf}_{|z-y|\leqslant h} V(y) < r$. Since z is in D, however,

$$\lim_{h\to 0^+} [\operatorname*{ess\,inf}_{|z-y|\leqslant h} V(y)] > s > r;$$

this contradiction proves (2.12) and completes the proof of Theorem 4.

3. Maximal function sufficiency theorems.

The following theorems are typical of the sufficiency theorems of this paper. Variations in the hypothesis $U(x) \leqslant AV(y)$ for $x/4 \leqslant y \leqslant 4x$ are discussed in Section 4 as are comments about the applicability to intervals other than $[0, \infty)$. Theorem 5 is the basic one; Theorem 6 is useful when combining intervals on which different sufficient conditions are satisfied. An interesting feature of Theorem 6 is that it requires only the necessary condition derived in Section 2 as an hypothesis. This section consists primarily of the proof of Theorem 5; the proof of Theorem 6 is similar and is sketched at the end of the section.

THEOREM 5. *Suppose that* $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$, $f(x) = 0$ *for* $x < 0$ *and there exists a constant* A *such that* $U(x) \leqslant AV(y)$ *for* $x/4 \leqslant y \leqslant 4x$ *and* $x > 0$. *Assume also that for every interval* $I \subset [0, \infty)$

$$(3.1)\qquad \left(\int_0^\infty \frac{|I|^{p-1}U(x)\,dx}{(|I|+|x-x_I|)^p}\right)\left(\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}dx\right)^{p-1} \leqslant B,$$

where x_I *denotes the center of* I *and* B *is a constant independent of* I. *Then there is a constant* C, *independent of* f, *such that*

$$(3.2)\qquad \int_0^\infty [f^*(x)]^p U(x)\,dx \leqslant C\int_0^\infty |f(x)|^p V(x)\,dx.$$

THEOREM 6. *Suppose that* $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$, $f(x) = 0$ *for* $x < 0$ *and there exists a constant* B *such that*

$$(3.3)\qquad \left[\int_{-\infty}^0 \frac{|I|^{p-1}U(x)\,dx}{(|I|+|x-x_I|)^p}\right]\left[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}dx\right]^{p-1} \leqslant B$$

for all subintervals I *of* $[0, \infty)$. *Then there is a constant* C, *independent of* f, *such that*

$$(3.4)\qquad \int_{-\infty}^0 [f^*(x)]^p U(x) \leqslant C\int_0^\infty |f(x)|^p V(x)\,dx.$$

To prove Theorem 5, let χ_I denote the characteristic function of the interval I. Then the left-hand side of (3.2) is bounded by 3^p times the sum of

$$(3.5)\qquad \sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [(f(x)\chi_{[0,2^{n-1}]}(x))^*]^p U(x)\,dx,$$

$$(3.6)\qquad \sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [(f(x)\chi_{[2^{n-1},2^{n+2}]}(x))^*]^p U(x)\,dx$$

and

$$(3.7)\qquad \sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} [(f(x)\chi_{[2^{n+2},\infty)}(x))^*]^p U(x)\,dx.$$

The proof of Theorem 5 will be completed by showing that (3.5), (3.6) and (3.7) are bounded by the right-hand side of (3.2).

First, since 2^{n-1} is less than the length of any interval about an x in $[2^n, 2^{n+1}]$ for which the integral of $f\chi_{[0,2^{n-1}]}$ is not 0 and since $\int_0^{2^{n-1}} |f(t)|\,dt$ is an upper bound for the absolute value of any integral of $f\chi_{[0,2^{n-1}]}$, it follows that the right-hand side of (3.5) is bounded by

$$\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} \left[2^{1-n}\int_0^{2^{n-1}} |f(t)|\,dt\right]^p U(x)\,dx.$$

This in turn is bounded by a constant times

$$(3.8)\qquad \sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} \left[\frac{1}{x}\int_0^x |f(t)|\,dt\right]^p U(x)\,dx = \int_0^\infty \left[\int_0^x |f(t)|\,dt\right]^p \frac{U(x)\,dx}{x^p}.$$

Now let r be any positive number and take $I = [0, r]$ in hypothesis (3.1). It follows that

$$\left[\int_r^\infty \frac{U(x)}{x^p}\,dx\right]\left[\int_0^r [V(x)]^{-1/(p-1)}dx\right]^{p-1} \leqslant B.$$

By Theorem A, (3.8) is bounded by the right-hand side of (3.2). This completes the proof for (3.5).

