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All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1+\varepsilon$ biorthogonal sequences

by

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Abstract. It is proved that in every infinite dimensional Banach space X for every $\varepsilon > 0$ there exists a biorthogonal sequence (x_n, x_n^*) such that (i) linear combinations of the x_n's are dense in X, (ii) if $x \in X$ and $x_n^*(x) = 0$ for all n, then $x = 0$, (iii) $\|x_n\|\,\|x_n^*\| < 1+\varepsilon$ for all n.

The proof bases upon the following result due to Milman:

(M) *If E is a finite dimensional subspace of an infinite dimensional Banach space X, then, for every $\varepsilon > 0$, there exists a subspace F of X with $\dim F > \varepsilon^{-1}$ such that*

$$\|e+f\| \geqslant (1-\varepsilon)\max(\|e\|, \|f\|) \quad \textit{for every } e \textit{ in } E \textit{ and } f \textit{ in } F.$$

A proof of (M) is included.

Introduction. It is known (cf., e.g. [1], p. 238 or [15]) that if X is a finite dimensional Banach space (say, $\dim X = m$), then X admits a biorthogonal sequence $(x_n, x_n^*)_{n=1}^m$ with $\|x_n\| \cdot \|x_n^*\| = 1$ for $n = 1, \ldots, m$. In the present paper we improve results of [3] and [11]. We establish the following.

THEOREM 1. *Let X be an infinite dimensional separable Banach space. Then, for every $\varepsilon > 0$, there exists in X a fundamental total and bounded by $1+\varepsilon$ biorthogonal sequence.*

Recall that a sequence (e_n, e_n^*), where e_n are elements of X and e_n^* are elements of X^* — the dual of X, is biorthogonal if $e_n^*(e_m) = \delta_n^m$ for $n, m = 1, 2, \ldots$, is total if, for every $x \in X$, the condition $e_n^*(x) = 0$ for every $n = 1, 2, \ldots$ implies $x = 0$, is fundamental if, for every $x^* \in X^*$, the condition $x^*(e_n) = 0$ for every $n = 1, 2, \ldots$ implies $x^* = 0$, is bounded by a $c \geqslant 1$ if $\|e_n\| \cdot \|e_n^*\| \leqslant c$ for every $n = 1, 2, \ldots$

The paper consists of two sections. Theorem 1 is proved in the first section. The proof bases on the following result due to Dvoretzky [5] and Milman [10], Theorem 5.8:

(D–M). *Given $\delta > 0$ and positive integers n, m, N. There exists a positive integer $K = K(n, m, N, \delta)$ such that if Y is a Banach space of dimension greater than K, then, for every n-dimensional linear subspace E of Y,* *and for every linear subspace Y_1 of Y of codimension m, there exists a linear subspace F of Y such that*

(i) $\dim F = N$, $F \subset Y_1$, $F \cap E = \{0\}$,

(ii) *if $G = E+F$, the direct sum of subspaces E and F, then*

$$\max(\|P\|, \|I_G - P\|) < 1+\delta,$$

where $P\colon G \xrightarrow[\text{onto}]{} F$ is the projection with $\ker P = E$,

(iii) *there exists an isomorphism $T\colon l^2_N \xrightarrow[\text{onto}]{} F$ such that*

$$\max(\|T\|, \|T^{-1}\|) < 1+\delta.$$

Here by I_G we denote the identity operator on G and by l^2_N the Hilbert space of dimension N.

The second section of the paper contains a proof of (D–M). We concentrate ourselves on the case of complex Banach spaces which is not treated by Milman [8], [10].

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1. Proof of Theorem 1. Since X is separable, there exist sequences (x_j) of elements of X and (x_j^*) of elements of X^* such that

$$(1)\quad \begin{array}{l} \text{if } x \in X \text{ and } x_j^*(x) = 0 \text{ for all } j, \text{ then } x = 0, \\ \text{if } x^* \in X^* \text{ and } x^*(x_j) = 0 \text{ for all } j, \text{ then } x^* = 0. \end{array}$$

After fixing such sequences (x_j) and (x_j^*) and ε with $0 < \varepsilon < 1$ we define inductively a sequence $(e_m)_{m\geq 0}$ of elements of X, a sequence $(e_m^*)_{m\geq 0}$ of elements of X^* and an increasing sequence $(n_m)_{m\geq 0}$ of the indices so that

