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DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY BLOOMINGTON, INDIANA

Received June 25, 1974 (850)

STUDIA MATHEMATICA, T. LV. (1976)

(L_p, L_q) mapping properties of convolution transforms

by

G. SAMPSON, A. NAPARSTEK and V. DROBOT (Buffalo, N. Y.)

Abstract. Let k and f be two Lebesgue measurable functions on $\boldsymbol{R}^n$. Then the equation

$$k*f(x) = \int_{\boldsymbol{R}^n} k(x-t)f(t)\,dt$$

defines the convolution transform of k and f. Let $T(f) = k*f$. In this paper we give necessary as well as sufficient conditions for T to map $L_p \to L_q$ continuously. We show that our results are sharp in the sense that we exhibit a class of functions k such that the mapping interval we obtain is maximal except for endpoints. For example, for $k(t) = e^{i|t|^a}/|t|^b$ we give the exact mapping properties. We also give the exact mapping properties for a class of kernels in $\boldsymbol{R}^n$.

Introduction. Let k and f be two Lebesgue measurable functions on $\boldsymbol{R}^n$. Then the equation

$$k*f(x) = \int_{\boldsymbol{R}^n} k(x-t)f(t)\,dt$$

defines the convolution transform of k and f. Let $T(f) = k*f$. In this paper we give necessary as well as sufficient conditions for T to map $L_p \to L_q$ continuously. We show that our results are sharp in the sense that we exhibit a class of functions k such that the mapping interval we obtain is maximal except for endpoints. For example, for $k(t) = e^{i|t|^a}/|t|^b$ we give the exact mapping properties (see Cors. 3.22 and 4.29).

The most basic result in this direction is Young's inequality [4]. It states

$$(*) \qquad \|T(f)\|_q = \|k*f\|_q \leqslant \|k\|_{1/\lambda}\|f\|_p,$$

where $1/p - 1/q = 1-\lambda$, $0 \leqslant \lambda \leqslant 1$.

Hardy and Littlewood [1] extended this theorem to include the functions $k(x) = 1/|x|^\lambda$, $0 < \lambda < 1$, as well as the Hilbert transform. Riesz, Thorin and then Marcinkiewicz [5] proved a general mapping theorem that not only included all the previous cases but also gave other proofs that the Hilbert transform maps $L_p(\boldsymbol{R}) \to L_p(\boldsymbol{R})$ for $1 < p < \infty$.

Hörmander [3] has weakened the condition $k \in L^{1/\lambda}$ (see $(*)$) by giving a strictly larger class K^λ (see Def. 1.3) which includes the Hardy–Littlewood kernels and the Hilbert transform. His theorems apply also in $\boldsymbol{R}^n$. For example, if $k \in K^\lambda$ and $T: L_{p_0} \to L_{q_0}$ for some p_0 and q_0, then T maps L_p into L_q for all p and q such that

$$1/p - 1/q = 1/p_0 - 1/q_0 = 1-\lambda \quad \text{with} \quad 1 < p \leqslant q < \infty.$$

For the case $L_p(\boldsymbol{R}^n) \to L_p(\boldsymbol{R}^n)$, Hirschman [2] has given conditions on k and $\hat{k}$ which are sufficient for T to map $L_p \to L_p$ over finite intervals of p, i.e., $1 < p_0 < p < p_0' < \infty$. He shows by an example that this p interval cannot be extended.

We extend Hirschman's result; see Corollaries 1.13 and 1.14. Further, we obtain theorems for the $L_p \to L_q$ case when $1/p - 1/q = 1-\lambda$; the methods of proof depend on λ. That is, the method used for $1/2 < \lambda \leqslant 1$ differs from the one for $0 \leqslant \lambda \leqslant 1/2$.

We also give, in a systematic way, necessary conditions on k in order that T maps.

1. Sufficient conditions on k such that $T(f) = k*f$ maps $L_p(\boldsymbol{R}^n) \to L_p(\boldsymbol{R}^n)$ continuously. All the functions $f, g, \ldots, k$ that appear in this paper are locally integrable on $\boldsymbol{R}^n$. Unless otherwise specified, we will use f to denote a function with compact support, whose Fourier transform $\hat{f} \in L(\boldsymbol{R}^n)$.

We begin with convolutions defined through limiting processes, such as the Hilbert transform. A suitable way to define the Hilbert transform $H(f) = \dfrac{1}{t} * f$ in $\boldsymbol{R}$ is to approximate $1/t$ by locally integrable functions $g_n(t)$ that are zero in a symmetric neighborhood of both the origin and infinity and coincide with $1/t$ everywhere else. Then we define

$$\frac{1}{t} * f = \lim g_n * f.$$

To show it maps $L_p \to L_p$ for $1 < p < \infty$ it is essential to get good estimates of $|\{x: |H(f)| > y\}|$ for each $y > 0$. We will start by looking at all transforms that are defined in this manner.

DEFINITION 1.1. Given a sequence of bounded functions $\{g_n\}$ with compact support, then if $g_n * f$ converges in measure for each f, we define

$$H(f) = \lim g_n * f,$$

where here the convergence is in measure.

DEFINITION 1.2. For $0 < \lambda \leqslant 1$ we define the weak $1/\lambda$ norm of a function g by

$$\|g\|_{1/\lambda}^* = \sup_{y>0} y\,|\{x: |g(x)| > y\}|^\lambda.$$

DEFINITION 1.3 (Hörmander [3]). We say $g \in K^\lambda(M, N)$ if there exists a compact set $M \subset \boldsymbol{R}^n$ and a neighborhood of the origin $N(0) \subset \boldsymbol{R}^n$ such that

$$A(g;\lambda) = \sup_{\substack{t>0 \\ y \in N(0)}} \left\{ \int_{C(M)} \left| t^{-n\lambda} \left[g\left(\frac{x-y}{t}\right) - g\left(\frac{x}{t}\right)\right]\right|^{1/\lambda} dx \right\}^\lambda < \infty.$$

A class of functions on $\boldsymbol{R}$ which are in $K = K^1(M, N)$ is defined as follows: Let g be such that $|g'(t)| \leqslant 1/t^2$, and for each n $g(t)$ is absolutely continuous for $1/n \leqslant |t| \leqslant n$.

DEFINITION 1.4. We will say $g \in L_p^q(\boldsymbol{R}^n, C)$ if

$$C = \sup_{f \in L_p} \frac{\|g*f\|_q}{\|f\|_p} < \infty.$$

It will sometimes be more convenient to write $g \in L_p^q(\boldsymbol{R}^n)$ without showing the constant C.

THEOREM 1.5 (Hörmander [3]). *If*

(i) $g_m \in K^\lambda(I_m^*, I_m)$, $m = 1, 2, \ldots,$ *and*

(ii) $g_m \in L_{p_0}^{q_0}(\boldsymbol{R}^n, C_m)$, $C_m \leqslant C$, $m = 1, 2, \ldots$, $1/p_0 - 1/q_0 = 1-\lambda$,

then

$$\|g_m * f\|_{1/\lambda}^* \leqslant 2\,|||g_m|||_{1/\lambda} \|f\|_1,$$

and

$$\|H(f)\|_{1/\lambda}^* \leqslant 2 \sup |||g_m|||_{1/\lambda} \|f\|_1,$$

for $m = 1, 2, \ldots,$ *where*

$$|||g_m|||_{1/\lambda} = 2^{n\lambda(q_0 - q_0/p_0)} C_m^{q_0\lambda} + (|I_m^*|/|I_m|)^\lambda + 2A(g_m;\lambda).$$

The proof is adapted from Hörmander ([3], Theorem 2.1). Condition (ii) of Theorem 1.5 implies $\|H(f)\|_{q_0} \leqslant C\|f\|_{p_0}$, and combining this with Marcinkiewicz's interpolation theorem [5], we obtain the following:

COROLLARY 1.6 (Hörmander). *If*

(i) $g_m \in K^\lambda(I_m^*, I_m)$, $m = 1, 2, \ldots,$ *and*

(ii) $g_m \in L_{p_0}^{q_0}(\boldsymbol{R}^n, C_m)$, $C_m \leqslant C$, $m = 1, 2, \ldots$, $1/p_0 - 1/q_0 = 1-\lambda$,

then $H \in L_p^q(\boldsymbol{R}^n)$ *for all* $1 < p \leqslant q < \infty$ *with* $1/p - 1/q = 1-\lambda$.

By means of Theorem 1.5 and Corollary 1.6 it is easy to show the full mapping properties of the Hilbert transform and the Hardy–Littlewood kernels. But there are other classes of kernels to which this theorem does not apply. For example, it does not apply to kernels which map only in a partial range.

Hirschman [2] introduced the idea of decomposing the kernel k and applying the Riesz–Thorin interpolation theorem. Naturally, other decompositions and estimates could be used. For example, one may replace the Riesz–Thorin estimates by Marcinkiewicz or Hörmander or other suitable estimates.

We first decompose

$$k(t) = \sum_{m=1}^{\infty} U_m(t), \quad t \in \boldsymbol{R}^n,$$

in which we assume that the series converges almost everywhere. In addition, for each $A \geqslant 0$ let

$$\sum_{m=1}^{\infty} \int_{|t| \leqslant A} |U_m(x-t)|\, dt < \infty$$

for almost all $x \in \boldsymbol{R}^n$. Also, we assume that $\hat{U}_m$ exists for each m, and

$$U_m * f = \mathfrak{F}^{-1}(\hat{U}_m \hat{f}),$$

where

$$\hat{U}(x) = \mathfrak{F}(U)(x) = \frac{1}{(2\pi)^{n/2}} \lim_{\omega \to \infty} \int_{|x| \leqslant \omega} e^{-ix \cdot t}\ U(t)\, dt.$$

Remark. If $U_m \in L_1$ for $m = 1, 2, \ldots$, then the latter properties would easily be satisfied.

Lemma 1.7. *If* $k(t) = \sum_{m=1}^{\infty} U_m(t)$, *then*

(i) $\|U_m * f\|_2 \leqslant \|\hat{U}_m\|_\infty \|f\|_2$, *and*

(ii) $\|k * f\|_p \leqslant \sum_{m=1}^{\infty} \|U_m * f\|_p$ *for* $1 \leqslant p \leqslant \infty$.

Definition 1.8. Let $\|U\|_* = \sup_{\|f\|_1 \leqslant 1} \|U * f\|_1^*$.

