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Some ideals of operators on Hilbert space

by

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Abstract. For q an even integer and $q < p < \infty$, it is shown that $\Pi_{p,q}$, the class of (p, q)-absolutely summing operators on Hilbert space, coincides with the ideal generated by the Lorentz sequence space $l_{2p/q,p}$. This differs from all previously known results (wherein $\Pi_{p,q}$ turns out to be a Schatten r-class for some $r = r(p, q)$) and settles negatively a conjecture of Kwapień and a problem of Pietsch.

1. Introduction. Following Mitiagin and Pełczyński [13], we say that a bounded linear operator T between Banach spaces X and Y is *(p, q)-absolutely summing*, $1 \leqslant q \leqslant p \leqslant \infty$, provided that the following condition holds.

(1) There exists a constant M (independent of n) such that, for all finite subsets $\{x_1, \ldots, x_n\}$ of X, we have

$$\Big(\sum_{j=1}^{n} \|Tx_j\|_Y^p\Big)^{1/p} \leqslant M \sup_{\|f\|_{X^*} \leqslant 1} \Big(\sum_{j=1}^{n} |\langle x_j, f\rangle|^q\Big)^{1/q}.$$

Condition (1) is clearly equivalent to

(2) $\sum_{j=1}^{\infty} \|Tx_j\|_Y^p < \infty$ whenever $(x_j)_{j=1}^{\infty}$ is a sequence of elements of X with the property that $\sum_{j=1}^{\infty} |\langle x_j, f\rangle|^q < \infty$ for each $f \in X^*$.

Such operators have received a good deal of attention in recent years, but their totality, $\Pi_{p,q}(X, Y)$, has not been characterized even when X and Y are Hilbert spaces. In this case the known results are described below. For the statement of these results, we denote by $\mathfrak{S}_r$ the so-called *Schatten r-class* [19]. Thus $\mathfrak{S}_r$ is the set of all bounded linear operators on l_2 which admit a factorization, UVW, where W is a unitary operator, U is an isometry on the range of V (i.e. $\|UVx\| = \|Vx\|$ for each $x \in l_2$), and V is a diagonal operator from l_r (i.e. $Vx = (\lambda_k x_k)_{k=1}^{\infty}$ for some fixed $\lambda \in l_r$).

(A) *If $p = q < \infty$, then $\Pi_{p,q} = \mathfrak{S}_2$.*

(B) *If $p = \infty$ or $\dfrac{1}{q} - \dfrac{1}{p} \geqslant \dfrac{1}{2}$, then every bounded linear operator on l_2 belongs to $\Pi_{p,q}$.*

(C) *If* $\frac{1}{q}-\frac{1}{p}<\frac{1}{2}$ *and* $q\leqslant 2$, *then* $\Pi_{p,q}=\mathfrak{S}_r$, *where* $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}+\frac{1}{2}$.

(D) *If* $2<q<p<\infty$, *then* (i) $\mathfrak{S}_{2p/q}\subseteq\Pi_{p,q}$, *and* (ii) $\Pi_{p,q}\subseteq\mathfrak{S}_p$.

(A) is due to Pełczyński [15]; (B) to Kwapień [10]; (C) and (D) (i) to Mitiagin [10]; and (D) (ii) to Pietsch–Triebel [18]. Special cases were discussed earlier by Grothendieck [8] ((A) when $p=1$); Pietsch [16] ((A) when $1\leqslant p\leqslant 2$); and Orlicz [14] ((B) when $p=2$, $q=1$).

To complete these results, Kwapień [10] has suggested the following

Conjecture. *If* $2<q<p<\infty$, *then* $\Pi_{p,q}\subseteq\mathfrak{S}_{2p/q}$.

Part (I) of our main result shows that this conjecture is false. Indeed, denoting by $\mathfrak{S}_{p,q}$ the ideal of operators generated by the Lorentz sequence space $l_{p,q}$ (defined below in Section 2), we have:

Theorem. (I) *If* $2<q<p<\infty$, *then* $\mathfrak{S}_{2p/q,p}\subseteq\Pi_{p,q}$.

(II) *There is equality in* (I) *if* q *is an even integer.*

Part (II) answers negatively a question raised by Pietsch ([17], Problem 14.5.5), and gives new results for every value of $q>2$. It seems likely that $\Pi_{p,q}$ should coincide with $\mathfrak{S}_{2p/q,p}$ whenever $2<q<p<\infty$, but this I have been unable to prove.

The proof of (I) borrows techniques from interpolation theory and is given in Section 3, following a brief discussion of the Lorentz sequence spaces $l_{p,q}$. The proof of (II) is given in Section 6 and relies on the concept of multiple orthogonality, developed in Section 5. This concept in turn relies on certain number theoretical considerations which are treated in Section 4.

2. The spaces $l_{p,q}$.

In this section we give a brief account of the sequence spaces $l_{p,q}$. These spaces were introduced by Lorentz in [11], special cases having appeared earlier in connection with certain problems in harmonic analysis. We begin with a technical result, which, for convenience, is not stated in the most general form possible.