(2) the sequence $(e_1, e_2, \ldots, e_{n_s};\ e_1^*, e_2^*, \ldots, e_{n_s}^*)$ is biorthogonal for $s = 1, 2, \ldots$,

(3) $\|e_j\|\cdot\|e_j^*\| < 1+\varepsilon$ for $1 \leqslant j \leqslant n_s$ and for $s = 1, 2, \ldots$,

(4) $\operatorname{span}(e_j)_{j\leqslant n_{2q-1}} \supset \operatorname{span}(x_p)_{p\leqslant q}$ for $q = 1, 2, \ldots$,

(5) $\operatorname{span}(e_j^*)_{j\leqslant n_{2q}} \supset \operatorname{span}(x_p^*)_{p\leqslant q}$ for $q = 1, 2, \ldots$

(By $\operatorname{span}(a_j)_{j\leqslant k}$ we denote the smallest linear subspace spanned by the vectors $a_1, a_2, \ldots, a_k$.)

We put $e_0 = 0$, $e_0^* = 0$ and $n_0 = 0$. Assume that, for some $t \geqslant 0$, elements $e_0, e_1, \ldots, e_{n_t}$, functionals $e_0^*, e_1^*, \ldots, e_{n_t}^*$ and indices $n_0, n_1, \ldots, n_t$ have been defined to satisfy conditions (2)–(5) for all $s \leqslant t$. We shall define the next index n_{t+1}, the elements e_j and the functionals e_j^* for $n_t < j \leqslant n_{t+1}$. We consider separately two cases.

1° $t+1 = 2q-1$ for some $q = 1, 2, \ldots$ First using the standard Schmidt biorthogonalization procedure and the inductive hypothesis, we pick a $y \in X$ and $y^* \in X^*$ so that

$$(6)\qquad \operatorname{span}(e_1, e_2, \ldots, e_{n_t}, y) \supset \operatorname{span}(x_p)_{1\leqslant p\leqslant q}$$

and

(7) the sequence $(e_1, e_2, \ldots, e_{n_t}, y;\ e_1^*, e_2^*, \ldots, e_{n_t}^*, y^*)$ is biorthogonal.

Next we pick an integer $r \geqslant 1$ so that

$$(8)\qquad 2^{-r/2}(\|y\| + \|y^*\|) < \varepsilon/4$$

and we put $n_{t+1} = n_t + 2^r$. Now applying (D–M) we pick, for $Y = X$, $N = 2^r - 1$, $E = \operatorname{span}(e_1, e_2, \ldots, e_{n_t}, y)$, $\delta = \varepsilon/4$, $Y_1 = \ker y^* \cap \bigcap_{1\leqslant j\leqslant n_t} \ker e_j^*$, a linear subspace F of X which satisfies conditions (i)–(iii). Let $(v_j)_{1\leqslant j\leqslant N}$ denote the unit vector basis of l^2_N and let $(v_j^*)_{1\leqslant j\leqslant N}$ denote the coordinate functionals on l^2_N. Let us set $f_j = T(v_j)$ and $f_j^* = (T^{-1})^*(v_j^*)$ for $1 \leqslant j \leqslant N$. (By S^* we denote the adjoint of a linear operator S.) Furthermore, let $(w^r_{i,j})_{1\leqslant i,j\leqslant 2^r}$ be the $2^r\times 2^r$ Walsh orthogonal matrix, i.e. $w^r_{i,j} = 2^{-r/2} w_i\big(2^{-r-1}(2j-1)\big)$, where $(w_i)_{1\leqslant i<\infty}$ denotes the Walsh orthonormal system (cf. [7], Kapitel IV, § 6). For $i = 1, 2, \ldots, 2^r$ we put

$$e_{n_t+i} = w^r_{i,1} y + \sum_{j=2}^{2^r} w^r_{i,j} f_{j-1},$$

and we define $e^*_{n_t+i} \in X^*$ to be any norm preserving extension of the linear functional g_i^* defined on $G = E+F$ by

$$g_i^* = w^r_{i,1} y_G^* + \sum_{j=2}^{2^r} w^r_{i,j} P^*(f^*_{j-1}),$$

where y_G^* denotes the restriction of y^* to G.