To estimate (3.6), let U_n be the essential supremum of $U(x)$ on $[2^n, 2^{n+1}]$. Then (3.6) is bounded by

$$(3.9)\qquad \sum_{n=-\infty}^{\infty} U_n \int_{2^n}^{2^{n+1}} [(f(x)\chi_{[2^{n-1},2^{n+2}]}(x))^*]^p\,dx.$$

By the unweighted norm theorem for the maximal function, [11], Vol. I, p. 32, (3.9) is bounded by a constant times

$$(3.10)\qquad \sum_{n=-\infty}^{\infty}\int_{2^{n-1}}^{2^{n+2}} U_n |f(x)|^p\,dx.$$

The hypothesis that $U(x) \leqslant BV(y)$ for $x/4 \leqslant y \leqslant 4x$ implies that $U_n \leqslant BV(x)$ for $2^{n-1} \leqslant x \leqslant 2^{n+2}$. This completes the proof that (3.6) is bounded by the right-hand side of (3.2).

To estimate (3.7), it will first be shown that if x is in $[2^n, 2^{n+1}]$, then

$$(3.11)\qquad [f(x)\chi_{[2^{n+2},\infty)}(x)]^* \leqslant 4\sup_{k\geqslant n+2} 2^{-k}\int_{2^k}^{2^{k+1}} |f(t)|\,dt.$$

If the right-hand side of (3.11) is infinite, the inequality is trivial. If the right-hand side is finite, call its value $4S$. Then

$$\int_{2^k}^{2^{k+1}} |f(t)|\,dt \leqslant S2^k \tag{3.12}$$

for $k \geqslant n+2$. Given $y \geqslant 2^{n+2}$, let j be the integer such that $2^j \leqslant y < 2^{j+1}$. Then if x is in $[2^n, 2^{n+1}]$.

$$\frac{1}{y-x}\int_{2^{n+2}}^{y} |f(t)|\,dt \leqslant \frac{1}{2^j-2^{n+1}}\sum_{k=n+2}^{j}\int_{2^k}^{2^{k+1}} |f(t)|\,dt. \tag{3.13}$$

Applying (3.12) to the right-hand side of (3.13) shows that the right-hand side of (3.13) is bounded by $4S$. Inequality (3.11) follows immediately from this.

Now (3.11) shows that (3.7) is bounded by a constant times

$$\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}}\Big[\sup_{k\geqslant n+2} 2^{-k}\int_{2^k}^{2^{k+1}} |f(t)|\,dt\Big]^p U(x)\,dx. \tag{3.14}$$

Since the inner expression in (3.14) does not depend on x, (3.14) is bounded by

$$\sum_{n=-\infty}^{\infty}\sup_{k\geqslant n+2}\Big[2^{-k}\int_{2^k}^{2^{k+1}} |f(t)|\,dt\Big]^p \int_{2^n}^{2^{n+1}} U(x)\,dx. \tag{3.15}$$

Now replace $\sup\limits_{k\geqslant n+2}$ by $\sum\limits_{k=n+2}^{\infty}$ and reverse the order of summation; this shows that (3.15) is bounded by

$$\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{k-2}\Big[2^{-k}\int_{2^k}^{2^{k+1}} |f(t)|\,dt\Big]^p \int_{2^n}^{2^{n+1}} U(x)\,dx. \tag{3.16}$$