Lemma 1. *Let* $(\lambda_k)_{k=1}^{\infty}$ *be a decreasing sequence of non-negative real numbers. Then, for* $2<p<\infty$, $2<q<\infty$, *the following conditions are equivalent.*

(i) $\sum_{k=1}^{\infty}k^{q/p-1}(\lambda_k)^q<\infty$.

(ii) $\sum_{t=1}^{\infty}t^{q/p-1-q}\Big(\sum_{k=1}^{t}\lambda_k\Big)^q<\infty$.

(iii) $\sum_{t=1}^{\infty}t^{q/p-1-q/2}\Big(\sum_{k=1}^{t}\lambda_k^2\Big)^{q/2}<\infty$.

(iv) $\sum_{t=1}^{\infty}2^{tq(1/p-1/2)}\Big(\sum_{k=2^t}^{2^{t+1}-1}\lambda_k^2\Big)^{q/2}<\infty$.

Proof. (i)$\Rightarrow$(ii). This is nothing more than a discrete version of Hardy's inequality [4], valid even when (λ_k) is not decreasing, and when $1<p<\infty$, $1<q<\infty$.

(ii)$\Rightarrow$(i). This is valid whenever $0<p<\infty$, $0<q<\infty$, and follows easily from the fact that $t\lambda_t\leqslant\sum_{k=1}^{t}\lambda_k$, $t=1,2,\ldots$, for (λ_k) decreasing.

(i)$\Leftrightarrow$(iii). Apply the equivalence, just established, of (i) and (ii) with (p,q) replaced by $(p/2,q/2)$ and (λ_k) replaced by (λ_k^2).

(i)$\Leftrightarrow$(iv). This follows easily from the monotonicity of (λ_k) by appropriate regrouping of the terms.

The importance of the lemma for us lies in the equivalence of the conditions (ii), (iii) and (iv). We use (ii) in the definition of the spaces $l_{p,q}$; the equivalent forms (iii) and (iv) being used in Sections 3 and 6. Condition (i), though interesting in itself, is essentially used only to establish the equivalence of (ii), (iii), and (iv).

For a sequence $\lambda=(\lambda_k)_{k=1}^{\infty}$ of complex numbers with $\lim_{k\to\infty}\lambda_k=0$, we define the *decreasing rearrangement*, $\hat{\lambda}$, of λ by

$$\hat{\lambda}_n=\min_{|K|<n}\max_{k\notin K}|\lambda_k|\qquad(n=1,2,\ldots),$$

where $|K|$ denotes the cardinality of the set $K\subseteq\{1,2,\ldots\}$. The *Lorentz sequence space* $l_{p,q}$, $1<p<\infty$, $1<q<\infty$, is defined to be the set of all λ for which

$$\|\lambda\|_{p;q}^q=\sum_{t=1}^{\infty}t^{q/p-1-q}\Big(\sum_{k=1}^{t}\hat{\lambda}_k\Big)^q<\infty.$$

The mapping $\lambda\to\|\lambda\|_{p;q}$ is a norm, under which $l_{p,q}$ becomes a (reflexive) Banach space.

The following well-known inclusion relationships will be of interest in subsequent sections.

Proposition 1. *Let* $1<p,q,r<\infty$. *Then*

(i) $l_{p,p}=l_p$;

(ii) $l_{p,q}\subset l_{p,r}$ *if* $q<r$, *and the inclusion is proper*;

(iii) $l_{p,q}\subset l_{r,q}$ *if* $p<r$, *and the inclusion is proper.*

Proof. The first part follows trivially from the definition and Lemma 1, (i)$\Leftrightarrow$(ii). To establish the inclusions given in (ii) and (iii) use the fact, easily proved, that $\hat{\lambda}_k=O(k^{-1/p})$ whenever $\lambda\in l_{p,q}$. To see that these inclusions are proper consider sequences of the form $\lambda_k=k^{\alpha}(\log k)^{\beta}$ for suitably chosen α and β.

3. Extension of Mitiagin's result. For $1 < p < \infty$, $1 < q < \infty$, $l_{p,q}$ is a solid, symmetric sequence space in the sense of [6], and $l_{p,q} \subset c_0$, so we may consider the ideal, say $\mathfrak{S}_{p,q}$, of operators generated by $l_{p,q}$. For $2 < q < p < \infty$, Proposition 1 shows that $\mathfrak{S}_{2p/q,p}$ properly contains $\mathfrak{S}_{2p/q}$, so the following result is a genuine improvement on the one obtained by Mitiagin ((D) (i)).

THEOREM (I). *For $2 < q < p < \infty$ we have $\mathfrak{S}_{2p/q,p} \subseteq \Pi_{p,q}$.*

Proof. Since $l_{2p/q,p} \subset c_0$, it follows that every member of $\mathfrak{S}_{2p/q,p}$ is compact. The argument (spectral- and polar-representation theorems) of Mitiagin ([10], p. 335) therefore applies, and it suffices to prove the theorem for diagonal operators. Thus, according to (1), we must establish the following inequality.