To complete the induction in case 1° it remains to verify that the sequences $(e_j)_{0\leqslant j\leqslant n_{t+1}}$ and $(e_j^*)_{1\leqslant j\leqslant n_{t+1}}$ satisfy conditions (2)–(5).

By the inductive hypothesis, $e_i^*(e_k) = \delta_i^k$ for $i, k \leqslant n_t$. If $i \leqslant n_t$ and $n_t < k \leqslant n_{t+1}$, then $e_i^*(e_k) = 0$ because $e_i^*(y) = 0$ (by (7)) and for $1 \leqslant j \leqslant N$, $e_i^*(f_j) = 0$ (because $f_j \in F \subset \ker e_i^*$). If $n_t < i \leqslant n_{t+1}$ and $k \leqslant n_t$, then $e_i^*(e_k) = g^*_{i-n_t}(e_k) = 0$ because $y^*(e_k) = 0$ (by (7)) and for $1 \leqslant j \leqslant N$, $P^*(f_j^*)(e_k) = f_j^*(P(e_k)) = 0$ (because $e_k \in E = \ker P$). If $n_t < i, k \leqslant n_{t+1}$, then $e_i^*(e_k) = g^*_{i-n_t}(e_k) = \delta_i^k$ because the sequence

$$(y, f_1, f_2, \ldots, f_N;\ y_G^*, P^*(f_1^*), P^*(f_2^*), \ldots, P^*(f_N^*))$$

is biorthogonal and the matrix $(w^r_{i,j})_{1\leqslant i,j\leqslant 2^r}$ is orthogonal. This proves (2).

To check (3) observe that $|w^r_{i,j}| = 2^{-r/2}$ for $i, j = 1, 2, \ldots, 2^r$. Thus, by (8),

$$\|e_{n_t+i}\| \leqslant \|y\| 2^{-r/2} + \Big\| T\Big(\sum_{j=2}^{2^r} w^r_{i,j} v_{j-1}\Big)\Big\| < \varepsilon/4 + \|T\|(1-2^{-r})^{1/2}$$
$$< \varepsilon/4 + 1 + \varepsilon/4 < 1+\varepsilon.$$

Similarly,

$$\|e^*_{n_t+i}\| = \|g^*_i\| \leqslant \|y^*\| 2^{-r/2} + \|P^*\| \, \|(T^{-1})^*\| \cdot \Big\|\sum_{j=2}^{2^r} w^r_{i,j} v^*_{j-1}\Big\|$$
$$< \varepsilon/4 + (1+\varepsilon/4)^2 < 1+\varepsilon.$$

The above inequalities together with the inductive hypothesis prove (3).

Finally, since the matrix $(w^r_{i,j})_{1\leqslant i,j\leqslant 2^r}$ is invertible and since the elements $y, f_1, f_2, \ldots, f_N$ are linearly independent, we infer that

$$\operatorname{span}(y, f_1, f_2, \ldots, f_N) = \operatorname{span}(e_{n_t+1}, e_{n_t+2}, \ldots, e_{n_{t+1}}).$$

Hence, remembering that $t+1 = 2q-1$, we get, by (6),

$$\operatorname{span}(e_j)_{1\leqslant j\leqslant n_{2q-1}} \supset \operatorname{span}(x_p)_{1\leqslant p\leqslant q}.$$

This proves (4) for $t+1 = 2q-1$. Condition (5) for $t+1 = 2q-1$ follows directly from the inductive hypothesis.

2° $t+1 = 2q$ for some $q = 1, 2, \ldots$ Similarly as in case 1°, we first pick $y \in X$ and $y^* \in X^*$ so that

(6*) $\quad \operatorname{span}(e^*_1, e^*_2, \ldots, e^*_{n_t}, y^*) \supset \operatorname{span}(x^*_p)_{1\leqslant p\leqslant q}$

and

(7*) $\quad$ the sequence $(e_1, e_2, \ldots, e_{n_t}, y; e^*_1, e^*_2, \ldots, e^*_{n_t}, y^*)$ is biorthogonal.