Performing the inner summation shows that (3.16) is bounded by

$$\sum_{k=-\infty}^{\infty} 2^{-kp}\Big[\int_{2^k}^{2^{k+1}} |f(t)|\,dt\Big]^p \int_{0}^{2^{k-1}} U(x)\,dx. \tag{3.17}$$

Hölder's inequality shows that (3.17) is bounded by

$$\sum_{k=-\infty}^{\infty}\Big[\int_{2^k}^{2^{k+1}} |f(t)|^p V(t)\,dt\Big]\Big[\int_{2^k}^{2^{k+1}} [V(t)]^{-1/(p-1)}\,dt\Big]^{p-1}\Big[\int_{0}^{2^{k-1}} 2^{-kp}\,U(x)\,dx\Big];$$

special arguments show that (3.17) is bounded by this even if $V(x)$ equals 0 or ∞ on a set of positive measure.

With I taken to be $[2^k, 2^{k+1}]$ in (3.1), it is immediate that the product of the last two integrals is bounded by a constant independent of k and f. The whole expression then is clearly bounded by the right-hand side of (3.2). This completes the proof of Theorem 5.

Theorem 6 is proved by first observing that the left-hand side of (3.4) is bounded by 2^p times the sum of

$$\sum_{n=-\infty}^{\infty}\int_{-2^{n+1}}^{-2^n}\big[\big(f(x)\chi_{[0,2^n]}(x)\big)^*\big]^p U(x)\,dx \tag{3.18}$$

and

$$\sum_{n=-\infty}^{\infty}\int_{-2^{n+1}}^{-2^n}\big[\big(f(x)\chi_{[2^n,\infty)}(x)\big)^*\big]^p U(x)\,dx. \tag{3.19}$$

With the same procedure as that used on (3.5), (3.18) can easily be shown to be bounded by a constant times

$$\int_0^\infty\Big[\int_0^x |f(t)|\,dt\Big]^p x^{-p}\,U(-x)\,dx,$$

and this is clearly bounded by the right-hand side of (3.4) by use of Theorem A. Similarly, (3.19) is treated in the same way that (3.7) was; $\big(f(x)\chi_{[2^n,\infty)}(x)\big)^*$ is bounded by a constant times $\sup\limits_{k\geqslant n}\int_{2^k}^{2^{k+1}} |f(t)|\,dt$ if $-2^{n+1}\leqslant x \leqslant -2^n$, and (3.19) is bounded by a constant times (3.14) with $U(x)$ replaced by $U(-x)$. The rest of the estimation is the same as that of (3.14) except that $U(x)$ is replaced throughout by $U(-x)$.

4. Variations of the sufficiency theorem. Various versions of Theorems 5 and 6 can be proved; some follow directly from Theorems 5 and 6 and one is based on different principles. The principal variations are given in Theorem 7.

THEOREM 7. *Suppose that $U(x)\geqslant 0$, $V(x)\geqslant 0$, $1<p<\infty$, $f(x)=0$ for $x<0$ and for every interval $I\subset(-\infty,\infty)$*

$$\Big[\int_{-\infty}^{\infty}\frac{|I|^{p-1}U(x)\,dx}{(|I|+|x-x_I|)^p}\Big]\Big[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\,dx\Big]^{p-1}\leqslant B, \tag{4.1}$$

where B is independent of I and x_I denotes the center of I. Suppose in addition that one of the following holds:

(a) *There exists $A>0$ such that $U(y)\leqslant AU(x)$ and $V(y)\leqslant AV(x)$ for $x\leqslant y\leqslant 2x$ and $x>0$.*

(b) *There exists* $A > 0$ *such that* $U(y) \geqslant AU(x)$ *and* $V(y) \geqslant AV(x)$ *for* $x \leqslant y \leqslant 2x$ *and* $x > 0$.

(c) *There exists* $A > 0$ *and* $d > 1$ *such that* $U(x) \leqslant AV(y)$ *for* $x/d \leqslant y \leqslant dx$ *and* $x > 0$.