Definition 1.9. Let ϱ be any one of the following functionals: $\|\cdot\|_*$, $|||\cdot|||_1$, or $\|\cdot\|_1$. Set

$$\mathfrak{S}_p(\{U_m\}, \varrho) = \sum_{m=1}^{\infty} \|\hat{U}_m\|_\infty^{(2p-2)/p} [\varrho(U_m)]^{(2-p)/p}.$$

Theorem 1.10. *Let* $1 \leqslant p \leqslant 2$. *If* $\mathfrak{S}_p(\{U_m\}, \varrho) < \infty$, *then* $k \in L_p^p(\boldsymbol{R}^n)$. *In fact,* $k \in L_r^r(\boldsymbol{R}^n)$ *for* $p \leqslant r \leqslant p'$.

Proof. We shall prove the theorem in the cases $\varrho = \|\cdot\|_*, |||\cdot|||_1$; the proof of the other case is similar and will be omitted.

We have

$$\|U_m * f\|_1^* \leqslant \varrho(U_m)\|f\|_1 \quad \text{and} \quad \|U_m * f\|_2 \leqslant \|\hat{U}_m\|_\infty \|f\|_2.$$

By a theorem of Marcinkiewicz, this implies

$$\|U_m * f\|_p \leqslant C_p \|\hat{U}_m\|_\infty^{(2p-2)/p} [\varrho(U_m)]^{(2-p)/p},$$

where C_p depends only on p. Therefore, by Lemma 1.7,

$$\|k * f\|_p \leqslant \sum_{m=1}^{\infty} \|U_m * f\|_p \leqslant C_p \|f\|_p \mathfrak{S}_p(\{U_m\}, \varrho).$$

One can, in fact, prove that

$$\mathfrak{S}_p(\{U_m\}, \|\cdot\|_1) < \infty \Rightarrow \mathfrak{S}_p(\{U_m\}, |||\cdot|||_1) < \infty \Rightarrow \mathfrak{S}_p(\{U_m\}, \|\cdot\|_*) < \infty.$$

Lemma 1.11. *If* $A_i \geqslant B_i \geqslant 0$, *then*

$$A_1 A_2 \ldots A_n - B_1 B_2 \ldots B_n \leqslant \sum_{i=1}^{n} \Big[(A_i - B_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} A_j\Big].$$

Proof. By induction we can show

$$\begin{aligned} &A_1 A_2 \ldots A_n - B_1 B_2 \ldots B_n \\ &= \tfrac{1}{2}(A_1 - B_1)(A_2 A_3 \ldots A_n + B_2 B_3 \ldots B_n) + \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ &\quad + \frac{1}{2^{n-1}}(A_{n-1} - B_{n-1})(A_1 + B_1)(A_2 + B_2) \ldots (A_{n-2} + B_{n-2})(A_n + B_n) + \\ &\quad + \frac{1}{2^{n-1}}(A_n - B_n)(A_1 + B_1)(A_2 + B_2) \ldots (A_{n-1} + B_{n-1}) \\ &\leqslant \sum_{i=1}^{n} \Big[(A_i - B_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} A_j\Big]. \end{aligned}$$

A useful decomposition [2] of k is given by

$$U_m(t) = k(t)\{R(t/S_m) - R(t/S_{m-1})\} \quad \text{for} \quad m > 1,$$
$$U_1(t) = k(t) R(t) \in L_1,$$

with $R(0) = 1$, R continuous at the origin and $S_m \nearrow \infty$. For Corollaries 1.13–1.15 we will set

$$\begin{aligned} R(t) &= R(t_1, t_2, \ldots, t_n) = 1/(1 + t_1^{2a})(1 + t_2^{2a}) \ldots (1 + t_n^{2a}) \\ &= R_1(t_1) R_1(t_2) \ldots R_1(t_n). \end{aligned}$$

Lemma 1.12. *Let* $t \in \boldsymbol{R}$. *If* $R_1(t) = 1/(1 + t^{2m})$, *then*

(i) *there exists an* A *and* $b > 0$ *such that* $|\hat{R}_1(t)| \leqslant A e^{-b|t|}$, *and*

(ii) $\int_{-\infty}^{\infty} t^j \hat{R}_1(t)\, dt = 0$ *for* $1 \leqslant j \leqslant 2m - 1$.

Proof. For (i) we use contour integration to show that there exists an A and $b > 0$ such that $|\hat{R}_1(t)| \leqslant A e^{-b|t|}$. To show (ii) we use the facts

$t^j \hat{R}_1(t) \in L_1 \cap L_\infty$ for all j and $R_1(t) \in L_1 \cap L_\infty$, which imply

$$R_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itv} \hat{R}_1(v)\, dv$$

and

$$R_1^{(j)}(t) = \frac{i.i\ldots i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itv} v^j \hat{R}_1(v)\, dv.$$

But

$$R_1^{(j)}(0) = \frac{i.i\ldots i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v^j \hat{R}_1(v)\, dv = 0 \quad \text{for} \quad 1 \leqslant j \leqslant 2m-1.$$

COROLLARY 1.13. *Let* $S_m \nearrow \infty$,

$$\varphi_m(t) = \begin{cases} |t|/S_m, & |t| \leqslant S_{m-1}, \\ 1, & S_{m-1} \leqslant |t| \leqslant S_m, \\ S_m/|t|, & |t| \geqslant S_m, \end{cases}$$

and suppose for some positive integer a

(1) $$\int_{\mathbf{R}^n} [\varphi_m(t)]^{2a} |k(t)|\, dt \leqslant a_m;$$

(2) $$\Big\| \int_{\mathbf{R}^n} e^{ix\cdot t} k(t) \{R(t/S_m) - R(t/S_{m-1})\}\, dt \Big\|_\infty \leqslant b_m,$$

and

(3) $$\sum_{m=1}^{\infty} a_m^{(2-p)/p} b_m^{(2p-2)/p} < \infty,$$

then $k \in L_p^p(\mathbf{R}^n)$.

Proof. Since $k(t) = \sum_{m=1}^{\infty} U_m(t)$, then using (2) together with Theorem 1.10, it is sufficient to show $\|U_m\|_1 \leqslant C a_m$, where C is an absolute constant. Let

$$A_i = 1 + (t_i/S_{m-1})^{2a} \quad \text{and} \quad B_i = 1 + (t_i/S_m)^{2a}.$$

Therefore, by Lemma 1.11,

$$0 \leqslant R(t/S_m) - R(t/S_{m-1}) = \Big(\prod_{j=1}^{n} A_j - \prod_{j=1}^{n} B_j\Big) \Big/ \prod_{j=1}^{n} A_j B_j$$

$$\leqslant \sum_{i=1}^{n} \Big[(A_i - B_i) \prod_{\substack{j=1 \\ j\neq i}}^{n} A_j\Big] \Big/ \prod_{j=1}^{n} A_j B_j \leqslant \sum_{i=1}^{n} t_i^{2a} \Big/ S_{m-1}^{2a} \leqslant (|t|/S_{m-1})^{2a}.$$

Therefore,

$$\|U_m\|_1 = \int_{\mathbf{R}^n} |k(t)|\, |R(t/S_m) - R(t/S_{m-1})|\, dt$$

$$\leqslant 2n^a \int_{\mathbf{R}^n} [\varphi_m(t)]^{2a} |k(t)|\, dt \leqslant 2a_m n^a.$$

In the following corollary, for the sake of simplicity, we assume

$$\sup_{\omega \geqslant M} \Big| \int_{|t| \leqslant \omega} e^{ix\cdot t} k(t)\, dt \Big| \leqslant |P(x)|,$$

where P is a polynomial and M is sufficiently large.

COROLLARY 1.14. *Let* $\hat{k} \in \mathrm{Lip}(a)$, $a \geqslant [a]+1$, *and* $S_m = 2^m$. *If*

$$\int_{\mathbf{R}^n} [\varphi_m(t)]^{2a} |k(t)|\, dt \leqslant A 2^{m(\frac{n}{2}-\beta)},$$

where A *is independent of* m, *then* $k \in L_p^p(\mathbf{R}^n)$ *for*

$$\frac{2a + 2\left(\frac{n}{2} - \beta\right)}{2a + \left(\frac{n}{2} - \beta\right)} < p < \frac{2a + 2\left(\frac{n}{2} - \beta\right)}{\left(\frac{n}{2} - \beta\right)}, \qquad 0 < a,\ 0 < \beta \leqslant \frac{n}{2}.$$

Proof. From the proof of Corollary 1.13 we see

$$\|U_m\|_1 \leqslant A 2^{m(\frac{n}{2}-\beta)}.$$

Now we show $\hat{k} \in \mathrm{Lip}(a)$ implies $\|\hat{U}_m\|_\infty \leqslant A 2^{-ma}$.

$$\hat{U}_m(x) = \int_{\mathbf{R}^n} e^{-ix\cdot t} k(t) \{R(t/S_m) - R(t/S_{m-1})\}\, dt$$

$$= \int_{\mathbf{R}^n} \hat{k}(x+t) \{S_m^n \check{R}(S_m t) - S_{m-1}^n \check{R}(S_{m-1} t)\}\, dt$$

$$= \int_{\mathbf{R}^n} \{\hat{k}(x + t/S_m) - \hat{k}(x + t/S_{m-1})\} \check{R}(t)\, dt.$$

Further,

$$\hat{k}(x + t/S_m) = \hat{k}(x) + \sum_{1 \leqslant |\mu| \leqslant [a]} a_\mu(x) (t/S_m)^\mu + \varphi(x, t, S_m) |t|^a / S_m^a,$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a multi-index, $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_n$, $t^\mu = t_1^{\mu_1} t_2^{\mu_2} \ldots t_n^{\mu_n}$, and for a suitable constant A, $|\varphi(x, t, S_m)| \leqslant A$ for all x, t and S_m. Therefore,

$$|\hat{U}_m(x)| \leqslant \Big| \sum_{1 \leqslant |\mu| \leqslant [a]} a_\mu(x) \int_{\mathbf{R}^n} \{(t/S_m)^\mu - (t/S_{m-1})^\mu\} \check{R}(t)\, dt \Big| +$$

$$+ 2A \int_{\mathbf{R}^n} |\check{R}(t)|\, |t|^a / S_{m-1}^a\, dt.$$

Since $\check{R}(t) = \check{R}_1(t_1)\check{R}_1(t_2)\dots\check{R}_1(t_n)$,

$$\int_{\boldsymbol{R}^n} t^\mu \check{R}(t)\,dt = 0, \quad 1 \leqslant |\mu| \leqslant [\alpha].$$

Therefore,

$$\|\hat{U}_m\|_\infty \leqslant (A/S^\alpha_{m-1}) \int_{\boldsymbol{R}^n} |t|^\alpha \check{R}(t)\,dt \leqslant A2^{-m\alpha}.$$

Now to show (3) of Corollary 1.13

$$A\sum_{m=1}^\infty 2^{-m\alpha(2-2/p)} 2^{m\left(\frac{n}{2}-\beta\right)(2-p)/p} = A\sum_{m=1}^\infty 2^{-m\{\alpha(2-2/p)+(\beta-n/2)(2-p)/p\}}.$$

This series converges if $\alpha(2-2/p)+(\beta-n/2)(2-p)/p > 0$, or

$$p > \frac{2\left(\alpha+\frac{n}{2}-\beta\right)}{2\alpha+\left(\frac{n}{2}-\beta\right)}.$$

Corollary 1.15 (Hirschman [2]). *If* $\hat{k}\in\operatorname{Lip}(\alpha)$ *and* $\hat{k}\in\operatorname{Lip}(2,\beta)$, *then* $k\in L^p_p(\boldsymbol{R}^n)$ *for*

$$\frac{2\alpha+2\left(\frac{n}{2}-\beta\right)}{2\alpha+\left(\frac{n}{2}-\beta\right)} < p < \frac{2\alpha+2\left(\frac{n}{2}-\beta\right)}{\left(\frac{n}{2}-\beta\right)}, \quad 0<\alpha, \quad 0<\beta\leqslant\frac{n}{2}.$$

Proof. The proof follows from Corollary 1.14 by using the condition $\hat{k}\in\operatorname{Lip}(2,\beta)$.