Next we put $n_{t+1} = n_t + 2^r$, where $r \geqslant 1$ is an integer satisfying (8). Now, by (D–M), we pick for $Y = X^*$, $N = 2^r - 1$, $E = \operatorname{span}(e^*_1, e^*_2, \ldots, e^*_{n_t}, y^*)$, $Y_1 = \ker y \cap \bigcap_{1\leqslant j\leqslant n_t} \ker e_j$ (where $\ker x = \{x^* \in X^*\colon x^*(x) = 0\}$) and $\delta = \varepsilon/4$ a linear subspace F of X^* which satisfies conditions (i)–(iii). Let us set, as in case 1°, $T(v_j) = f_j \in F \subset X^*$ and $(T^{-1})^*(v^*_j) = f^*_j \in F^*$ for $j = 1, 2, \ldots, N$ and define for $i = 1, 2, \ldots, 2^r$ the functionals $e^*_{n_t+i}$ by

$$e^*_{n_t+i} = w^r_{i,1} y^* + \sum_{j=2}^{2^r} w^r_{i,j} \cdot f_{j-1}.$$

Let $e^{**}_{n_t+i} \in X^{**}$ be any norm preserving extension of the linear functional g^*_i defined on G by

$$g^*_i = w^r_{i,1} y_G + \sum_{j=2}^{2^r} w^r_{i,j} P^*(f^*_{j-1})$$

(where $y_G \in G^*$ is defined by $y_G(g) = g(y)$ for $g \in G \subset X^*$). Finally, for $i = 1, 2, \ldots, 2^r$, using Goldstine's Theorem we define e_{n_t+i} to be an arbitrary element of X such that

$$e^*_k(e_{n_t+i}) = e^{**}_{n_t+i}(e^*_k) \quad \text{ for } k = 1, 2, \ldots, n_{t+1},$$
$$\|e_{n_t+i}\| \leqslant \|e^{**}_{n_t+i}\|(1+\varepsilon/16).$$

We omit the verification (similar to that of case 1°) that such defined sequences $(e_j)_{0\leqslant j\leqslant n_{t+1}}$ and $(e^*_j)_{0\leqslant j\leqslant n_{t+1}}$ satisfy conditions (2)–(5). This completes the induction and the proof of the theorem.

Remarks. A. The proof of Theorem 1 gives also:

If X^* is separable, then for every $\varepsilon > 0$ there exists in X a biorthogonal sequence (e_n, e^*_n) bounded by $1+\varepsilon$ which is fundamental (equivalently the e_n's are linearly dense in X) and such that the e^*_n's are linearly dense in X^*.

B. We do not know any example of a separable Banach space which does not admit any fundamental and total biorthogonal sequence bounded by 1.

2. We begin with some notation (cf. Dvoretzky [5] and Milman [8], [9]). By S_K we denote the unit sphere of the K-dimensional real Hilbert space $\mathscr{E}_K$. By $\mu_{K,m}$ we denote the rotation invariant normalized Borel measure on the Grassman manifold $\mathscr{G}^m_K$ of all m-dimensional linear subspaces of $\mathscr{E}_K$. Clearly, $\mu_{K,1}$ can be regarded as the normalized Lebesgue measure of symmetric Boral subsets of S_K. Given a symmetric Borel subset A of S_K a $t \geqslant 0$ and $m = 1, 2, \ldots, K$, we denote by A_t the set of all points of S_K whose geodesic distance from A is $\leqslant t$ and we put

$$A^m = \{H \in \mathscr{G}^m_K\colon H \cap A \neq \emptyset\}.$$

Clearly, if A is measurable, so are A_t and $A^m_t = (A_t)^m$.

The basic tool for the proof of (D–M) is the following result due to Dvoretzky [5], Theorem 2(B) (for some details of the proof cf. [6]).

For every $t > 0$ and for every symmetric Borel subset A of S_K, we have

$$(\mathrm{D}_1) \quad \mu_{K,1}(A_t) \geqslant [\mu_{K,2}(A^2)]^{1/2}\big(1 - e^{-2t((K-2)\mu_{K,2}(A^2)/2\pi)^{1/2}}\big)^2, \quad K = 3, 4, \ldots$$

$$(\mathrm{D}_2) \quad \mu_{K,2}(A^2_t) \geqslant \mu_{K,m}(A^m)\big(1 - e^{-\frac{t}{m-2}((K-m)/2\pi)^{1/2}}\big)^{m-2}, \quad K = 4, 5, \ldots, \; m = 3, 4, \ldots, K-1.$$

The next lemma shows how the above inequalities work.

