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On the hyperbolic metric on Harnack parts

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Abstract. If B is a C^*-algebra with unit element e, S a subspace of B which contains e, and H a Hilbert space, then a compact Hausdorff space Ω and a subspace $\mathcal{M} \subset C_R(\Omega)$ are constructed in such manner that there is one-to-one correspondence between completely positive maps from S into $L(H)$ and positive functionals on $\mathcal{M}$. Using this "scalarization", a new proof of Arveson's extension theorem [1] is given and a relation between Harnack parts [14] and Gleason parts [5] is established. Hyperbolic metric on Harnack parts is introduced and a characterization of convergence in this metric is given.

Introduction. Our initial goal was to find a "Bear schema" (see [5]) in which the Harnack equivalence for the completely positive linear maps on a subspace of C^*-algebra B into $L(H)$ (see [14]) turns into the Gleason equivalence for the positive functionals on a suitable function space. Looking for this "scalarization", we remarked that it permits us to study completely positive maps on a subspace of B as positive functionals on a linear space of real valued continuous functions on a compact Hausdorff space. As the first step we present in Section 1 a new proof for the Arveson extension theorem ([1], [2]) (in a slightly more general form). More about implications of Choquet theory in the study of boundary representations, Šilov boundary for noncommutative case and other problems related to Arveson's papers [2], [3] we will take up in a subsequent paper.

In Section 2 we introduce the Harnack equivalence and establish its relation with Gleason equivalence. In analogy with scalar case we define in Section 3 the hyperbolic metric on Harnack parts and prove the equivalence between metric convergence and norm convergence of the corresponding (generalized) Radon–Nikodym derivatives.

Finally we make some remarks on continuous selection of mutually absolutely continuous spectral dilations and integral kernels, but the main facts in this direction remain for the further studies.

1. Let B be a C^*-algebra and $S \subset B$ a linear subspace such that the identity e of B belongs to S. For H, a Hilbert space, $\mathscr{B}(B; H)$ (respectively $\mathscr{B}(S; H)$) will denote the vector space of all bounded linear maps of B (respectively S) into $L(H)$. For $r>0$, let $\mathscr{B}_r(B;H)$ denote the closed ball of radius r in $\mathscr{B}(B;H)$.

We shall endow $\mathscr{B}(B;H)$ with the BW-topology. The *BW-topology* is the strongest locally convex topology on $\mathscr{B}(B;H)$ which coincides on each ball $\mathscr{B}_r(B;H)$, $r>0$, with the topology given by the following convergence: a net $\varphi_\nu\in\mathscr{B}_r(B;H)$ converges to $\varphi\in\mathscr{B}_r(B;H)$ if $\varphi_\nu(b)\to\varphi(b)$ in the weak operator topology, for every $b\in B$.

By standard arguments $\mathscr{B}_r(B;H)$ is BW-compact for any $r>0$ (see [11], [2]).

An element $\mu\in\mathscr{B}(B;H)$ is *positive* if $\mu(b)$ is a positive operator on H for any positive $b\in B$.

Let n be a positive integer. M_n will denote the C^*-algebra of all scalar $n\times n$ matrices. $B\oplus M_n$ denotes the C^*-algebra of all $n\times n$ matrices over B with the usual involution $(u_{ij})^*=(u_{ji}^*)$. For a matrix (u_{ij}) over B we put

$$\operatorname{Re}(u_{ij})=\tfrac{1}{2}[(u_{ij})+(u_{ij})^*].$$

The matrix (u_{ij}) over B is *positive*, if there exists a matrix (v_{ij}) over B such that $(u_{ij})=(v_{ij})^*(v_{ij})$.

We say that an element $\varphi\in\mathscr{B}(S;H)$ is *completely positive* if for each integer $n\geqslant 1$, for each $n\times n$ matrix (u_{ij}) over S for which the matrix $\operatorname{Re}(u_{ij})$ is positive and for each n-tuple $h_1,\ldots,h_n$ of elements of H we have

$$\operatorname{Re}\sum_{i,j}\big(\varphi(u_{ij})h_j,h_i\big)\geqslant 0.$$

If S is self-adjoint then φ is *completely positive* if and only if for each $n\geqslant 1$, for each positive $n\times n$ matrix (u_{ij}) over S and for each n-tuple $h,\ldots,h$ of element of H we have:

$$\sum_{i,j}\big(\varphi(u_{ij})h_j,h_i\big)\geqslant 0.$$

This is the definition of complete positivity used by W.F. Stinespring in [13] and W.B. Arveson in [1], [2].

Let μ be an element of $\mathscr{B}(B;H)$. Stinespring theorem [13] says that μ is completely positive if and only if there is a Hilbert space K, a bounded linear operator $V\colon H\to K$, and a representation π of B on K such that $\mu(h)=V^*\pi(b)V$ for every $b\in B$. For every completely positive μ we have:

$$\|\mu(b)\|=\|V^*\pi(b)V\|\leqslant\|V^*V\|\,\|b\|=\|\mu(e)\|\,\|b\|,$$

hence $\|\mu\|=\|\mu(e)\|$.

In what follows we denote by $\Omega=\Omega(B;H)$ the set of all completely positive linear maps $\mu\colon B\to L(H)$ such that $\|\mu(e)\|=1$.

It is easy to see that Ω is BW-closed, and because $\Omega\subset\mathscr{B}_1(B;H)$ it follows that Ω is BW-compact.