Corollary 1.16. *If there exists a sequence* $S_m\nearrow\infty$ *such that*

$$\int_{S_{m-1}\leqslant|t|\leqslant S_m} |k(t)|\,dt \leqslant a_m, \tag{1}$$

$$\Big\|\int_{S_{m-1}\leqslant|t|\leqslant S_m} k(t)e^{it\cdot x}\,dt\Big\|_\infty \leqslant b_m, \tag{2}$$

and

$$\sum_{m=1}^\infty a_m^{(2-p)/p} b_m^{(2p-2)/p} < \infty, \tag{3}$$

then $k\in L^p_p(\boldsymbol{R}^n)$.

Proof. Take $U_m(t)$ with $R(t)$ the characteristic function on the closed unit ball. Then we find

$$\|U_m\|_1 \leqslant a_m \quad \text{and} \quad \|\hat{U}_m\|_\infty \leqslant b_m.$$

The result follows by Theorem 1.10.

Remark. In Corollary 1.16 we could set

$$R(t) = R_1(t_1)R_1(t_2)\dots R_1(t_n),$$

where

$$R_1(t_i) = \begin{cases} 1, & |t_i|\leqslant 1,\\ 0, & \text{elsewhere}.\end{cases}$$

Theorem 1.17. *If there exists a continuous* $f\geqslant 0$ *and* $f\nearrow\infty$ *such that*

$$\int_{|t|\leqslant S} |k(t)|\,dt = O([f(S)]^\alpha) \tag{1}$$

and

$$\Big\|\int_{|t|\geqslant S} e^{it\cdot x}k(t)\,dt\Big\|_\infty = O\left(\left[\frac{1}{f(S)}\right]^\beta\right), \tag{2}$$

where $\alpha>0$, $\beta>0$, *then* $k\in L^p_p(\boldsymbol{R}^n)$ *for*

$$\frac{2(\alpha+\beta)}{\alpha+2\beta} < p < \frac{2(\alpha+\beta)}{\alpha}.$$

Proof. Choose S_m such that $f(S_m) = 2^m$. Then there is a constant A such that

$$\int_{S_{m-1}\leqslant|t|\leqslant S_m} |k(t)|\,dt \leqslant A2^{m\alpha}$$

and

$$\Big\|\int_{S_{m-1}\leqslant|t|\leqslant S_m} e^{it\cdot x}k(t)\,dt\Big\|_\infty \leqslant A/(2^{m-1})^\beta.$$

But here Corollary 1.16 applies and gives the result.

Remark. It follows from Theorem 1.17 that the functions $(\sin t^2)/t$ and $1/t\log(|t|+e)$ are in $L^p_p(\boldsymbol{R})$ for $1<p<\infty$; and e^{it^3} is in $L^p_p(\boldsymbol{R})$ for $3/2<p<3$.

2. Necessary conditions on k such that $T(f) = f*k$ maps $L_p(\boldsymbol{R}^n)\to L_q(\boldsymbol{R}^n)$ continuously. It is well known that $T(f)=k*f$ maps $L_p\to L_p$ continuously for $1\leqslant p\leqslant\infty$ if and only if $k\in L_1$. Thus in this section, we will only look at k's $\notin L_1$. As a matter of fact, the basic idea in this section is to study the interplay between the partial derivatives of k and the way the L_1-norm of k goes to infinity.

For a given function $k(t)$, $t\in\boldsymbol{R}^n$, and with $\varepsilon_j = \pm 1$, $T_j>0$, we set

$$\sigma_k(t) = \begin{cases} e^{-i\arg k(t)}, & T_j\leqslant\varepsilon_j t_j\leqslant 2T_j \text{ for } 1\leqslant j\leqslant n,\\ 0, & \text{elsewhere}.\end{cases}$$

We also set

$$I(g) = \int\limits_{T_1 \leqslant s_1 t_1 \leqslant 2T_1} dt_1 \dots \int\limits_{T_n \leqslant s_n t_n \leqslant 2T_n} dt_n \, |g(t)|.$$

LEMMA 2.18. *If*

(1) *there exist positive constants A and B $(B \leqslant 1/2)$, and functions ω_j such that*

$$I\big(k(\cdot) - k(\cdot - v)\big) \leqslant A \sum_{j=1}^{n} v_j I(\omega_j) \quad \text{for} \quad 0 \leqslant v_j \leqslant BT_j, \ 1 \leqslant j \leqslant n,$$

and

(2) *there exists a positive constant $C \leqslant 2ABn$ such that*

$$\overline{\lim_{T_s \to \infty}} \frac{I(k)}{T_j I(\omega_j)} < C \quad \text{for} \quad 1 \leqslant j \leqslant n,$$

and

(3) $$\overline{\lim_{T_s \to \infty}} \frac{\big(I(k)\big)^{1+n/q}}{\Big(\prod_{r \in Z(s)} T_r^{1/p}\Big) \prod_{j=1}^{n} \big(I(\omega_j)\big)^{1/q}} = \infty,$$

where $Z(s)$ is a subset of $\{1, 2, \dots, n\}$ which includes s, and $T_r \to \infty$ as $T_s \to \infty$, $r \in Z_s$, while T_i remains fixed for $i \in \{1, 2, \dots, n\} \setminus Z_s$. Here A and B are absolute constants and s is exactly one of the integers $1, 2, \dots, n$.

Then $k \notin L_p^q(\boldsymbol{R}^n)$.

Proof. By (2), this implies for T_s large

$$\frac{I(k)}{I(\omega_j)} \leqslant CT_j \quad \text{for} \quad 1 \leqslant j \leqslant n.$$

Now consider only those v's in $\boldsymbol{R}^n$ for which

$$0 \leqslant v_j \leqslant \frac{I(k)}{2AnI(\omega_j)}.$$

Then for T_s large,

$$\begin{aligned} v_j &\leqslant \frac{C}{2An} T_j \quad \text{for} \quad 1 \leqslant j \leqslant n \\ &\leqslant BT_j. \end{aligned}$$

Therefore for these v's we have

$$|I(k) - k * \sigma_k(v)| \leqslant I\big(k(\cdot) - k(\cdot - v)\big) \leqslant A \sum_{j=1}^{n} v_j I(\omega_j) \leqslant \tfrac{1}{2} I(k).$$

This implies

$$\left|\left\{v \in \boldsymbol{R}^n \colon |k * \sigma_k(v)| > \frac{I(k)}{2}\right\}\right| \geqslant D \frac{\big(I(k)\big)^n}{\prod_{j=1}^{n} I(\omega_j)}.$$

Therefore,

$$\frac{I(k)}{2} \left|\left\{v \colon |k * \sigma_k(v)| > \frac{I(k)}{2}\right\}\right|^{1/q} \geqslant D \frac{\big(I(k)\big)^{1+n/q}}{\prod_{j=1}^{n} \big(I(\omega_j)\big)^{1/q}}.$$

But if k is to map $L_p \to L_q$ this would imply the existence of an absolute constant C such that

$$CT_1^{1/p} \dots T_n^{1/p} \geqslant D \frac{\big(I(k)\big)^{1+n/q}}{\prod_{j=1}^{n} \big(I(\omega_j)\big)^{1/q}}.$$

But on letting $T_s \uparrow \infty$ this contradicts the hypothesis (3) and thus $k \notin L_p^q(\boldsymbol{R}^n)$.

THEOREM 2.19. *If*

(1) $$0 \leqslant a_i \leqslant 1 + \beta_{ji} \quad \text{for} \quad 1 \leqslant i, j \leqslant n,$$

(2) $$C_1 T_1^{a_1} \dots T_n^{a_n} \geqslant I(k) \geqslant C_2 T_1^{a_1} \dots T_n^{a_n}$$

and

(3) $$\left|\frac{\partial k}{\partial t_j}\right| \leqslant C_3 |t_1|^{\beta_{j1}} \dots |t_n|^{\beta_{jn}} \quad \text{for} \quad 1 \leqslant j \leqslant n,$$

where these estimates hold for large t_j and T_j, and C_1, C_2, C_3 are absolute constants.

Then $k \notin L_p^p(\boldsymbol{R}^n)$ for

$$(*) \qquad p > \frac{n + \sum_{i,j=1}^{n} (1 + \beta_{ji} - a_i)}{\sum_{i=1}^{n} a_i}.$$

Proof. We shall apply Lemma 2.18 with $T_1 = T_2 = \dots = T_n$. Set

$$\omega_j(t) = t_1^{\beta_{j1}} \dots t_n^{\beta_{jn}}.$$

Then by (3) there exists an A such that

$$I\big(k(\cdot) - k(\cdot - v)\big) \leqslant A \sum_{j=1}^{n} v_j I(\omega_j) \quad \text{for} \quad 0 \leqslant v_j \leqslant \frac{T_1}{2}, \ 1 \leqslant j \leqslant n.$$

Since $0 \leqslant a_i \leqslant 1 + \beta_{ji}$ for $1 \leqslant i, j \leqslant n$, then

$$\frac{I(k)}{I(\omega_j)} \leqslant C_4,$$

and therefore for T_1 large, $1 \leqslant j \leqslant n$, we find

$$\lim_{T_1 \to \infty} \frac{I(k)}{T_1 I(\omega_j)} = 0.$$

Finally,

$$\varlimsup_{T_1\to\infty} \frac{\big(I(k)\big)^{1+n/p}}{T_1^{n/p}\prod\limits_{j=1}^{n}\big(I(\omega_j)\big)^{1/p}} \geqslant C_5 \varlimsup_{T_1\to\infty} \frac{T_1^{\alpha_1}\dots T_1^{\alpha_n}\prod\limits_{j=1}^{n} T_1^{\frac{\alpha_1-\beta_{j1}-1}{p}}\dots T_1^{\frac{\alpha_n-\beta_{jn}-1}{p}}}{T_1^{n/p}}.$$

This implies $k\notin L_p^p(\boldsymbol{R}^n)$ for p satisfying $(*)$.