If $\lambda \in l_{2p/q,p}$ and $x^{(1)}, \ldots, x^{(n)} \in l_2$, then

$$\Big(\sum_{j=1}^{n}\Big(\sum_{k=1}^{\infty}|\lambda_k x_k^{(j)}|^2\Big)^{p/2}\Big)^{1/p} \leqslant M \sup_{\|f\|_2\leqslant 1}\Big(\sum_{j=1}^{n}\Big|\sum_{k=1}^{\infty} x_k^{(j)} f_k\Big|^q\Big)^{1/q},$$

where $M = M(\lambda, p, q)$ is a constant depending only on λ, p and q.

We notice (by rearranging the coordinates of $x^{(1)}, \ldots, x^{(n)}$) that the last statement is independent of the order of the terms λ_k, and so, without any loss of generality, we may assume that $(|\lambda_k|)_{k=1}^{\infty}$ is a decreasing sequence.

Suppose then that $x^{(1)}, \ldots, x^{(n)} \in l_2$ are fixed and that

$$\sup_{\|f\|_2\leqslant 1}\Big(\sum_{j=1}^{n}\Big|\sum_{k=1}^{\infty} x_k^{(j)} f_k\Big|^q\Big)^{1/q} = N. \tag{3}$$

We must show that

$$\Big(\sum_{j=1}^{n}\Big(\sum_{k=1}^{\infty}|\lambda_k x_k^{(j)}|^2\Big)^{p/2}\Big)^{1/p} \leqslant M(\lambda, p, q)\, N. \tag{4}$$

To do this, choose and fix a positive integer t. For $j = 1, 2, \ldots, n$, let

$$\mu_j = \sum_{k=1}^{\infty}|\lambda_k x_k^{(j)}|^2 = \sum_{k=1}^{t}|\lambda_k x_k^{(j)}|^2 + \sum_{k=t+1}^{\infty}|\lambda_k x_k^{(j)}|^2 = v_j + w_j \ \text{(say)}.$$

Now

$$\begin{aligned} w_j &\leqslant \sup_{k>t}|\lambda_k|^2 \sum_{k=t+1}^{\infty}|x_k^{(j)}|^2 \\ &\leqslant |\lambda_{t+1}|^2 \max_{1\leqslant j\leqslant n}\sum_{k=1}^{\infty}|x_k^{(j)}|^2 \quad \text{since } |\lambda_k| \text{ is decreasing} \\ &= |\lambda_{t+1}|^2 \max_{1\leqslant j\leqslant n}\sup_{\|f\|_2\leqslant 1}\Big|\sum_{k=1}^{\infty}x_k^{(j)}f_k\Big|^2 \quad \text{by Landau's theorem [9]} \\ &\leqslant |\lambda_{t+1}|^2 N^2 \quad \text{by Jensen's inequality and (3),} \end{aligned}$$

so that

$$\max_{1\leqslant j\leqslant n} w_j \leqslant \frac{N^2}{t}\sum_{k=1}^{t}|\lambda_k|^2. \tag{5}$$

On the other hand,

$$\sum_{j=1}^{n} v_j^{q/2} = \sum_{j=1}^{n}\Big(\sum_{k=1}^{t}|\lambda_k x_k^{(j)}|^2\Big)^{q/2} \leqslant \Big(\sum_{k=1}^{t}\Big(\sum_{j=1}^{n}|\lambda_k x_k^{(j)}|^q\Big)^{2/q}\Big)^{q/2}$$

by Minkowski's inequality in the space $l_{q/2}$

$$\leqslant \Big(\sum_{k=1}^{t}|\lambda_k|^2\Big)^{q/2}\max_{1\leqslant k\leqslant t}\sum_{j=1}^{n}|x_k^{(j)}|^q = \Big(\sum_{k=1}^{t}|\lambda_k|^2\Big)^{q/2}\sup_{\|f\|_1\leqslant 1}\sum_{j=1}^{n}\Big|\sum_{k=1}^{\infty}x_k^{(j)}f_k\Big|^q$$

by three applications of Landau's theorem

$$\leqslant \Big(\sum_{k=1}^{t}|\lambda_k|^2\Big)^{q/2} N^q$$

by Jensen's inequality and (3), so that

$$\hat{v}_j \leqslant \frac{N^2}{j^{2/q}}\sum_{k=1}^{t}|\lambda_k|^2. \tag{6}$$

(5) and (6) show that

$$\hat{v}_{[t^{q/2}]} + \hat{w}_{[t^{q/2}]} \leqslant \frac{2N^2}{t}\sum_{k=1}^{t}|\lambda_k|^2,$$

where $[s]$ denotes the least positive integer greater than or equal to s. Now at most $2[t^{q/2}]-2$ values of $\mu_j = v_j + w_j$ can exceed $\hat{v}_{[t^{q/2}]-1} + \hat{w}_{[t^{q/2}]}$, so that

$$\hat{\mu}_{2[t^{q/2}]-1} \leqslant \frac{2N^2}{t}\sum_{k=1}^{t}|\lambda_k|^2. \tag{7}$$