LEMMA 2.1. *Let $\varepsilon > 0$ and let positive integers s and $m \geqslant 3$ be given. Then there exists a $C = C(\varepsilon, s, m)$ such that, for every integer $K > C$ and for every symmetric Borel subset A of S_K with $A^m = \mathscr{G}^m_K$, there exists an s-dimensional linear subspace H of $\mathscr{E}_K$ such that $S_K \cap H \subset A_s$.*

Proof (¹). First observe that, given $\varepsilon > 0$ and a positive integer s, there exists an $\eta = \eta(\varepsilon, s) < 1$ such that if B is a symmetric Borel subset of S_s with $\mu_{s,1}(B) > 1-\eta$, then B is an $\varepsilon/3$-net for S_s. This is obvious because the Lebesgue measure of a geodesic sphere (in S_s) of radius $\varepsilon/3$ is positive. Since $A^m = \mathscr{G}_K^m$, it follows from (D_2) that for every t with $\varepsilon/3 > t > 0$ and for K large enough (precisely $K > C_1(\eta, m)$),

$$\mu_{K,2}(A_t^2) \geqslant \left(1 - e^{-\frac{t}{m-2}((K-m)/2\pi)^{1/2}}\right)^{m-2} > (1-\eta)^2.$$

Thus, by (D_1) applied for A_t, we get

$$\mu_{K,1}\big((A_t)_t\big) > (1-\eta)\left(1 - e^{-2t((K-2)(1-\eta)/2\pi)^{1/2}}\right)^2$$

for K large enough. Since $(A_t)_t = A_{2t}$, we infer that $\mu_{K,1}(A_{2t}) > 1-\eta$ for K large enough (precisely for $K > C_2(\eta, m)$).

Now, for each $H \in \mathscr{G}_K^s$, let $\mu_{s,1}^H$ be the normalized rotation invariant Borel measure on $S_K \cap H$. By the uniqueness of the normalized rotation invariant Borel measure on S_K, for every symmetric Borel subset B of S_K, we have

$$\mu_{K,1}(B) = \int_{\mathscr{G}_K^s} \mu_{s,1}^H(B \cap H)\cdot \mu_{K,s}(dH).$$

Applying this formula to $B = A_{2t}$, for $K > C_2(\eta, m)$, we get

$$1-\eta < \int_{\mathscr{G}_K^s} \mu_{s,1}^H(A_{2t} \cap H)\, \mu_{K,s}(dH).$$

Hence, for some $H \in \mathscr{G}_K^s$, $1-\eta < \mu_{s,1}^H(A_{2t} \cap H)$. Thus, by the choise of η. $A_{2t} \cap H$ is an $\varepsilon/3$-net for $H \cap S_K$. Thus $S_K \cap H \subset A_\varepsilon \cap H$ because $t < \varepsilon/3$. Hence $S_K \cap H \subset A_\varepsilon$ for $K > C(\varepsilon, s, m) = C_2(\eta(\varepsilon, s), m)$.

Our next lemma goes back to Krasnosel'skii, Krein, Milman [16] (cf. also [3]). We state it for a strictly convex Banach space, i.e. such a space whose unit sphere does not contain intervals.

LEMMA 2.2. *Let $1 \geqslant \varepsilon > 0$ and let s and $m \geqslant 3$ be positive integers. Let Y be a finite dimensional strictly convex Banach space which is a direct sum of its subspaces E and Z. Assume that* $\dim E = m$ *and Z is isometrically isomorphic to the Hilbert space l_L^2 with $L > C(\varepsilon/6, s, m+1)$ in the case of real scalars and $L > C(\varepsilon/6, 2s, 2m+1)$ in the case of complex scalars, where $C(\cdot,\cdot,\cdot)$ is the function of Lemma 2.1. Then there exists a linear subspace H of Z with* $\dim H = s$ *such that the projection $P\colon E+H \xrightarrow[\text{onto}]{} H$ with* $\ker P = E$ *has the norm $\|P\| \leqslant 1+\varepsilon$.*

(¹) The author has learned this proof from T. Figiel.