To see how Theorem 2.19 works, take $k(t) = e^{i|t|^r}$, $r\geqslant 2$ and $n=1$. At this point we could also make some straight-forward observations. It is well known that $T(f) = k*f$ maps $L_p\to L_p$ continuously implies T maps $L_2\to L_2$ continuously. Therefore, if the right-hand side of $(*)$ is <2, then $k\notin L_p^p(\boldsymbol{R}^n)$ for any p.

A way of showing $k\in L_p^p(\boldsymbol{R}^n)$ is to first localize k and then study both the L_1-norm as well as the Fourier transform of this localized version. We could give a theorem which is in a sense a partial converse, by investigating the L_1-norm of the local version of k and the Fourier transform of k to determine when $k\notin L_p^p(\boldsymbol{R}^n)$. Its proof would be similar to that of Theorem 2.19.

3. Examples to show our theorems are sharp. To show our theorems are best possible we give the exact mapping properties of a certain class of functions. For $\boldsymbol{R}^n$ $(n\geqslant 1)$ we consider the class of functions $e^{if(t)}/g(t)$; while in $\boldsymbol{R}$ we consider other classes as well. We shall divide this section into parts A and B. In Part A we shall consider $\boldsymbol{R}$-examples and in Part B we shall consider $\boldsymbol{R}^n$-examples, $n\geqslant 2$.

A. Examples in $\boldsymbol{R}$.

Theorem 3.20. *Let* $k(t) = e^{if(t)}/g(t)$ *with* $t\in\boldsymbol{R}$, *where* $f(t)$ *is real-valued and* $g(t)\geqslant 0$. *Also,*

$$|f''(t)|\uparrow,\quad g(t)\uparrow \quad \text{for} \quad t>0,$$

$$|f''(t)|\downarrow,\quad g(t)\downarrow \quad \text{for} \quad t<0,$$

and $|f''(t)|$ *is larger than a fixed positive constant outside a compact set. Finally, we will assume* $1/g(t)$ *is locally integrable and*

$$\varliminf_{T\to\infty} T\int_T^{2T}\frac{1}{g(t)}\,dt = \infty \quad \text{and} \quad \varliminf_{T\to\infty} T\int_{-2T}^{-T}\frac{1}{g(t)}\,dt = \infty.$$

If

$$\varlimsup_{|S|\to\infty}\frac{|S|^{1-2/p'}}{g(S)\,|f''(S)|^{1/p'}}<\infty \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{2^{m(1-2/p')}}{g(\varepsilon_j 2^m)\,|f''(\varepsilon_j 2^m)|^{1/p'}} = \infty,$$

$j=1,2$, *where* $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, *then*

$$k\in L_q^q(\boldsymbol{R}) \quad \text{for} \quad p<q<p'$$

and

$$k\notin L_q^q(\boldsymbol{R}) \quad \text{for} \quad q<p \text{ and } q>p'.$$

Proof. First we will show $k\in L_q^q(\boldsymbol{R})$ for $p<q<p'$. We define for $m\geqslant 1$,

$$V_{2m}(t) = \begin{cases}\dfrac{e^{if(t)}}{g(t)}, & 2^{m-1}\leqslant t< 2^m,\\ 0, & \text{elsewhere},\end{cases}$$

$$V_{2m-1}(t) = \begin{cases}\dfrac{e^{if(t)}}{g(t)}, & -2^{m}\leqslant t< -2^{m-1},\\ 0, & \text{elsewhere},\end{cases}$$

and

$$k(t) = \sum_{m=0}^{\infty} V_m(t).$$

Now

$$\|V_{2m}\|_1\leqslant \frac{2^{m-1}}{g(2^{m-1})}, \qquad \|V_{2m-1}\|_1\leqslant \frac{2^{m-1}}{g(-2^{m-1})}$$

and

$$\|\hat V_{2m}\|_\infty \leqslant \frac{1}{g(2^{m-1})\,|f''(2^{m-1})|^{1/2}}, \qquad \|\hat V_{2m-1}\|_\infty \leqslant \frac{1}{g(-2^{m-1})\,|f''(-2^{m-1})|^{1/2}}.$$

Thus, for $2\leqslant q<p'$ consider

$$\sum_{m=1}^{\infty}\|\hat V_{2m}\|_\infty^{2/q}\|V_{2m}\|_1^{1-2/q}\leqslant \sum_{m=0}^{\infty}\frac{2^{m(1-2/p')}2^{m(2/p'-2/q)}}{g(2^m)\,|f''(2^m)|^{1/p'}\,|f''(2^m)|^{1/q-1/p'}}.$$

But this series converges since

$$\varlimsup_{m\to\infty}\frac{2^{m(1-2/p')}}{g(2^m)\,|f''(2^m)|^{1/p'}}<\infty.$$

Similarly, we get the same estimates for the pair $\|V_{2m-1}\|_1$, $\|\hat V_{2m-1}\|_\infty$. Hence, by Theorem 1.10, this implies $k\in L_q^q(\boldsymbol{R})$.

To complete the theorem we shall show $k\notin L_q^q(\boldsymbol{R})$ for $q>p'$. To do this we shall use Lemma 2.18. Without loss of generality we could assume $k(t)=0$ for $t<0$, and for t large $f''(t)>C$.

Since $f''(t)>C$ for t large we have

$$|f'(t)-f'(A)|\geqslant C\,|t-A| \qquad (A - \text{fixed}).$$

Therefore,

$$|f'(t)| \geqslant \frac{C}{2}T \quad \text{for} \quad t > T.$$

Since $f''(t) > 0$ and $|f'(t)| \to \infty$ this implies for t large $f'(t) > 0$ and in fact

$$f'(t) \geqslant \frac{C}{2}T \quad \text{for} \quad t > T.$$

Now we shall employ Lemma 2.18 and hence assume $0 \leqslant v \leqslant T/2$, and T sufficiently large.

$$\begin{aligned} I\big(k(\cdot)-k(\cdot-v)\big) &= \int_{T\leqslant t\leqslant 2T} |k(t)-k(t-v)|\,dt \\ &= \int_{T\leqslant t\leqslant 2T} dt \left| \frac{e^{if(t)}}{g(t)} - \frac{e^{if(t-v)}}{g(t)} + \frac{e^{if(t-v)}}{g(t)} - \frac{e^{if(t-v)}}{g(t-v)} \right| \\ &\leqslant \int_{T\leqslant t\leqslant 2T} dt \frac{1}{g(t)} |e^{if(t)} - e^{if(t-v)}| + \int_{T-v\leqslant t\leqslant T} \frac{dt}{g(t)} + \int_{2T-v\leqslant t\leqslant 2T} \frac{dt}{g(t)} \\ &\leqslant |v| \int_{T\leqslant t\leqslant 2T} \frac{dt}{g(t)} |e^{if(\zeta)}\cdot f'(\zeta)| + \frac{|v|}{g(T-v)} + \frac{|v|}{g(2T-v)} \\ &\leqslant v \int_T^{2T} \frac{f'(t)}{g(t)}\,dt + \frac{v}{g(T-v)} + \frac{v}{g(2T-v)} \\ &\leqslant 2v \int_T^{2T} \frac{f'(t)}{g(t)}\,dt. \end{aligned}$$

Hence,

$$I\big(k(\cdot)-k(\cdot-v)\big) \leqslant 2v \int_T^{2T} \omega(t)\,dt \quad \text{with} \quad \omega(t) = f'(t)/g(t)$$
$$\text{and } 0 \leqslant v \leqslant T/2.$$

Now we consider

$$\frac{I(k)}{TI(\omega)} = \frac{\displaystyle\int_T^{2T} \frac{dt}{g(t)}}{\displaystyle T\int_T^{2T} \frac{f'(t)}{g(t)}\,dt}.$$

But since $f'(t) \geqslant \frac{C}{2}T$ for $t > T$, this implies

$$\lim_{T\to\infty} \frac{I(k)}{TI(\omega)} \leqslant \lim_{T\to\infty} \frac{1}{CT^2} = 0.$$

Finally, we must show for $q > p'$,

$$\overline{\lim_{T\to\infty}} \frac{\big(I(k)\big)^{1+1/q}}{T^{1/q}\big(I(\omega)\big)^{1/q}} = \infty.$$

Now,

$$\begin{aligned} |f'(t)-f'(A)| &= |f''(\zeta)|\,|t-A| \quad (A \text{ is fixed}) \ T \leqslant t \leqslant 2T \\ &\leqslant 4Tf''(2T). \end{aligned}$$

Therefore,

$$f'(t) \leqslant 5Tf''(2T) \quad \text{for} \quad T \leqslant t \leqslant 2T.$$

Hence,

$$\begin{aligned} &\overline{\lim_{T\to\infty}} \frac{\left(\displaystyle\int_T^{2T} \frac{dt}{g(t)}\right)^{1+1/q}}{T^{1/q}\left(\displaystyle\int_T^{2T} \frac{f'(t)}{g(t)}\,dt\right)^{1/q}} \\ &\geqslant C\,\overline{\lim_{T\to\infty}} \frac{\left(\displaystyle\int_T^{2T} \frac{dt}{g(t)}\right)^{1+1/q}}{T^{2/q}\big(f''(2T)\big)^{1/q}\left(\displaystyle\int_T^{2T} \frac{dt}{g(t)}\right)^{1/q}} \geqslant C\,\overline{\lim_{T\to\infty}} \frac{T^{1-2/q}}{g(2T)\big(f''(2T)\big)^{1/q}}. \end{aligned}$$

But by hypothesis

$$\sum_{m=1}^{\infty} \frac{2^{m(1-2/p')}}{g(2^m)\big(f''(2^m)\big)^{1/p'}} = \infty,$$

which implies for $q > p'$,

$$\overline{\lim_{T\to\infty}} \frac{T^{1-2/q}}{g(T)\big(f''(T)\big)^{1/q}} = \infty$$

and hence our result.