(d) $f(x)$ *is monotone on* $[0, \infty)$.

Then there is a constant C, *independent of* f, *such that*

$$\int_{-\infty}^{\infty} [f^*(x)]^p U(x)\,dx \leqslant C \int_0^{\infty} |f(x)|^p V(x)\,dx. \tag{4.2}$$

By translation and truncation or splicing, a version of this theorem can be proved in which the set where $f(x) \neq 0$ is any interval. Furthermore, if a finite number of intervals can be found such that on each one an appropriate version of one of the conditions (a)–(d) is true, then a norm inequality like (4.2) also follows. Such results are routine variations of Theorem 7 and will not be discussed further here. It should be noted that condition (a) includes the case of both U and V being monotone decreasing; condition (b) includes the case of both U and V being monotone increasing. Note also that parts (a) and (b) with the 2 replaced by another constant greater than 1 are equivalent to the stated versions, while condition (c) for a given d does not imply condition (c) for a larger d.

To prove Theorem 7 with assumption (a), fix an $x > 0$ and observe that by condition (a), $V(4x) \leqslant A^4 V(y)$ for $x/4 \leqslant y \leqslant 4x$. Because of this and Theorems 5 and 6, (4.2) will follow by showing that there is a constant C, independent of x, such that $U(x) \leqslant CV(4x)$. Now, taking $I = [4x, 8x]$ in (4.1) and reducing the interval of integration in the first integral shows that

$$\left[\int_{x/2}^{x} \frac{U(t)\,dt}{2^p 4x}\right]\left[\frac{1}{4x}\int_{4x}^{8x} [V(t)]^{-1/(p-1)}\,dt\right]^{p-1} \leqslant B. \tag{4.3}$$

Condition (a) implies that $V(t) \leqslant AV(4x)$ for $4x \leqslant t \leqslant 8x$ and $U(t) \geqslant U(x)/A$ for $x/2 \leqslant t \leqslant x$. Using these facts in (4.3) shows that

$$\left[\frac{x}{2}\,\frac{U(x)}{A2^p 4x}\right]\left[[AV(4x)]^{-1/(p-1)}\right]^{p-1} \leqslant B, \tag{4.4}$$

and this easily reduces to the inequality $U(x) \leqslant CV(4x)$.

The proof of Theorem 7 with assumption (b) is similar to the proof given above for assumption (a). In this case it is sufficient to prove that $U(x) \leqslant CV(x/4)$, and this is done by taking $I = [x/8, x/4]$ in (4.1) and using assumption (b).

Theorem 7 with assumption (c) follows from Theorem 6 and a modification of the proof of Theorem 5. The modified proof is the same as the original proof of Theorem 5 except that the decomposition in (3.5), (3.6) and (3.7) is done with 2 replaced by $d^{1/2}$, and 2 is replaced by $d^{1/2}$ in appropriate places thereafter.

To prove Theorem 7 with condition (d), write f as the sum of its positive and negative parts. Each part is monotone and it is sufficient to prove the theorem for each part separately. Without loss of generality, therefore, assume that $f(x) \geqslant 0$ and monotone. If $f(x)$ is decreasing, then $f^*(x) = \frac{1}{x}\int_0^x f(t)\,dt$ and (4.2) follows immediately from Theorem A and Theorem 6. If $f(x)$ is increasing, then (4.2) is easily seen to be implied by the following lemma, which implies that either the left-hand side of (4.2) is 0 or the right-hand side is ∞.

LEMMA. *If* $U(x) \geqslant 0$, $V(x) \geqslant 0$, $1 < p < \infty$ *and* (4.1) *holds with* B *independent of* I *for every* $I \subset [0, \infty)$, *then either* $U(x) = 0$ *almost everywhere in* $(-\infty, \infty)$ *or* $\int_r^{\infty} V(x)\,dx = \infty$ *for all* $r > 0$.