(7) holds for $t = 1, 2, \ldots$, so we have

$$\begin{aligned} \sum_{t=1}^{n}\mu_t^{p/2} &= \sum_{t=1}^{n}\hat{\mu}_t^{p/2} = \sum_{t}\ \sum_{s=2[t^{q/2}]-1}^{2[(t+1)^{q/2}]-2}\hat{\mu}_s^{p/2} = C(q)\sum_{t} t^{q/2-1}\hat{\mu}^{p/2}_{2[t^{q/2}]-1} \\ &\leqslant 2N^p C(q)\sum_{t=1}^{\infty} t^{q/2-1-p/2}\Big(\sum_{k=1}^{t}|\lambda_k|^2\Big)^{p/2} \quad \text{by (7)} \\ &\leqslant N^p C(p, q)\,\|\lambda\|^p_{2p/q;p} \quad \text{by Lemma 1, (ii)}\Leftrightarrow\text{(iii)}. \end{aligned}$$

Thus (4) holds with $M(\lambda, p, q) = C(p, q)^{1/p}\|\lambda\|_{2p/q;p}$.

The observant reader will perhaps have noticed that the above proof could be shortened by using a generalized form of the Marcinkiewicz interpolation theorem ([4], p. 189). However, we have not required anything like the full force of this deep theorem, and for this reason have preferred the self-contained treatment given above. The idea of looking at decreasing rearrangements in this general context goes back, of course, to the original paper of Marcinkiewicz [12].

4. A combinatorial lemma. A (finite or infinite) sequence $(a_k)_k$ of positive integers is called an *r-sequence*, $r = 1, 2, \ldots$, provided that

(8) $$a_{j_1}+\ldots+a_{j_r} = a_{k_1}+\ldots+a_{k_r}$$

only when $\{j_1, \ldots, j_r\} = \{k_2, \ldots, k_r\}$, i.e., r-fold sums of terms from $(a_k)_k$ are distinct.

We turn now to a powerful combinatorial lemma of Bose and Chowla [3]. Their result itself is striking, its (short) proof even more so. For this reason, and for the sake of completeness, we provide the details.

LEMMA 2. *Let r be a fixed positive integer and let n be any prime number. Then there exists an r-sequence $(a_k)_{k=1}^n$ with*

(9) $$1 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant n^r.$$

Proof. Let $\beta_1, \ldots, \beta_n$ be a listing of the elements of the Galois field $GF(n)$. The non-zero elements of the extended field, $GF(n^r)$, form a cyclic group under multiplication ([5], p. 248). Letting γ denote a generating element for this group, we may choose n positive integers $1 \leqslant a_1, \ldots, a_n < n^r$ such that

$$\gamma^{a_k} = \gamma+\beta_k \quad (k = 1, 2, \ldots, n).$$

$(a_k)_{k=1}^n$ is then an r-sequence, which satisfies (9) after relabelling. To see this, suppose that

$$a_{j_1}+\ldots+a_{j_r} \equiv a_{k_1}+\ldots+a_{k_r} \bmod (n^r-1).$$

Then

$$(\gamma+\beta_{j_1})\ldots(\gamma+\beta_{j_r}) = (\gamma+\beta_{k_1})\ldots(\gamma+\beta_{k_r}).$$

After cancelling the highest power of γ from both sides we are left with an equation of degree $r-1$ in γ, with coefficients from $GF(n)$. This contradicts the fact that γ is a generator, and the lemma is established.

We remark that the above proof works whenever n is a prime power, and even shows that equality in (8) may be replaced by congruence modulo n^r-1. We shall use these facts in the proofs of Proposition 2 and Theorem (II).

Given a sequence $(a_k)_k$ of positive integers, let us denote by $c_n(a)$, $n = 1, 2, \ldots$, the number of terms of $(a_k)_k$ that do not exceed n. Then we have the following

COROLLARY.

$$\liminf_{n\to\infty} \max_a \frac{c_n(a)}{n^{1/r}} \geqslant 1 \quad (r = 1, 2, \ldots),$$

the maximum being taken over all r-sequences a.

Proof. Lemma 2 shows that

(10) $$\max_a c_{p^r}(a) \geqslant p$$

for every prime p. If n is an arbitrary positive integer ($\geqslant 2^r$), we may choose consecutive primes p and q so that $p^r \leqslant n < q^r$. Then, by (10), we have

$$\max_a \frac{c_n(a)}{n^{1/r}} \geqslant \max_a \frac{c_{p^r}(a)}{q} \geqslant \frac{p}{q}.$$

Since the quotient of consecutive primes tends to one, we obtain the desired result.