THEOREM 3.21. *Let* $k(t) = \dfrac{e^{if(t)}}{g(t)}$ *with* $t \in \mathbf{R}$, *where* f *is real-valued,* $g(t) > 0$, *and* $1/g(t)$ *is locally integrable. Also,*

$$|f''(t)|\downarrow,\ g(t)\uparrow \text{ for } t > 0, \qquad |f''(t)|\uparrow,\ g(t)\downarrow \text{ for } t < 0.$$

Finally, we assume

$$\frac{1}{g(S/2)} \leqslant \big(|S|\,|f''(S)| + |f'(S)|\big) \left| \int_S^{2S} \frac{dt}{g(t)} \right| \quad \text{for} \quad |S| \text{ large}.$$

If

$$\overline{\lim_{|S|\to\infty}} \frac{|S|^{1-2/p'}}{g(S)\,|f''(2S)|^{1/p'}} < \infty, \quad \lim_{|S|\to\infty} |S|^{\beta}|f''(2S)| = \infty \quad \textit{for some } \beta < 2,$$

and

$$\sum_{m=1}^{\infty} \frac{2^{m(1-1/p')}}{g(\varepsilon_j 2^m)\{2^{m-1}|f''(\varepsilon_j 2^{m-1})|+|f'(\varepsilon_j 2^{m-1})|\}^{1/p'}} = \infty,$$

$j=1,2,$ *where* $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, *then*

$$k \in L_q^q(\boldsymbol{R}) \quad \textit{for} \quad p < q < p'$$

and

$$k \notin L_q^q(\boldsymbol{R}) \quad \textit{for} \quad q < p \quad \textit{and} \quad q > p'.$$

Proof. First we shall show $k \in L_q^q(\boldsymbol{R})$ for $p < q < p'$. With the same decomposition as in Theorem 3.20,

$$\|V_{2m}\|_1 \leqslant \frac{2^{m-1}}{g(2^{m-1})}, \qquad \|V_{2m-1}\|_1 \leqslant \frac{2^{m-1}}{g(-2^{m-1})},$$

and

$$\|\hat{V}_{2m}\|_\infty \leqslant \frac{1}{g(2^{m-1})\,|f''(2^m)|^{1/2}}, \qquad \|\hat{V}_{2m-1}\|_\infty \leqslant \frac{1}{g(-2^{m-1})\,|f''(-2^m)|^{1/2}}.$$

Thus, for $2 \leqslant q < p'$

$$\sum_{m=1}^{\infty} \|\hat{V}_{2m}\|_\infty^{2/q}\|V_{2m}\|_1^{1-2/q} \leqslant \sum_{m=0}^{\infty} \frac{2^{m(1-2/p')}2^{m(2/p'-2/q)}}{g(2^m)\,|f''(2^{m+1})|^{1/p'}|f''(2^{m+1})|^{1/q-1/p'}}.$$

Now for m large enough,

$$\frac{2^{m(1-2/p')}}{g(2^m)\,|f''(2^{m+1})|^{1/p'}} \leqslant C$$

and

$$\frac{1}{\big(2^{m\beta}|f''(2^{m+1})|\big)^{1/q-1/p'}} \leqslant C.$$

Therefore,

$$\frac{2^{m(1-2/q)}}{g(2^m)|f''(2^{m+1})|^{1/q}} \leqslant C^2/2^{m(2-\beta)(1/q-1/p')}.$$

Hence,

$$\sum_{m=1}^{\infty} \|\hat{V}_{2m}\|_\infty^{2/q}\|V_{2m}\|_1^{1-2/q} < \infty,$$

and we get the same estimates for the pair $\|V_{2m-1}\|_1$, $\|\hat{V}_{2m-1}\|_\infty$. Hence, by Theorem 1.10, this implies $k \in L_q^q(\boldsymbol{R})$. To complete the theorem we shall show $k \notin L_q^q(\boldsymbol{R})$ for $q > p'$. To do this we shall use Lemma 2.18. Without

loss of generality we assume $k(t) = 0$ for $t < 0$ and $f''(t) > 0$. Now in applying Lemma 2.18 we can assume $0 \leqslant v \leqslant T/2$.

$$I\big(k(\cdot)-k(\cdot-v)\big) \leqslant \int_T^{2T} |k(t)-k(t-v)|\,dt$$

$$\leqslant |v| \int_T^{2T} \frac{dt}{g(t)}\, |e^{if(t)}\cdot f'(\zeta)| + \frac{2|v|}{g(T/2)}.$$

But

$$|f'(t) - f'(T)| = |f''(\zeta)|\,|t-T|;$$

hence

$$|f'(t)| \leqslant T|f''(T)| + |f'(T)| \quad \text{for} \quad T \leqslant t \leqslant 2T.$$

Therefore

$$I\big(k(\cdot)-k(\cdot-v)\big) \leqslant 3|v|\big(T|f''(T)|+|f'(T)|\big)\int_T^{2T}\frac{dt}{g(t)}.$$

Setting

$$\omega(t) = \frac{T|f''(T)|+|f'(T)|}{g(t)} \quad \text{for} \quad T \leqslant t \leqslant 2T,$$

we have

$$I\big(k(\cdot)-k(\cdot-v)\big) \leqslant 3v\int_T^{2T} dt\,\omega(t).$$

Now we consider

$$\frac{I(k)}{TI(\omega)} = \frac{\displaystyle\int_T^{2T}\frac{dt}{g(t)}}{\big(T^2|f''(T)|+T|f'(T)|\big)\displaystyle\int_T^{2T}\frac{dt}{g(t)}} \leqslant \frac{1}{T^2|f''(T)|} \to 0 \quad \text{as} \quad T\to\infty.$$

Finally, we must show for $q > p'$,

$$\overline{\lim_{T\to\infty}} \frac{\big(I(k)\big)^{1+1/q}}{T^{1/q}\big(I(\omega)\big)^{1/q}} = \infty.$$

Now

$$\overline{\lim_{T\to\infty}} \frac{\Big(\displaystyle\int_T^{2T}\frac{dt}{g(t)}\Big)^{1+1/q}}{T^{1/q}\{T|f''(T)|+|f'(T)|\}^{1/q}\Big(\displaystyle\int_T^{2T}\frac{dt}{g(t)}\Big)^{1/q}}$$

$$\geqslant \overline{\lim_{T\to\infty}} \frac{T^{1-1/q}}{g(2T)\{T|f''(T)|+|f'(T)|\}^{1/q}}.$$

But

$$\sum_{m=1}^{\infty}\frac{2^{m(1-1/p')}}{g(2^m)\{2^{m-1}|f''(2^{m-1})|+|f'(2^{m-1})|\}^{1/p'}}=\infty$$

and

$$\lim_{T\to\infty} T^{\beta}|f''(T)|=\infty \quad \text{for some } \beta<2$$

implies our result. For, if

$$\varlimsup_{T\to\infty}\frac{T^{1-1/q}}{g(2T)\{T|f''(T)|+|f'(T)|\}^{1/q}}=C<\infty,$$

then

$$\frac{2^{m(1-1/p')}2^{m(1/p'-1/q)}}{g(2^{m+1})\{2^m|f''(2^m)|+|f'(2^m)|\}^{1/p'}\{2^m|f''(2^m)|+|f'(2^m)|\}^{1/q-1/p'}} \leqslant C+1 \quad \text{for } m \text{ large.}$$

Hence,

$$\begin{aligned}\frac{2^{m(1-1/p')}}{g(2^{m+1})\{2^m|f''(2^m)|+|f'(2^m)|\}^{1/p'}} &\leqslant (C+1)\frac{\{2^m|f''(2^m)|+|f'(2^m)|\}^{1/q-1/p'}}{2^{m(1/p'-1/q)}}\\ &\leqslant \frac{(C+1)}{2^{m(2/p'-2/q)}|f''(2^m)|^{1/p'-1/q}}\\ &= \frac{(C+1)}{2^{m(2-\beta)(1/p'-1/q)}(2^{m\beta}|f''(2^m)|)^{1/p'-1/q}},\end{aligned}$$

which is a contradiction and hence our result follows.

Corollary 3.22. *Let* $k(t)=e^{i|t|^a}/|t|^b$ *with* $t\in\mathbf{R}$, $a\neq 1$, $b<1$ *and* $(1/2)a+b>1$, *then*

$$k\in L_q^q(\mathbf{R}) \quad \textit{for} \quad \frac{a}{a+b-1}<q<\frac{a}{1-b}$$

and

$$k\notin L_q^q(\mathbf{R}) \quad \textit{for} \quad q>\frac{a}{1-b} \ \textit{and for}\ q<\frac{a}{a+b-1}.$$

Proof. Case 1. Here we apply Theorem 3.20. $f(t)=|t|^a$, $a\geqslant 2$, $g(t)=|t|^b$ implies

$$|f''(t)|=a(a-1)|t|^{a-2}$$

and since $(1-b)/a<1/2$,

$$\sum_{m=1}^{\infty}\frac{2^{m\left(1-\frac{2(1-b)}{a}\right)}}{2^{mb}2^{m\frac{(a-2)(1-b)}{a}}}=\sum_{m=1}^{\infty}\frac{2^{m\left(\frac{a+2b-2}{a}\right)}}{2^{m\left(\frac{a+2b-2}{a}\right)}}=\sum_{m=1}^{\infty}1=\infty.$$

Also,

$$\varlimsup_{T\to\infty}\frac{|T|^{1-\frac{2(1-b)}{a}}}{|T|^b|T|^{\frac{(a-2)(1-b)}{a}}}=\varlimsup_{T\to\infty}1=1,$$

and hence we get our result.

Case 2. $f(t)=|t|^a$, $0<a<2$, $a\neq 1$, and $g(t)=|t|^b$;

$$|f''(t)|=a|a-1||t|^{a-2}.$$

Now we will apply Theorem 3.21:

$$\frac{1}{g(T/2)}=(2/|T|)^b\leqslant\left(|a(a-1)||T|^{a-1}+a|T|^{a-1}\right)\frac{1}{(1-b)}(2^{1-b}-1)|T|^{1-b},$$

since

$$2^b/|T|^b\leqslant C|T|^{a-b} \quad \text{as} \quad |T|\to\infty.$$

Also,

$$\varlimsup_{T\to\infty}\frac{|T|^{1-\frac{2(1-b)}{a}}}{|T|^b|T|^{\frac{(a-2)(1-b)}{a}}}=\varlimsup_{T\to\infty}1=1$$

and

$$\lim_{T\to\infty}|T|^{\beta}|T|^{a-2}=\infty \quad \text{if} \quad \beta>2-a.$$

Finally,

$$\sum_{m=1}^{\infty}\frac{2^{m\left(1-\frac{(1-b)}{a}\right)}}{2^{mb}\{2^{m-1}(2^{m-1})^{a-2}+2^{(m-1)(a-1)}\}^{(1-b)/a}}\geqslant\left(\frac{1}{2}\right)^{\frac{1-b+a}{a}}\sum_{m=1}^{\infty}\frac{2^{m\left(\frac{a+b-1}{a}\right)}}{2^{m\frac{(a+b-1)}{a}}}=\infty.$$

When $b<0$, a similar argument applies.