Proof. If B is a Banach space, then $\operatorname{rdim} B$ denotes the dimension of B over the reals. It will be convenient for us to identify the unit sphere of Z with the metric space $S_{\operatorname{rdim} Z}$. A point $z \in S_{\operatorname{rdim} Z}$ is E-supporting if $\|z+e\| \geqslant 1$ for every $e \in E$. We denote by A the (closed and symmetric) subset of $S_{\operatorname{rdim} Z}$ consisting of all E-supporting points of $S_{\operatorname{rdim} Z}$. First we show that, for every $H \in \mathscr{G}_{\operatorname{rdim} Z}^{\operatorname{rdim} E+1}$, the intersection $A \cap H$ is non-empty. To this end, fix a normalized basis for E over the real scalars, say $e_1, e_2, \ldots, e_{\operatorname{rdim} E}$, and a real subspace H of Z with $\operatorname{rdim} H = \operatorname{rdim} E+1$. For $j = 1, 2, \ldots, \operatorname{rdim} E$ and for $z \in H \cap S_{\operatorname{rdim} Z}$ we put

$$a_j(z) = \begin{cases} 0, & \text{if } \|te_j+z\| \geqslant 1 \quad \text{for every real } t, \\ \text{the unique real number } t \neq 0 \text{ such that } \|te_j+z\| = 1, & \text{otherwise.} \end{cases}$$

Since Y is strictly convex, all $a_j(\cdot)$ are well-defined continuous functions on $S_{\operatorname{rdim} Z} \cap H$. Now we define $\Phi\colon S_{\operatorname{rdim} Z} \cap H \to \mathscr{E}_{\operatorname{rdim} E}$ by

$$\Phi(z) = \big(a_1(z), a_2(z), \ldots, a_{\operatorname{rdim} E}(z)\big) \quad \text{for } z \in S_{\operatorname{rdim} Z} \cap H.$$

Clearly, Φ is a continuous antipodic function. Thus, by the Borsuk Theorem [2], there exists a $z_0 \in S_{\operatorname{rdim} Z} \cap H$ such that $\Phi(z_0) = 0$. Evidently, the point z_0 is E-supporting because there are $\operatorname{rdim} E$ straight lines passing through z_0 supporting the sphere $S_{\operatorname{rdim} Z} \cap H$ at z_0 and parallel to the linearly independent vectors $e_1, e_2, \ldots, e_{\operatorname{rdim} E}$, respectively.

We have just proved that $A^{\operatorname{rdim} E+1} = \mathscr{G}_{\operatorname{rdim} Z}^{\operatorname{rdim} E+1}$. Thus, by Lemma 2.1, the assumption imposed on L yields that there exists a real subspace H_R of Z such that $H_R \cap S_{\operatorname{rdim} Z} \subset A_{\varepsilon/6}$, and $\operatorname{rdim} H_R = s$ in the real case and $\operatorname{rdim} H_R = 2s$ in the complex case. Since in the case of complex scalars A is circled, it follows that if H is the smallest complex subspace containing H_R, then $H \cap S_{\operatorname{rdim} Z} \subset A_{\varepsilon/6}$. Thus in both cases of the real and of the complex scalars there exists an s-dimensional subspace H of Z such that $H \cap S_{\operatorname{rdim} Z} \subset A_{\varepsilon/2}$. To complete the proof we show that such chosen H has the desired property. To this end, pick $z \in H$ with $\|z\| = 1$. Since $H \cap S_{\operatorname{rdim} Z} \subset A_{\varepsilon/2}$, there exists a $z_0 \in A$ such that $\|z - z_0\| \leqslant \varepsilon/2$. Thus, for every $e \in E$, we have

$$\|z+e\| \geqslant \|z_0+e\| - \|z-z_0\| \geqslant 1 - \varepsilon/2 > (1+\varepsilon)^{-1}.$$

Hence, by the homogeneity of the norm, we get

$$\|z+e\| > (1+\varepsilon)^{-1}\|z\| \quad \text{for every } 0 \neq z \in H \text{ and every } e \in E.$$

The last inequality implies that the projection $P\colon H+E \xrightarrow[\text{onto}]{} H$ with $\ker P = E$ has the norm $\|P\| < 1+\varepsilon$.