Though we shall not need the following observation, it is interesting to note that the corollary is best possible in a certain sense. To see this, suppose that a is an arbitrary r-sequence. Then it is easy to check that all combinations of the form

$$a_{k_1}-a_{k_2}+\ldots+(-1)^{r+1}a_{k_r},$$

with $1 \leqslant k_r < k_{r-1} < \ldots < k_1 \leqslant c_n(a)$, must be distinct. Moreover, there are $\binom{c_n(a)}{r}$ such combinations, and each takes its values from $\{1, 2, \ldots, n-1\}$. Therefore $\binom{c_n(a)}{r} \leqslant n-1$, and we have $c_n(a) < (r!\,n)^{1/r}+r-1$.

5. Multiply orthogonal matrices. An $m\times n$ matrix $A = (a_{jk})$ with complex coefficients is called *r-orthogonal*, $r = 1, 2, \ldots$, if, whenever $1 \leqslant j_1, \ldots, j_r, k_1, \ldots, k_r \leqslant n$, we have

(11) $$\sum_{j=1}^{m}\prod_{h=1}^{r} a_{j,j_h}\bar{a}_{j,k_h} = \begin{cases} 1 & \text{for } \{j_1, \ldots, j_r\} = \{k_1, \ldots, k_r\}, \\ 0 & \text{otherwise.} \end{cases}$$

(Thus a 1-orthogonal matrix is precisely one whose columns are pairwise orthogonal in the usual sense.) If, in addition to (11), we have

(12) $$|a_{jk}| = m^{-1/2r} \quad (j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n),$$

then A is called *r-orthonormal*. Given positive integers r and n it is clear (by choosing $m = m(r, n)$ sufficiently large) that we can find an r-orthonormal matrix of order $m\times n$. Denoting by $m_r(n)$ the smallest value of m for which such a matrix exists, we have

LEMMA 3.

$$\limsup_{n\to\infty}\frac{m_r(n)}{n^r}\leqslant r \qquad (r=1,2,\ldots).$$

Proof. Let r and $\varepsilon>0$ be fixed. By the corollary to Lemma 2, we may choose a positive integer $N_0\,(\geqslant 1/\varepsilon)$ so that

$$\max c_N(a)\geqslant N^{1/r}(1-\varepsilon) \tag{13}$$

whenever $N\geqslant N_0$. Let n be any positive integer $\geqslant N_0^{1/r}(1-\varepsilon)$ and choose $N\geqslant N_0$ so that

$$n\leqslant N^{1/r}(1-\varepsilon)\leqslant n(1+\varepsilon). \tag{14}$$

This last step is possible since

$$(1-\varepsilon)\big((N+1)^{1/r}-N^{1/r}\big)\leqslant n\varepsilon \quad \text{whenever } N\geqslant N_0.$$

Using (13) and (14) let $(a_k)_{k=1}^n$ be an r-sequence with

$$a_1<\ldots<a_n\leqslant N, \tag{15}$$

and define the matrix $A=(a_{jk})$ by

$$a_{jk}=(rN)^{-1/2r}\exp\left(\frac{2\pi i j a_k}{rN}\right) \quad (j=1,2,\ldots,rN;\ k=1,2,\ldots,n). \tag{16}$$

Then we have

$$\sum_{j=1}^{rN}\prod_{h=1}^{r}a_{j,j_h}\bar a_{j,k_h}=(rN)^{-1}\sum_{j=1}^{rN}\exp\frac{2\pi i}{rN}(a_{j_1}+\ldots+a_{j_r}-a_{k_1}-\ldots-a_{k_r})$$
$$=\begin{cases}1 & \text{if}\quad a_{j_1}+\ldots+a_{j_r}\equiv a_{k_1}+\ldots+a_{k_r}\,(\mathrm{mod}\, rN)\\ 0 & \text{otherwise}\end{cases}$$
$$=\begin{cases}1 & \text{if}\quad a_{j_1}+\ldots+a_{j_r}= a_{k_1}+\ldots+a_{k_r}\\ 0 & \text{otherwise (by (15))}\end{cases}$$
$$=\begin{cases}1 & \text{if}\quad \{j_1,\ldots,j_r\}=\{k_1,\ldots,k_r\}\\ 0 & \text{otherwise (since } a \text{ is an } r\text{-sequence).}\end{cases}$$

This, together with (16), shows that A is an r-orthonormal matrix, and it follows from (14) that

$$m_r(n)\leqslant rN\leqslant r\left(\frac{n(1+\varepsilon)}{1-\varepsilon}\right)^r.$$

Therefore $\limsup_{n\to\infty}\frac{m_r(n)}{n^r}\leqslant r$, as desired.