We would like to point out that these methods show

$$k(t)=\begin{cases}\dfrac{1}{t\log(|t|)}, & |t|>e,\\ 0, & |t|\leqslant e\end{cases}$$

maps $L_p\to L_p$ for $1<p<\infty$ (apply Theorem 1.17), but $\hat{k}\notin\mathrm{Lip}(\delta)$ for any $\delta>0$. This extends a result of Hirschman ([2], Theorem 4e).

If $f\in L_p^p(\mathbf{R})$ for $1<p<\infty$, then it follows that if $\sum|C_n|<\infty$, then $\sum C_nf_n(t)\in L_p^p(\mathbf{R})$ for $1<p<\infty$. One can then ask whether $\sum C_nf_n(t)\in L_p^p(\mathbf{R})$ for $1<p<\infty$ when $\sum|C_n|=\infty$.

Consider the example

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{t} * \chi_{[-n^{1/2}, n^{1/2}]} = E(\cdot),$$

where

$$\frac{1}{t} * \chi_{[-n^{1/2}, n^{1/2}]} = f_n$$

is the Hilbert transform applied to the characteristic function of the interval $[-n^{1/2}, n^{1/2}]$. Setting

$$U_n = \frac{(-1)^{2n-1}}{(2n-1)} \frac{1}{t} * \chi_{[-(2n-1)^{1/2}, (2n-1)^{1/2}]} + \frac{(-1)^{2n}}{2n} \frac{1}{t} * \chi_{[-(2n)^{1/2}, (2n)^{1/2}]},$$

we were able to show, by employing Theorem 1.10, that $E(\cdot) \in L_p^p(\boldsymbol{R})$ for $1 < p < \infty$.

B. Examples in $\boldsymbol{R}^n$.

Corollary 3.23. *Let $t \in \boldsymbol{R}^n$, $n \geqslant 2$, and $0 < a < 1$, then*

$$k(t) = \frac{e^{i|t|^2}}{|t_1|^{2a} + |t_2|^{2a} + \ldots + |t_n|^{2a}} \in L_p^p(\boldsymbol{R}^n) \quad \text{for} \quad \frac{2n}{n+2a} < p < \frac{2n}{n-2a}$$

and

$$k \notin L_p^p(\boldsymbol{R}^n) \quad \text{for} \quad p > \frac{2n}{n-2a} \text{ and } p < \frac{2n}{n+2a}.$$

Proof. We apply the remark following Corollary 1.16; thus we need to estimate

$$\Big| \int\limits_{|t_1| \leqslant S_m} dt_1 \ldots \int\limits_{|t_n| \leqslant S_m} dt_n e^{it\cdot x} k(t) - \int\limits_{|t_1| \leqslant S_{m-1}} dt_1 \ldots \int\limits_{|t_n| \leqslant S_{m-1}} dt_n e^{it\cdot x} k(t) \Big| =$$

$$\Big| \sum_{j=1}^{n} \int\limits_{|t_1| \leqslant S_{m-1}} \ldots \int\limits_{|t_{j-1}| \leqslant S_{m-1}} dt_{j-1} \int\limits_{S_{m-1} \leqslant |t_j| \leqslant S_m} dt_j \int\limits_{|t_{j+1}| \leqslant S_m} dt_{j+1} \ldots \int\limits_{|t_n| \leqslant S_m} e^{it\cdot x} k(t) \Big|$$

$$\leqslant \sum_{j=1}^{n} \Big| \int\limits_{|t_1| \leqslant S_{m-1}} dt_1 \ldots \int\limits_{S_{m-1} \leqslant |t_j| \leqslant S_m} dt_j \ldots \int\limits_{|t_n| \leqslant S_m} dt_n e^{it\cdot x} k(t) \Big|$$

$$\leqslant \frac{C}{S_{m-1}^{2a}}$$

and

$$\sum_{j=1}^{n} \int\limits_{|t_1| \leqslant S_{m-1}} dt_1 \ldots \int\limits_{S_{m-1} \leqslant |t_j| \leqslant S_m} dt_j \ldots \int\limits_{|t_n| \leqslant S_m} dt_n |k(t)| \leqslant C S_m^{n-2a}.$$

Now set $S_m = 2^m$. Then we just need to find those $p \geqslant 2$ such that

$$\sum_{m=1}^{\infty} 2^{-m\left(\frac{4a}{p} - (1-2/p)(n-2a)\right)} = \sum_{m=1}^{\infty} 2^{-m\left(\frac{4a}{p} - n + 2a + \frac{2n}{p} - \frac{4a}{p}\right)} = \sum_{m=1}^{\infty} 2^{-m\left(\frac{2n}{p} + 2a - n\right)} < \infty.$$

Hence we need

$$\frac{2n}{p} + 2a - n > 0 \quad \text{or} \quad p < \frac{2n}{n-2a}.$$

We apply Theorem 2.19 to show $k \notin L_p^p(\boldsymbol{R}^n)$ for $p > \dfrac{2n}{n-2a}$.

We set

$$\sigma_k(t) = \begin{cases} e^{-i|t|^2}, & T^{1/n} \leqslant t_j \leqslant 2T^{1/n}, \\ 0, & \text{elsewhere}, \end{cases}$$

$$I(k) = \int\limits_{T^{1/n}}^{2T^{1/n}} dt_1 \ldots \int\limits_{T^{1/n}}^{2T^{1/n}} dt_n \frac{1}{|t_1|^{2a} + \ldots + |t_n|^{2a}}.$$

Therefore

$$D_a T^{1-2a/n} \leqslant I(k) \leqslant C_a T^{1-2a/n}.$$

Also,

$$\frac{\partial k}{\partial t_j} = \frac{i 2 t_j e^{i|t|^2}}{t_1^{2a} + t_2^{2a} + \ldots + t_n^{2a}} - \frac{2a t_j^{2a-1} e^{i|t|^2}}{(t_1^{2a} + t_2^{2a} + \ldots + t_n^{2a})^2} \quad \text{and} \quad \left| \frac{\partial k}{\partial t_j} \right| \leqslant 3 |t_j|^{1-2a},$$

and thus with $\alpha_1 = \ldots = \alpha_n = 1 - 2a/n$

$$\beta_{jS} = \begin{cases} 1-2a, & S = j, \\ 0, & S \neq j, \end{cases}$$

we get $k \notin L_p^p(\boldsymbol{R}^n)$ for $p > 2n/(n-2a)$. By the same methods we can also show $e^{i|t|^2} \notin L_p^p(\boldsymbol{R}^n)$ for $p \neq 2$, while for $p = 2$, $e^{i|t|^2} \in L_2^2(\boldsymbol{R}^n)$.

4. Conditions on k such that $T(f) = f * k$ maps $L_p(\boldsymbol{R}^n) \to L_q(\boldsymbol{R}^n)$. In this section we shall obtain some analogues of the results of the first chapter, without going into complete generality. However, we believe that these methods could be used to obtain essentially all of the results which are analogous to the L_p^p case. It turns out that the problem breaks into two parts. We will first study the cases where $1/p - 1/q = 1 - \lambda$ and $\lambda > 1/2$, and then the case where $0 \leqslant \lambda \leqslant 1/2$. The reason $\lambda = 1/2$ is the dividing line depends heavily on the behavior of $\|\hat{U}\|_{1/(1-\lambda)}$ as a function of λ. That is, in the examples we study, where $\lambda > 1/2$, $\|\hat{U}\|_{1/(1-\lambda)}$ is reasonably small; while for $0 \leqslant \lambda \leqslant 1/2$ it is too large. Intuitively, one can see this by means of the Riesz–Thorin Theorem.

Part A. $1/2 < \lambda < 1$.

LEMMA 4.24. *Let* $1/2 < \lambda < 1$. *If*

$$k(t) = \sum_{m=1}^{\infty} U_m(t) \quad \textit{with} \quad 1/p - 1/q = 1-\lambda,$$

then

(i) $\|U_m * f\|_2 \leqslant C_\lambda \|\hat{U}_m\|_{1/(1-\lambda)} \|f\|_{2/(3-2\lambda)}$ *and*

(ii) $\|k*f\|_q \leqslant \sum_{m=1}^{\infty} \|U_m * f\|_q$ *for* $1 \leqslant q \leqslant \infty$.

Proof. Part (ii) of the theorem (which also appears as part (ii) in Lemma 1.7 of the first chapter) follows from Minkowski's inequality. To show (i) we note,

$$\begin{aligned}\|U_m * f\|_2^2 &= \|\hat{U}_m \cdot \hat{f}\|_2^2 \\ &\leqslant \Big\{\int |\hat{U}_m|^{\frac{1}{1-\lambda}}\Big\}^{2(1-\lambda)} \Big\{\int |\hat{f}|^{\frac{2}{2\lambda-1}}\Big\}^{2\lambda-1} \\ &\leqslant C_\lambda \Big\{\int |\hat{U}_m|^{\frac{1}{1-\lambda}}\Big\}^{2(1-\lambda)} \Big\{\int |f|^{\frac{2}{3-2\lambda}}\Big\}^{3-2\lambda},\end{aligned}$$

and hence our result.

COROLLARY 4.25. *Let* $1/2 < \lambda < 1$. *If*

$$\sum_{m=1}^{\infty} \|\hat{U}_m\|_{1/(1-\lambda)}^{t} \|U_m\|_{1/\lambda}^{1-t} < \infty,$$

then

$$k \in L_p^q(\boldsymbol{R}^n) \quad \textit{with} \quad \frac{1}{q} = \frac{t(1-2\lambda)+2\lambda}{2} \quad \textit{and} \quad \frac{1}{p} - \frac{1}{q} = 1-\lambda.$$

Proof. We know

$$\|U_m * f\|_2 \leqslant C_\lambda \|\hat{U}_m\|_{1/(1-\lambda)} \|f\|_{2/(3-2\lambda)}$$

and

$$\|U_m * f\|_{1/\lambda} \leqslant \|U_m\|_{1/\lambda} \|f\|_1 .$$

Therefore,

$$\|U_m * f\|_q \leqslant C_\lambda \|\hat{U}_m\|_{1/(1-\lambda)}^{t} \|U_m\|_{1/\lambda}^{1-t} \|f\|_p$$

with

$$\frac{1}{q} = \frac{t(1-2\lambda)+2\lambda}{2} \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} = 1-\lambda .$$

But

$$\|k*f\|_q \leqslant \sum_{m=1}^{\infty} \|U_m * f\|_q \leqslant C_\lambda \|f\|_p \sum_{m=1}^{\infty} \|\hat{U}_m\|_{1/(1-\lambda)}^{t} \|U_m\|_{1/\lambda}^{1-t},$$

and hence our result.