To prove the lemma, assume that $U(x) > 0$ on a set of positive measure. Given $r > 0$, choose an $h > r$ such that $\int_{-h}^{h} U(x)\,dx > 0$. If $n \geqslant 0$ and $I = [2^n h, 2^{n+1}h]$, then (4.1) implies that

$$\left[\frac{1}{2^n h 3^p}\int_{-h}^{h} U(x)\,dx\right]\left[\frac{1}{h2^n}\int_I [V(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqslant B. \tag{4.5}$$

By Hölder's inequality,

$$\left[\frac{1}{2^n h}\int_I V(x)\,dx\right]^{-1} \leqslant \left[\frac{1}{2^n h}\int_I [V(x)]^{-1/(p-1)}\,dx\right]^{p-1}. \tag{4.6}$$

Using (4.6) in (4.5) then implies that

$$\int_{2^n h}^{2^{n+1}h} V(x)\,dx \geqslant \frac{1}{3^p B}\int_{-h}^{h} U(x)\,dx.$$

Adding these inequalities for $n = 0, 1, 2, \ldots$ then proves that $\int_h^{\infty} V(x)\,dx = \infty$. Since $h > r$, the conclusion of the lemma follows.

5. Hilbert transform sufficiency results. The following will be proved in this section.

THEOREM 8. *Theorems* 5, 6 *and parts* (a), (b) *and* (c) *of Theorem* 7 *remain true with f^* replaced by $|\tilde{f}|$ provided that it is also assumed that there is a constant D such that*

$$\left[\frac{1}{|I|}\int_I U(x)\,dx\right]\left[\int_0^\infty \frac{|I|^{p'-1}[V(x)]^{-1/(p-1)}}{(|I|+|x-x_I|)^{p'}}\,dx\right]^{p-1} \leqslant D, \tag{5.1}$$

where $p' = p/(p-1)$ and x_I denotes the center of the interval I. Condition (5.1) is to be assumed for all $I\subset[0,\infty)$ in the case of Theorem 5, all $I\subset(-\infty,0]$ in the case of Theorem 6 and all $I\subset(-\infty,\infty)$ in the case of Theorem 7.

In the proof of this modification of Theorem 5, the analogues of (3.5) and (3.6) are treated in the same way that they were in the proof of Theorem 5; the unweighted norm theorem for the Hilbert transform [11], Vol. II, p. 256, is used instead of the norm theorem for the maximal function. The analogue of (3.7) is

$$\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} \left|\left(f(x)\chi_{[2^{n+2},\infty)}(x)\right)^{\sim}\right|^p U(x)\,dx.$$

This is bounded by a constant times

$$\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}}\left[\int_x^\infty \frac{|f(t)|}{t}\,dt\right]^p U(x)\,dx = \int_0^\infty\left[\int_x^\infty\frac{|f(t)|}{t}\,dt\right]^p U(x)\,dx. \tag{5.2}$$

By Theorem B, (5.2) is bounded by $\int_0^\infty |f(x)|^p V(x)\,dx$ provided that there is a constant C such that for $r>0$

$$\left[\int_0^r U(x)\,dx\right]\left[\int_r^\infty [x^pV(x)]^{-1/(p-1)}\,dx\right]^{p-1}\leqslant C.$$

This, however, follows easily from (5.1) by taking $I=[0,r]$.

The rest of Theorem 8 is proved in the same way that the original versions were proved.

The same procedures can be used to show that Theorem 8 is valid for the maximal Hilbert transform

$$\sup_{\varepsilon>0}\left|\int_{|x-t|>\varepsilon}\frac{f(t)}{x-t}\,dt\right|.$$

The analogues of (3.5) and (3.7) have the same estimates as the corresponding parts of Theorem 8. The analogue of (3.6) is treated by the unweighted norm theorem for the maximal Hilbert transform, Theorem 4, p. 42, of [10].

6. Sufficiency for $p=1$. In this case it is easy to show that the necessary condition obtained in Theorem 4 is also sufficient. The result is the following.