The importance of r-orthogonal matrices lies in the fact that their norms as operators from l_2 to l_{2r} can easily be estimated. To see this, let $A=(a_{jk})$ be such a matrix of order $m\times n$. Then, for any complex sequence $(x_k)_{k=1}^n$, we have

$$\sum_{j=1}^m\Big|\sum_{k=1}^n a_{jk}x_k\Big|^{2r}=\sum_{j=1}^m\prod_{h=1}^r\Big(\sum_{j_h=1}^n a_{j,j_h}x_{j_h}\sum_{k_h=1}^n\bar a_{j,k_h}\bar x_{k_h}\Big)$$
$$=\sum_{h=1}^r\sum_{j_h=1}^n\sum_{k_h=1}^n\Big(\prod_{l=1}^r x_{j_l}\bar x_{k_l}\Big)\Big(\sum_{j=1}^m\prod_{h=1}^r a_{j,j_h}\bar a_{j,k_h}\Big)$$
$$=\sum_{\{j_1,\ldots,j_r\}=\{k_1,\ldots,k_r\}}\prod_{l=1}^r x_{j_l}\bar x_{k_l}=r!\Big(\sum_{k=1}^n|x_k|^2\Big)^r.$$

It follows that $\|A\|_{2,2r}\leqslant(r!)^{1/2r}$.

It is convenient to summarize our results as follows.

PROPOSITION 2. *Let q be a fixed even integer. Then, for any positive integer n, it is possible to construct an $m\times n$ matrix with the following properties.*

(i) $\|A\|_{2,q}\leqslant\big((q/2)!\big)^{1/q}$,

(ii) $|a_{jk}|=m^{-1/q}$ $(j=1,\ldots,m;\ k=1,\ldots,n)$,

(iii) $m=\frac{q}{2}n^{q/2}+o(n)$.

Moreover, if n is a prime or a prime power, then (iii) *may be replaced by*

(iv) $m\leqslant n^{q/2}$.

Proof. For (i), (ii) and (iii), we take $q=2r$ and construct an r-orthonormal matrix as above. For (iv) we modify the above argument, replacing (16) by

$$a_{jk}=(n^r-1)^{-1/2r}\exp\left(\frac{2\pi i j a_k}{n^r-1}\right)\quad (j=1,2,\ldots,n^r-1;\ k=1,2,\ldots,n)$$

and noting, in Lemma 2, that equality in (8) may be replaced by congruence $\mathrm{mod}\,(n^r-1)$.

In the next section we shall be interested in constructing an infinite matrix from a given sequence of finite ones. For this purpose it will be convenient to first consider so-called "block diagonal" matrices. A matrix $A=(a_{jk})_{j,k=1}^\infty$ is called a *block diagonal matrix* if there exist sequences $(m_t)_{t=1}^\infty$ and $(n_t)_{t=1}^\infty$ of positive integers with $1=m_1<m_2<\ldots$, $1=n_1<n_2\ldots$, such that $a_{jk}=0$ if $(j,k)\notin[m_t,m_{t+1})\times[n_t,n_{t+1})$ for any t. Putting $J_t=[m_t,m_{t+1})$ and $K_t=[n_t,n_{t+1})$ for $t=1,2,\ldots$, the *blocks* $A^{(t)}$ are defined by

$$a_{jk}^{(t)}=\begin{cases}a_{jk} & \text{if}\quad (j,k)\in J_t\times K_t,\\ 0 & \text{otherwise.}\end{cases}$$

We then have $A = \sum_{t=1}^{\infty} A^{(t)}$, where the summation is performed coordinatewise.

Conversely, if a sequence of finite matrices, $(A^{(t)})_{t=1}^{\infty}$, is given, we can construct the matrix $A = \sum_{t=1}^{\infty} A^{(t)}$ as above. A will then be called the *block diagonal matrix associated with* $(A^{(t)})_{t=1}^{\infty}$. The norm of A is computed as follows.

PROPOSITION 3. *Let $(A^{(t)})_{t=1}^{\infty}$ be a given sequence of finite matrices, and let $\|\cdot\|$ denote the operator norm from l_p to l_q, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then we have*

$$\Big\|\sum_{t=1}^{\infty} A^{(t)}\Big\| = \Big(\sum_{t=1}^{\infty} \|A^{(t)}\|^r\Big)^{1/r},$$

where

$$r = \begin{cases} \dfrac{pq}{p-q} & \text{if} \quad p > q, \\ \infty & \text{if} \quad p \leqslant q. \end{cases}$$