Part B. $0 \leqslant \lambda \leqslant 1/2$. It will help us to study the following picture:

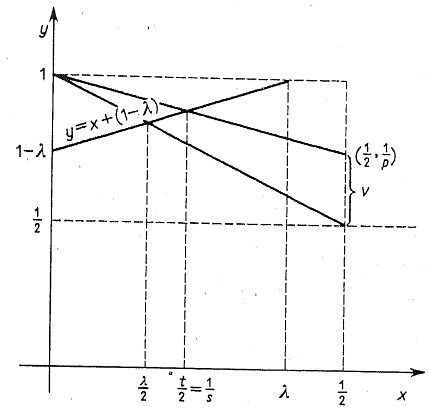

Fig. 1

Here we will think of y as $1/p$ and x as $1/q$, and hence study the estimates along the line $y = x+(1-\lambda)$.

LEMMA 4.26. *Let* $0 \leqslant \lambda \leqslant 1/2$ *with* $1/p = 1/2+v$ *and* $t = 2\lambda p/(3p-2)$. *If*

$$\sum_{m=1}^{\infty} \|U_m\|_{\infty}^{1-t} \|\hat{U}_m\|_{1/v}^{t} < \infty,$$

then

$$k \in L_r^s(\boldsymbol{R}^n) \quad \textit{with} \quad s = \frac{3p-2}{\lambda p} \quad \textit{and} \quad \frac{1}{r} - \frac{1}{s} = 1-\lambda .$$

Proof. Since $1/p = 1/2+v$, we have

$$\|U_m * f\|_2 \leqslant C_p \|\hat{U}_m\|_{1/v} \|f\|_p$$

and

$$\|U_m * f\|_\infty \leqslant \|U_m\|_\infty \|f\|_1 .$$

Thus, we are interpolating along the line

$$y = \Big(\frac{2-2p}{p}\Big)x + 1 \quad \text{(see Fig. 1)}.$$

We are interested in that x where

$$\Big(\frac{2-2p}{p}\Big)x + 1 = x + (1-\lambda).$$

This implies

$$x = \frac{\lambda p}{3p-2} = \frac{t}{2} + \frac{1-t}{\infty} = \frac{t}{2},$$

and therefore by Riesz–Thorin,

$$\|U_m * f\|_s \leqslant C_p \|U_m\|_\infty^{1-t} \|\hat{U}_m\|_{1/v}^{t} \|f\|_r,$$

where

$$s = \frac{3p-2}{\lambda p} = \frac{2}{t} \quad \text{and} \quad \frac{1}{r} = \frac{1}{s} + 1 - \lambda.$$

Putting this together with part (ii) of Lemma 4.24, we get our result.

EXAMPLES IN $\boldsymbol{R}$.

LEMMA 4.27. *Let* $0 \leqslant \lambda \leqslant 1$ *and* $a > 0$. *If*

$$k(t) = \frac{e^{i}|t|^a}{|t|^b}, \qquad t \in \boldsymbol{R}, \ \textit{with } b < \lambda,$$

then

$$k \notin L_p^q \quad \textit{for} \quad q > \frac{a}{\lambda - b} \ \textit{with } \frac{1}{p} - \frac{1}{q} = 1 - \lambda.$$

Proof. Let

$$\sigma_k(-t) = e^{-i|t|^a} \chi_{[T,2T]}.$$

For

$$k(t) = \frac{e^{i|t|^a}}{|t|^b},$$

$$\left|\frac{dk}{dt}\right| \leqslant C|t|^{a-1-b}$$

for $|t|$ sufficiently large, where C depends only on a and b.

Then

$$|k*\sigma_k(0) - k*\sigma_k(v)| \leqslant \int_T^{2T} |k(t) - k(t-v)|\,dt \leqslant C|v|T^{a-b},$$

which implies

$$\left|\left\{v\colon\ |k*\sigma_k(v)| > \frac{T^{1-b}}{2}\right\}\right| \geqslant CT^{1-a}.$$

But, if $k \in L_p^q$, there exists $C_{p,q}$ independent of T such that

$$T^{1-b}\left|\left\{v\colon\ |k*\sigma_k(v)| > \frac{T^{1-b}}{2}\right\}\right|^{1/q} \leqslant C_{p,q} T^{1/p}$$

with $1/p - 1/q = 1 - \lambda$.

But that implies

$$T^{1-b} T^{(1-a)/q} \leqslant C_{p,q} T^{1/p}.$$

Thus, $k \notin L_p^q$ if $(1-b) + (1-a)/q > 1/p$ (letting $T \nearrow \infty$), or if $q > a/(\lambda - b)$ and $1/p - 1/q = 1 - \lambda$.

THEOREM 4.28. *Let* $0 < \lambda < 1$ *with* $t \in \boldsymbol{R}$ *and* $k(t) = e^{i|t|^a}/|t|^b$. *If* $a \neq 1$, $b < \lambda$ *and* $\frac{a}{2}\lambda + (b - \lambda) > 0$, *then*

$$k \in L_p^q \quad \textit{for} \quad \frac{a}{\lambda(a-1)+b} < q < \frac{a}{\lambda - b},$$

and

$$k \notin L_p^q \quad \textit{for} \quad q > \frac{a}{\lambda - b} \ \textit{with } \frac{1}{p} - \frac{1}{q} = 1 - \lambda.$$

Proof. By Lemma 4.27, we have $k \notin L_p^q$ for $q > a/(\lambda - b)$. Now we set

$$V_m(t) = \frac{e^{i|t|^a}}{|t|^b}\{\chi_{[2^{m-1},\,2^m]} + \chi_{[-2^m,\,-2^{m-1}]}\} \quad \text{for} \quad m \geqslant 1$$

and

$$V_0(t) = \frac{e^{i|t|^a}}{|t|^b}\chi_{[-1,\,1]}.$$

Case 1. Assume $1/2 < \lambda < 1$. To show $k \in L_p^q$ for $\frac{a}{\lambda(a-1)+b} < q < \frac{a}{\lambda - b}$ we will first use Corollary 4.25, which applies to $q < 2$ or $q > 2/(2\lambda - 1)$ We note

$$\|\hat{V}_0\|_{1/(1-\lambda)} \leqslant C_\lambda \|V_0\|_{1/\lambda} < \infty.$$

Now,

$$\|\hat{V}_m\|_{1/(1-\lambda)} \leqslant \mathrm{I} + \mathrm{II} + \mathrm{III},$$

and for $a > 1$ we estimate as follows:

$$\|\hat{V}_m\|_{1/(1-\lambda)} \leqslant \left\{\int_{a2^{(m-2)(a-1)}}^{a2^{(m+2)(a-1)}} dx \left|\int_{2^{m-1}}^{2^m} dt\,\frac{e^{i|t|^a}e^{-ixt}}{|t|^b}\right|^{1/(1-\lambda)}\right\}^{1-\lambda} +$$

$$+ \left\{\int_{2^{(m+2)(a-1)}}^{\infty} d(x) \left|\int_{2^{m-1}}^{2^m} \frac{d(e^{it^a}e^{-ixt})}{i(at^{a-1}-x)t^b}\right|^{1/(1-\lambda)}\right\}^{1-\lambda}$$

$$+ \left\{\int_{-\infty}^{a2^{(m-2)(a-1)}} d(x) \left|\int_{2^{m-1}}^{2^m} \frac{d(e^{it^a}e^{-ixt})}{i(at^{a-1}-x)t^b}\right|^{1/(1-\lambda)}\right\}^{1-\lambda}.$$

While for $0 < a < 1$ we decompose the x-axis into the intervals $(-\infty, 0)$, $[0, 2a2^{(m-1)(a-1)})$, and $[2a2^{(m-1)(a-1)}, \infty)$, and estimate the corresponding integrals.

Note that the estimates for the intervals $[-2^m, -2^{m-1}]$ are similar. Thus, we obtain

$$\mathrm{I} \leqslant \frac{C}{2^{mb}} \cdot \frac{1}{2^{m\frac{(a-2)}{2}}} 2^{m(a-1)(1-\lambda)} = C\, 2^{m\left\{(a-1)(1-\lambda)-b-\frac{(a-2)}{2}\right\}},$$

and

$$\mathrm{II}+\mathrm{III} \leqslant C\, 2^{m(-b-(a-1)\lambda)}.$$

Now,

$$-b-(a-1)\lambda < (a-1)(1-\lambda)-b-\left(\frac{a-2}{2}\right), \quad \text{since} \quad a>0.$$

Hence

$$\|\hat{V}_m\|_{1/(1-\lambda)} \leqslant C\, 2^{m\left\{(a-1)(1-\lambda)-\left(\frac{a-2}{2}\right)-b\right\}}, \qquad \|V_m\|_{1/\lambda} \leqslant C\, 2^{m(\lambda-b)},$$

where C depends on a, b and λ but not on m. Thus,

$$\sum_{m=1}^{\infty} \|\hat{V}_m\|_{1/(1-\lambda)}^{t} \|V_m\|_{1/\lambda}^{1-t} \leqslant \sum_{m=1}^{\infty} 2^{m\left\{(a-1)(1-\lambda)-\left(\frac{a-2}{2}\right)-b\right\}t}\, 2^{m(\lambda-b)(1-t)}.$$

We are interested in those t's for which

$$\left\{(a-1)(1-\lambda)-\left(\frac{a-2}{2}\right)-b\right\}t+(\lambda-b)(1-t)<0,$$

or

$$\left\{(a-1)(1-\lambda)-\left(\frac{a-2}{2}\right)-b-(\lambda-b)\right\}t+(\lambda-b)<0,$$

or

$$\left\{\frac{a}{2}-a\lambda\right\}t+(\lambda-b)<0.$$

Therefore,

$$(\lambda-b)<t\{a(\lambda-\tfrac{1}{2})\}, \quad \text{or} \quad t>\frac{\lambda-b}{a(\lambda-\frac{1}{2})}.$$

From Corollary 4.25 we see the q's we are looking for satisfy

$$\frac{1}{q}=\frac{t(1-2\lambda)+2\lambda}{2}=\frac{2\lambda-(2\lambda-1)t}{2},$$

$$\frac{1}{q}<\frac{2\lambda-\frac{\lambda-b}{a(\lambda-\frac{1}{2})}(2\lambda-1)}{2}=\lambda-\frac{(\lambda-b)}{a}=\frac{\lambda(a-1)+b}{a},$$

and

$$q>\frac{a}{\lambda(a-1)+b}.$$

For the remaining values of q, the proof follows the same argument as worked out below in Case 2, but with $\frac{1}{2}<\lambda<1$.