THEOREM 9. *Assume that $U(x)\geqslant 0$, $V(x)\geqslant 0$ and that there is a constant B such that for almost every x*

$$\int_{-\infty}^{\infty}\frac{U(t)\,dt}{|x-t|}\leqslant BV(x). \tag{6.1}$$

Then

$$\int_{-\infty}^{\infty} f^*(x)\,U(x)\,dx\leqslant B\int_{-\infty}^{\infty}|f(x)|\,V(x)\,dx$$

and

$$\int_{-\infty}^{\infty}|\tilde{f}(x)|\,U(x)\,dx\leqslant B\int_{-\infty}^{\infty}|f(x)|\,V(x)\,dx.$$

To prove this, observe first that by their definitions, $f^*(x)$ and $|\tilde{f}(x)|$ are both bounded above by $\int_{-\infty}^{\infty}\frac{|f(y)|\,dy}{|x-y|}$. It is sufficient, therefore, to prove that

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{|f(y)|\,dy}{|x-y|}\right)U(x)\,dx\leqslant B\int_{-\infty}^{\infty}|f(x)|\,V(x)\,dx.$$

This follows trivially by using Fubini's theorem on the left-hand side, and then using (6.1).

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All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1+\varepsilon$ biorthogonal sequences

by

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Abstract. It is proved that in every infinite dimensional Banach space X for every $\varepsilon > 0$ there exists a biorthogonal sequence (x_n, x_n^*) such that (i) linear combinations of the x_n's are dense in X, (ii) if $x \in X$ and $x_n^*(x) = 0$ for all n, then $x = 0$, (iii) $\|x_n\|\,\|x_n^*\| < 1+\varepsilon$ for all n.

The proof bases upon the following result due to Milman:

(M) *If E is a finite dimensional subspace of an infinite dimensional Banach space X, then, for every $\varepsilon > 0$, there exists a subspace F of X with $\dim F > \varepsilon^{-1}$ such that*

$$\|e+f\| \geqslant (1-\varepsilon)\max(\|e\|, \|f\|) \quad \textit{for every } e \textit{ in } E \textit{ and } f \textit{ in } F.$$

A proof of (M) is included.

Introduction. It is known (cf., e.g. [1], p. 238 or [15]) that if X is a finite dimensional Banach space (say, $\dim X = m$), then X admits a biorthogonal sequence $(x_n, x_n^*)_{n=1}^m$ with $\|x_n\|\cdot\|x_n^*\| = 1$ for $n = 1, \ldots, m$. In the present paper we improve results of [3] and [11]. We establish the following.

THEOREM 1. *Let X be an infinite dimensional separable Banach space. Then, for every $\varepsilon > 0$, there exists in X a fundamental total and bounded by $1+\varepsilon$ biorthogonal sequence.*

Recall that a sequence (e_n, e_n^*), where e_n are elements of X and e_n^* are elements of X^* — the dual of X, is biorthogonal if $e_n^*(e_m) = \delta_n^m$ for $n, m = 1, 2, \ldots$, is total if, for every $x \in X$, the condition $e_n^*(x) = 0$ for every $n = 1, 2, \ldots$ implies $x = 0$, is fundamental if, for every $x^* \in X^*$, the condition $x^*(e_n) = 0$ for every $n = 1, 2, \ldots$ implies $x^* = 0$, is bounded by a $c \geqslant 1$ if $\|e_n\|\cdot\|e_n^*\| \leqslant c$ for every $n = 1, 2, \ldots$

The paper consists of two sections. Theorem 1 is proved in the first section. The proof bases on the following result due to Dvoretzky [5] and Milman [10], Theorem 5.8:

(D–M). *Given $\delta > 0$ and positive integers n, m, N. There exists a positive integer $K = K(n, m, N, \delta)$ such that if Y is a Banach space of dimension greater than K, then, for every n-dimensional linear subspace E of Y,*