Proof of (D–M). First observe that given a positive integer n and $\delta > 0$, there exists an integer $d = d(n, \delta)$ such that the unit sphere of any n-dimensional Banach space admits a δ-net consisting of less than

d points. Thus the standard Mazur's technique of constructing basic sequences (cf. e.g. [13]) yields the existence of a subset B of the unit sphere of Y^* such that B consists of exactly d points and if $Y_B = \bigcap_{y^* \epsilon B} \ker y^*$, then the projection $Q: E + Y_B \xrightarrow[\text{onto}]{} E$ with $\ker Q = Y_B$ has the norm $\|Q\| < 1+\delta$. Now, by the Dvoretzky Theorem (cf. [5], [14]), for every positive integer L, there exists a $D = D(L, \delta)$ such that if $\dim Y > D + d + m \geqslant D + \operatorname{codim} Y_B + \operatorname{codim} Y_1$, then there exists a linear subspace W of $Y_B \cap Y_1$ which admits an isomorphism $T_W: W \xrightarrow[\text{onto}]{} l_L^2$ with $\|w\| \leqslant \|T_W(w)\| \leqslant (1+\delta)^{1/3}\|w\|$ for every $w \epsilon W$. Using Lemma 1 of [12] we can construct a Banach space Y_W which contains l_L^2 isometrically and has the property that the isomorphism T_W admits an extension to an isomorphism T from $E+W$ onto Y_W satisfying the condition $\|x\| \leqslant \|T(x)\| \leqslant (1+\delta)^{1/3}\|x\|$ for every $x \epsilon E+W$. The finite dimensional space Y_W admits a renorming such that in the new norm Y_W is strictly convex and the ratio of the new norm and the original norm of every element belongs to the interval $[1, (1+\delta)^{1/3}]$. Being a little bit careful we can do the renorming so that in addition the new norm coincides with the original norm on $T(W) = l_L^2$. In the sequel we denote by $\tilde{Y}$ the space Y_W under this new strictly convex norm and we regard T as an isomorphism from $E+W$ onto $\tilde{Y}$. Now, assuming that $L > C\big(((1+\delta)^{1/3}-1)/6,\ N,\ n+1\big)$ in the case of real scalars and $L > C\big(((1+\delta)^{1/3}-1)/6,\ 2N,\ 2n+1\big)$ in the case of complex scalars, we can apply Lemma 2.2 to choose an N-dimensional linear subspace of l_L^2 so that the projection $\tilde{P}$ from $T(E)+H$ onto H with $\ker \tilde{P} = T(E)$ has the norm $\|\tilde{P}\| < (1+\delta)^{1/3}$. Finally, we put $F = T^{-1}(H)$ and $P = T^{-1}\tilde{P}T$. We admit

$$K = K(n, m, N, \delta)$$
$$= d(n, \delta) + m + D\Big(1 + \big[C\big(((1+\delta)^{1/3}-1)/6,\, 2N,\, 2n+1\big)\big], \delta\Big).$$

Added in proof. After this paper was submitted for publication, the author discovered the following simple derivation of (D–M) from the Dvoretzky Theorem. The essential part of (D–M) is:

(*) *Given positive integers k and n and $1 > \varepsilon > 0$, there is an integer $N = N(k, n, \varepsilon)$ such that if X is a Banach space with $\dim X > N$ and E is a linear subspace of X with $\dim E = k$, then there exists an ε-Euclidean subspace F of X with $\dim F = n$ such that*

$$\|e+f\| \geqslant (1-\varepsilon)\|f\| \quad \textit{for every } e \epsilon E \textit{ and } f \epsilon F.$$

An n-dimensional Banach space F is *ε-Euclidean* iff there is an isomorphism $U: l_n^2 \to F$ with $\|U\|\,\|U\|^{-1}\| < 1+\varepsilon$.