Case 2. Assume $0<\lambda\leqslant 1/2$. Since $k(t)=\sum_{m=0}^{\infty} V_m(t)$ we have by part (ii) of Lemma 4.24,

$$\|k*f\|_s \leqslant \|V_0*f\|_s+\sum_{m=1}^{\infty}\|V_m*f\|_s$$

and by Lemma 4.26,

$$\|k*f\|_s \leqslant \|V_0\|_{1/\lambda}\|f\|_r+\sum_{m=1}^{\infty}\|V_m\|_\infty^{1-t}\|\hat{V}_m\|_{1/v}^{t}\|f\|_r$$

with $t=2\lambda p/(3p-2)$, $1/r-1/s=1-\lambda$ and $1/p=1/2+v$.

As we have shown in Case 1,

$$\|\hat{V}_m\|_{1/v} \leqslant \frac{C}{2^{mb}}\cdot\frac{1}{2^{m\frac{(a-2)}{2}}}2^{m(a-1)v} \quad \text{and} \quad \|V_m\|_\infty \leqslant \frac{C}{2^{mb}},$$

where C depends only on a, b and p but not on m. Now consider

$$\sum_{m=1}^{\infty}\left[\frac{1}{2^{mb}}\cdot\frac{1}{2^{m\frac{(a-2)}{2}}}2^{m(a-1)\left(\frac{1}{p}-\frac{1}{2}\right)}\right]^t\left(\frac{1}{2^{mb}}\right)^{1-t}$$
$$=\sum_{m=1}^{\infty}2^{m\left\{\left((a-1)\left(\frac{1}{p}-\frac{1}{2}\right)-\left(\frac{a-2}{2}\right)\right)t-b\right\}}.$$

Suppose

$$\frac{t}{2}<\frac{\lambda(a-1)+b}{a} \quad \text{(see Fig. 1)},$$

i.e. $2/t$ is a prospective q.

Set $t=2\left(\frac{\lambda(a-1)+b}{a}\right)-\delta$ with $\delta>0$ but δ small. Since

$$\frac{t}{2}=\frac{\lambda p}{3p-2} \quad \text{implies} \quad \frac{1}{2}-\frac{1}{2}=\frac{\lambda a-2\lambda+2b-\delta a}{2\lambda(a-1)+2b-\delta a},$$

then

$$\left\{(a-1)\left(\frac{1}{p}-\frac{1}{2}\right)-\left(\frac{a-2}{2}\right)\right\}t-b$$

$$=\left\{(a-1)\frac{(\lambda(a-2)+2b-\delta a)}{a}-\left(\frac{a-2}{2}\right)\left(\frac{2\lambda(a-1)+2b-\delta a}{a}\right)\right\}-b$$

$$=\frac{ba-\delta a^2/2}{a}-b=-\frac{\delta a}{2}<0.$$

Therefore,

$$k\in L_p^q \quad \text{for} \quad \frac{a}{\lambda(a-1)+b}<q<\frac{a}{\lambda-b} \text{ and } \frac{1}{p}-\frac{1}{q}=1-\lambda.$$

We note that by adding the Hardy–Littlewood kernels $1/|t|^\lambda$, we get the following:

Corollary 4.29. *Let* $0<\lambda<1$ *with* $t\in\mathbf{R}$ *and* $k(t)=e^{i|t|^a}/|t|^b$. *If* $a\neq 1$, $b\leqslant\lambda$ *and* $\frac{a}{2}\lambda+(b-\lambda)>0$, *then*

$$k\in L_p^q \quad \textit{for} \quad \frac{a}{\lambda(a-1)+b}<q<\frac{a}{\lambda-b}$$

and

$$k\notin L_p^q \quad \textit{for} \quad q>\frac{a}{\lambda-b},$$

with $1/p-1/q=1-\lambda$.

Examples in $\mathbf{R}^n$.

Example 4.30. *If*

$$k(t)=\frac{e^{i|t|^2}}{|t_1|^{2b}+\ldots+|t_n|^{2b}}, \quad n\geqslant 2 \textit{ and } 0<b<\lambda,$$

then

$$k\in L_p^q(\mathbf{R}^n) \quad \textit{for} \quad \frac{2n}{n\lambda+2b}<q<\frac{2n}{n\lambda-2b},$$

and

$$k\notin L_p^q(\mathbf{R}^n) \quad \textit{for} \quad q>\frac{2n}{n\lambda-2b},$$

with $1/p-1/q=1-\lambda$.

Proof. Since this argument is very similar to the ones found in Corollary 3.23 and Theorem 4.28, we shall be brief.

We define

$$V_1(t)=\frac{e^{i|t|^2}}{|t_1|^{2b}+\ldots+|t_n|^{2b}}\prod_{j=1}^{n}\chi_{[-1,1]}(t_j),$$

and for $m>1$,

$$V_m(t)=\frac{e^{i|t|^2}}{|t_1|^{2b}+|t_2|^{2b}+\ldots+|t_n|^{2b}}\left[\prod_{j=1}^{n}\chi_{[-2^m,2^m]}(t_j)-\prod_{j=1}^{n}\chi_{[-2^{m-1},2^{m-1}]}(t_j)\right].$$

We first do the case $1/2<\lambda<1$ and $q<2$ or $q>2/(2\lambda-1)$. We can show

$$\|\hat{V}_m\|_{1/(1-\lambda)}\leqslant C\frac{2^{mn(1-\lambda)}}{2^{2mb}} \quad \text{for} \quad m>1,$$

$$\|V_m\|_{1/\lambda}\leqslant C2^{m(n\lambda-2b)} \quad \text{for} \quad m\geqslant 1$$

and

$$\|\hat{V}_1\|_{1/(1-\lambda)}\leqslant C\|V_1\|_{1/\lambda},$$

then apply Corollary 4.25 to show that

$$k\in L_p^q \quad \text{for} \quad \frac{2n}{n\lambda+2b}<q<\frac{2n}{n\lambda-2b},$$

and $1/p-1/q=1-\lambda$. Here, C depends only on n, b and λ but not on m.

For the remaining cases, we show

$$\|V_m\|_\infty\leqslant\frac{C}{2^{m2b}},$$

$$\|\hat{V}_m\|_{1/v}\leqslant\frac{C2^{mn(1/p-1/2)}}{2^{m2b}} \quad \text{for} \quad m>1,$$

and

$$\|V_1*f\|_s\leqslant C\|V_1\|_{1/\lambda}\|f\|_r,$$

where $1/r-1/s=1-\lambda$, and here we apply the method of proof in Lemma 4.26 in conjunction with Lemma 4.24 to show

$$k\in L_p^q \quad \text{for} \quad \frac{2n}{n\lambda+2b}<q<\frac{2n}{n\lambda-2b}$$

and $1/p-1/q=1-\lambda$.

To show $k\notin L_p^q$ for $q>2n/(n\lambda-2b)$ and $1/p-1/q=1-\lambda$, we apply Lemma 2.18 in conjunction with Theorem 2.19.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO

Received June 25, 1974 (851)

STUDIA MATHEMATICA, T. LV. (1976)

L^p-Struktur in Banachräumen

von

EHRHARD BEHRENDS (Berlin)

Abstract. Let V be a Banach space, $1 \leqslant p \leqslant \infty$. Two closed subspaces $X, X^\perp$ are called L^p-summands, if (algebraically) $V = X \oplus X^\perp$ and for all $x \in X$, $x^\perp \in X^\perp$

$$\|x + x^\perp\|^p = \|x\|^p + \|x^\perp\|^p$$

(for $p = \infty$: $\|x + x^\perp\| = \max\{\|x\|, \|x^\perp\|\}$). L^p-projections (with respect to these decompositions), are projections onto L^p-summands.

It is shown that V has nontrivial L^p-summands for at most one p (the only exceptions: $(\boldsymbol{R}^2, \|\ \|_1)$ and $(\boldsymbol{R}^2, \|\ \|_\infty)$). For $p \neq 2$ L^p-projections commute.

1. Einleitung. Sei V ein $\boldsymbol{K}$-Banachraum, $1 \leqslant p \leqslant \infty$. Zwei abgeschlossene Unterräume $X, X^\perp$ von V heißen zueinander orthogonale L^p-*Summanden* (Schreibweise: $V = X \oplus_p X^\perp$), wenn algebraisch $V = X \oplus X^\perp$ gilt und für $x \in X$, $x^\perp \in X^\perp$ stets

$$\|x + x^\perp\|^p = \|x\|^p + \|x^\perp\|^p$$

(bzw. für $p = \infty$: $\|x + x^\perp\| = \max\{\|x\|, \|x^\perp\|\}$) ist. Projektionen e auf L^p-Summanden, die offensichtlich durch

$$\|v\|^p = \|ev\|^p + \|v - ev\|$$

(für $p = \infty$: $\|v\| = \max\{\|ev\|, \|v - ev\|\}$) für alle $v \in V$ charakterisiert sind, heißen L^p-*Projektionen*.

L^p-Summanden und L^p-Projektionen für $p = 1, \infty$ wurden — ausgehend von Arbeiten von Cunningham [4], [5] — in letzter Zeit besonders von [1], [2], [6], [8] untersucht. Die Arbeiten [9], [10], die den allgemeinen Fall $p \in [1, \infty]$ behandeln, setzen die Kommutativität von L^p-Projektionen voraus (etwa implizit dadurch, daß nur Räume betrachtet werden, in denen die Norm der Clarksonschen Ungleichung genügt) oder beschränken sich auf das Studium maximaler kommutierender Familien von L^p-Projektionen.

Untersuchungen der Menge sämtlicher L^p-Summanden im Falle klassischer Banachräume V ergaben ([7], [11a, e, f]), daß dort für $p \neq 2$ L^p-Projektionen kommutieren und daß für $\dim V > 2$ überhaupt nur ein Typ von nichttrivialen L^p-Projektionen existieren kann. In der vorlie-