Proof. Given a sequence $(x_k)_{k=1}^{\infty}$ of complex numbers and a subset I of the positive integers, we denote by $x(I)$ the *restriction of x to I* viz:

$$x_k(I) = \begin{cases} x_k & \text{if} \quad k \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\begin{aligned}
&\Big\|\sum_{t=1}^{\infty} A^{(t)}\Big\| \\
&= \sup\Big\{\Big|y\Big(\sum_{t=1}^{\infty} A^{(t)}\Big)x\Big| : \|x\|_p \leqslant 1, \|y\|_{q^*} \leqslant 1\Big\} \\
&= \sup\Big\{\sum_{t=1}^{\infty} |y(J_t) A^{(t)} x(K_t)| : \sum_{t=1}^{\infty} \|x(K_t)\|_p^p \leqslant 1, \ \sum_{t=1}^{\infty} \|y(J_t)\|_{q^*}^{q^*} \leqslant 1\Big\} \\
&= \sup\Big\{\sum_{t=1}^{\infty} \|y(J_t)\|_{q^*} \|A^{(t)}\| \|x(K_t)\|_p : \sum_{t=1}^{\infty} \|x(K_t)\|_p^p \leqslant 1, \ \sum_{t=1}^{\infty} \|y(J_t)\|_{q^*}^{q^*} \leqslant 1\Big\} \\
&= \Big(\sum_{t=1}^{\infty} \|A^{(t)}\|^r\Big)^{1/r},
\end{aligned}$$

as desired.

In this paper we shall only need Proposition 3 with $2 = p < q < \infty$, in which case $\|A\| = \sup_t \|A^{(t)}\|$. We have stated and proved the general case since other values of p and q are considered in [2].

6. Extension of the Pietsch–Triebel result. In this section we establish the converse inclusion, $\Pi_{p,q} \subseteq \mathfrak{S}_{2p/q,p}$, to that given in Section 3, at least when q is an even integer. It is easily seen from Proposition 1 that this is a substantial improvement on the Pietsch–Triebel result (D)(ii) — even for non-integral values of q. The proof relies heavily on the results of Sections 4 and 5.

THEOREM (II). *If q is an even integer, and $q < p < \infty$, then $\Pi_{p,q} \subseteq \mathfrak{S}_{2p/q,p}$.*

Proof. It follows from Theorem 1 (iii) of [1] that the identity operator on l_2 is not (p, q)-absolutely summing, and so, by Calkin's theorem [19], (the two-sided ideal) $\Pi_{p,q}$ consists entirely of compact operators. Thus the argument of Mitiagin again applies, and it suffices to prove the theorem for diagonal operators. According to (2), we must establish the following inequality.

If λ is a sequence of complex numbers with the property that

$$\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty} |\lambda_k a_{jk}|^2\Big)^{p/2} < \infty \tag{17}$$

whenever $A = (a_{jk})_{j,k=1}^{\infty}$ is a matrix satisfying

$$\sum_{j=1}^{\infty}\Big|\sum_{k=1}^{\infty} a_{jk} x_k\Big|^q < \infty \quad \textit{for each } x \in l_2, \tag{18}$$

hen $\lambda \in l_{2p/q,p}$.

Before proceeding with the proof, we notice (by changing the order of the columns of A) that the last statement is independent of the order of the terms λ_k. Moreover, by taking $a_{jk} = \delta_{jk}$, $j, k = 1, 2, \ldots$, it follows at once that $\lambda \in l_p$ (Pietsch–Triebel result), so that $\hat{\lambda}$, the decreasing rearrangement of λ, must exist. Thus, without any loss in generality, we shall assume that $(|\lambda_k|)_{k=1}^{\infty}$ is a decreasing sequence.

We construct a single matrix A satisfying (18), which, via (17), forces λ to belong to $l_{2p/q,p}$. A will be a block diagonal matrix, $A = \sum_{t=1}^{\infty} A^{(t)}$, where each $A^{(t)}$ is r $(= q/2)$-orthonormal. It transpires that the quantity (17) depends critically on the location of these blocks: if they stray too far from the "curve" $j = k^r$, the estimates given below deteriorate considerably. For this reason, we choose $A^{(t)}$ of order $2^{rt} \times 2^t$, $t = 1, 2, \ldots$, which choice is possible by Proposition 2 (iv). The block $A^{(t)}$ then occupies the rows J_t and the columns K_t, where

$$J_1 = K_1 = \{1\}, \qquad J_t = \Big[\frac{2^{r(t-1)} + 2^{rt}}{2^r - 1}, \ \frac{2^{rt} - 1}{2^r - 1}\Big].$$

and

$$K_t = [2^{t-2}, 2^{t-1}), \quad t = 2, 3, \ldots$$

Since each $A^{(t)}$ is r-orthonormal, it follows from Propositions 2 and 3 that A satisfies (18). In fact, we have

$$\sup_{\|x\|_2 \leqslant 1} \sum_{j=1}^{\infty} \Big|\sum_{k=1}^{\infty} a_{jk} x_k\Big|^q = \sup_t \|A^{(t)}\|_{2,q}^q \leqslant r!.$$

Consequently, by (17),

$$\infty > \sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty} |\lambda_k a_{jk}|^2\Big)^{p/2} = \sum_{t=1}^{\infty}\sum_{j\in J_t}\Big(\sum_{k\in K_t} |\lambda_k a_{jk}^{(t)}|^2\Big)^{p/2}$$
$$= \sum_{t=1}^{\infty} 2^{rt}(2^{rt})^{-p/2r}\Big(\sum_{k\in K_t}|\lambda_k|^2\Big)^{p/2} = \sum_{t=1}^{\infty} 2^{tp(q/2p-1/2)}\Big(\sum_{k\in K_t}|\lambda_k|^2\Big)^{p/2},$$

so that $\lambda \in l_{2p/q,p}$ by Lemma 1, (ii) $\Leftrightarrow$ (iv).