Proof. The Dvoretzky Theorem says: given $\delta > 0$ and a positive integer q, there exists a positive integer $d(q, \delta)$ such that every Banach space B with $\dim B \geqslant d(q, \delta)$ contains a δ-Euclidean subspace C with $\dim C = q$. Put $\delta = \min\big(\varepsilon, (1-\varepsilon)^{-1/4} - 1\big)$, $N = d\big(d((2n+2k-1), \delta) - k, \delta\big) + k$. Let $h: X \to X/E$ be the quotient map. Since

$\dim X/E > N - k$, there is a δ-Euclidean subspace C_1 of X/E with $\dim C_1 = d(2n+2k-1, \delta) - k$. Thus $\dim h^{-1}(C_1) = d(2n+2k-1, \delta)$. Hence there is a δ-Euclidean subspace C_2 of $h^{-1}(C_1)$ with $\dim C_2 = 2n+2k-1$. Since $\dim E = k$, there is a δ-Euclidean subspace C of C_2 with $\dim C = 2n+k-1$ and such that $C \cap E = \{0\}$, i.e. h restricted to C is an isomorphism. Thus there are isomorphisms $U: l_{2n+k-1}^2 \to C$ and $V: h(C) \to l_{2n+k-1}^2$ such that $\max(\|U\|\,\|U^{-1}\|, \|V\|\,\|V^{-1}\|) < 1+\delta$.

Next recall the Krasnosel'skii–Krein–Milman Lemma [16] (cf. Lemma 2.2 in this paper) which says: If E is a linear subspace of a Banach space X with $\dim E = k$, then for every linear subspace Z of X with $\dim Z = k+1$ and $E \cap Z = \{0\}$ there exists a $z \epsilon Z$ such that $\|z\| = \|h(z)\| = 1$ ($h: X \to X/E$ is the quotient map). The lemma yields that the operator $T = VhU: l_{2n+k-1}^2 \to l_{2n+k-1}^2$ satisfies the condition

(+) *If G is a linear subspace of l_{2n+k-1}^2 with $\dim G = k+1$, then there is a $g \epsilon G$ with $\|g\| = 1$ and $\|Tg\| \geqslant (1+\delta)^{-2}$.*

Using (+) and the observation that if H is a subspace of l_{2n+k-1}^2 with $\dim H = j-1$ ($1 \leqslant j \leqslant n$), then

$$\dim\big(H^\perp \cap T^{-1}(T(H)^\perp)\big) \geqslant 2n+k-1-2(j-1) \geqslant k+1$$

($A^\perp$ denotes the orthogonal complement of $A \subset l_{2n+k-1}^2$), we define inductively a sequence $(g_j)_{1\leqslant j\leqslant n}$ in l_{2n+k-1}^2 so that, for $j = 1, 2, \ldots, n$, $\|g_j\| = 1$, $\|T(g_j)\| \leqslant (1+\delta)^{-2} \cap g_j \epsilon H_{j-1}^\perp \cap T^{-1}(T(H_{j-1})^\perp)$, where $H_0 = \{0\}$, $H_j = \operatorname{span}\{g_1, g_2, \ldots, g_j\}$. Clearly, $(g_j)_{1\leqslant j\leqslant n}$ and $(T(g_j))_{1\leqslant j\leqslant n}$ are orthogonal sequences. Hence, for $g = \sum_{j=1}^n t_j g_j \epsilon H_n$,

$$\|T(g)\|^2 = \sum_{j=1}^n |t_j|^2 \|T(g_j)\|^2 \geqslant (1+\delta)^{-4} \sum_{j=1}^n |t_j|^2 = (1+\delta)^{-4}\|g\|^2.$$

Thus if $F = U(H_n)$ then, for every $f \epsilon F$, we have

$$\|h(f)\| = \|V^{-1}TU^{-1}(f)\| \geqslant \|V^{-1}\|^{-1}\,\|TU^{-1}(f)\| \geqslant \|V^{-1}\|^{-1}(1+\delta)^{-2}\|U^{-1}(f)\|$$
$$\geqslant \|V^{-1}\|^{-1}\,\|U^{-1}\|^{-1}(1+\delta)^{-2}\|f\| \geqslant (1+\delta)^{-4}\|f\|.$$

Thus, for every $e \epsilon E$, the definition of δ yields

$$\|e+f\| \geqslant \inf_{e \epsilon E}\|e+f\| = \|h(f)\| \geqslant (1-\varepsilon)\|f\|.$$

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