7. Closing remarks. (i) Throughout this paper we have considered only complex Hilbert space. The extension to the real case follows easily by considering real and imaginary components of the appropriate inequalities.

(ii) Theorem (II) can be used to obtain new information concerning $\Pi_{p,q}$ for *every* $q > 2$ — provided that p is suitably restricted. To see this, let s denote the smallest even integer exceeding q (i.e. $s \geqslant q$). If p satisfies $1/p \geqslant 1/q - 1/s$, and T is (p, q)-absolutely summing, then T is also (r, s)-absolutely summing, where $r = \dfrac{pqs}{pq - ps + qs}$ (see [10], 0.7). It follows from Theorem (II) that $T \in \mathfrak{S}_{2r/s,r}$. The foregoing remark, however, is probably redundant. Indeed, it should be possible to interpolate between even integral values and so extend Theorem (II) to arbitrary $q > 2$. This I have been unable to do.

(iii) The space $\Pi_{p,q}$ becomes a Banach space when topologized by means of the norm

$$\Pi_{p,q}(T) = \inf\{M:\ M \text{ satisfies } (1)\}.$$

In a recent paper [7] Garling has given a proof of (A) in which the norms $\Pi_{p,p}$ are calculated. The corresponding problem for $\Pi_{p,q}$ appears to be difficult.

(iv) Professor A. Pełczyński has kindly brought to my attention the paper *Trigonometric series with gaps*, J. Math. Mech. 9 (1960), pp. 203–227, by W. Rudin. Rudin uses combinatorial results of Erdös, Turán and Stöhr to construct certain trigonometric polynomials with small L_q-norms ($q = 2, 4, 6, \ldots$). The central result (Proposition 2) of the second half of our paper follows easily from Rudin's estimates. (Note, however, the Bose–Chowla result (Lemma 2) gives slightly sharper estimates for $\Pi_{p,q}(T)$ (cf. (iii)) than do the results of Erdös, Turán and Stöhr.)

(v) The techniques of this paper, and of [1], can be used to give many new results concerning diagonal operators between l_p spaces. We remark, however, that we are still far from solving the following general problem. Given real numbers p, q, r, s satisfying $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $1 \leqslant s \leqslant r \leqslant \infty$, what are necessary and sufficient conditions on λ so that $\lambda\colon l_p \to l_q$ be (r, s)-absolutely summing?

Added in proof. An affirmative solution to the conjecture of 7 (ii) is given in a paper *Norms of random matrices* (to appear in Pacific J. Math.), written jointly with V. Goodman and C. M. Newman. Using probabilistic techniques, an analogue of Proposition 2 is given for every $q > 2$. The arguments of Section 6 then show that $\Pi_{p,q} = \sigma_{2p/q,p}$ whenever $2 < q < p < \infty$.

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(L_p, L_q) mapping properties of convolution transforms

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Abstract. Let k and f be two Lebesgue measurable functions on $\boldsymbol{R}^n$. Then the equation

$$k*f(x) = \int_{\boldsymbol{R}^n} k(x-t)f(t)\,dt$$

defines the convolution transform of k and f. Let $T(f) = k*f$. In this paper we give necessary as well as sufficient conditions for T to map $L_p \to L_q$ continuously. We show that our results are sharp in the sense that we exhibit a class of functions k such that the mapping interval we obtain is maximal except for endpoints. For example, for $k(t) = e^{i|t|^a}/|t|^b$ we give the exact mapping properties. We also give the exact mapping properties for a class of kernels in $\boldsymbol{R}^n$.

Introduction. Let k and f be two Lebesgue measurable functions on $\boldsymbol{R}^n$. Then the equation

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The most basic result in this direction is Young's inequality [4]. It states

$$(*) \qquad \|T(f)\|_q = \|k*f\|_q \leqslant \|k\|_{1/\lambda}\|f\|_p,$$

where $1/p - 1/q = 1-\lambda$, $0 \leqslant \lambda \leqslant 1$.

Hardy and Littlewood [1] extended this theorem to include the functions $k(x) = 1/|x|^\lambda$, $0 < \lambda < 1$, as well as the Hilbert transform. Riesz, Thorin and then Marcinkiewicz [5] proved a general mapping theorem that not only included all the previous cases but also gave other proofs that the Hilbert transform maps $L_p(\boldsymbol{R}) \to L_p(\boldsymbol{R})$ for $1 < p < \infty$.

Hörmander [3] has weakened the condition $k \in L^{1/\lambda}$ (see $(*)$) by giv-