Let $\mathscr{M}$ be the set of all functions $w\colon\Omega\to\mathbf{R}$ for whic hthere exist an $n\times n$ matrix $(u_{ij})\in S\otimes M_n$, $n\geqslant 1$, and $h_1,\ldots,h_n\in H$ such that for every $\mu\in\Omega$ we have

$$w(\mu)=\operatorname{Re}\sum_{i,j}\big(\mu(u_{ij})h_j,h_i\big). \tag{1.1}$$

It is clear that any function w of the form (1.1) is a real valued continuous function on Ω.

LEMMA 1. *We have*

(i) *$\mathscr{M}$ is a linear subspace of $C_R(\Omega)$.*

(ii) *There exists $w\in\mathscr{M}$ such that for any $\mu\in\Omega$, $w(\mu)>0$.*

(iii) *For each $w\in\mathscr{M}$ there exist $n\geqslant 1$, a matrix $(\tilde u_{ij})\in S\otimes M_m$ and a orthonormal m-tuple $e_1,e_2,\ldots,e_m\in H$ such that*

$$w(\mu)=\operatorname{Re}\sum_{i,j=1}^{m}\big(\mu(\tilde u_{ij})e_j,e_i\big)\qquad(\mu\in\Omega).$$

Proof. (i) For $w_1,w_2\in\mathscr{M}$ of the form

$$w_1(\mu)=\operatorname{Re}\sum_{i,j=1}^{m}\big(\mu(u_{ij})h_j',h_i'\big),\qquad w_2(\mu)=\operatorname{Re}\sum_{i,j=1}^{p}\big(\mu(u_{ij}'')h_j'',h_i''\big)$$

and $\alpha,\beta\in\mathbf{R}$ the function $w_3=\alpha w_1+\beta w_2$ can be written as

$$w_3(\mu)=\operatorname{Re}\sum_{i,j}\big(\mu(u_{ij})h_j,h_i\big)$$

where the matrix (u_{ij}) is

$$u_{ij}=\begin{cases}\alpha u_{ij}', & 1\leqslant i,j\leqslant m,\\ \beta u_{i-m,j-m}'', & m+1\leqslant i,j\leqslant m+p,\\ 0, & \text{otherwise}\end{cases} \tag{1.2}$$

and

$$h_i=\begin{cases}h_i', & 1\leqslant i\leqslant m,\\ h_i'', & m+1\leqslant i\leqslant m+p.\end{cases}$$

Hence $w_3\in\mathscr{M}$, i.e. $\mathscr{M}$ is a linear subspace of $C_R(\Omega)$.

(ii) Let μ_0 be in Ω. Since $\|\mu_0(e)\|=1$, there exists $h_0\in H$ so that $\big(\mu_0(e)h_0,h_0\big)\neq 0$. The element $w_{\mu_0}\in\mathscr{M}$ defined by $w_{\mu_0}(\mu)=\big(\mu(e)h_0,h_0\big)$ is positive since μ is completely positive. We have $w_{\mu_0}(\mu_0)=\big(\mu_0(e)h_0,h_0\big)>0$. Since Ω is BW-compact, we can find a finite system $w_1,w_2,\ldots,w_n$ of positive functions in $\mathscr{M}$ such that for any $\mu\in\Omega$ there exists w_k, $1\leqslant k\leqslant n$, satisfying $w_k(\mu)>0$. If we put $w=\sum_{k=1}^{n}w_k$, then $w(\mu)>0$ for any $\mu\in\Omega$.

(iii) Let $w \in \mathcal{M}$ be of the form (1.1) and let $e_1, \ldots, e_m$ $(m \leqslant n)$ be an orthonormal base for linear subspace spanned by $h_1, \ldots, h_n$ in H.

Then

$$h_i = \sum_{k=1}^{m} c_{ki} e_k, \qquad i = 1, \ldots, n$$

and we have:

$$\begin{aligned} w(\mu) &= \operatorname{Re} \sum_{i,j} \big(\mu(u_{ij}) h_j, h_i\big) = \operatorname{Re} \sum_{i,j} \Big(\mu(u_{ij}) \sum_{p=1}^{m} c_{pj} e_p, \sum_{k=1}^{m} c_{ki} e_k\Big) \\ &= \operatorname{Re} \sum_{k,p} \Big(\sum_{i,j} \bar{c}_{ki} \mu(u_{ij}) c_{pj} e_p, e_k\Big) \\ &= \operatorname{Re} \sum_{k,p} \Big(\mu\Big(\sum_{i,j} \bar{c}_{ki} c_{pj} u_{ij}\Big) e_p, e_k\Big) \\ &= \operatorname{Re} \sum_{k,p} \big(\mu(\tilde{u}_{kp}) e_p, e_k\big). \end{aligned}$$

In the proof of the following lemma, which is the key lemma in our considerations, we have been inspired by some arguments used in Averson's proof of this extension theorem [2].

Lemma 2. *Let w be an element of $\mathcal{M}$:*

$$w(\mu) = \operatorname{Re} \sum_{i,j} \big(\mu(u_{ij}) e_j, e_i\big),$$

with orthonormal $e_1, \ldots, e_n \in H$. Then $w \geqslant 0$ if and only if the matrix $\operatorname{Re}(u_{ij})$ is positive.

Proof. We consider B embedded in $L(K)$, where K is a Hilbert space. Let $k_1, \ldots, k_n$ be an n-tuple of elements of K. Since $e_1, \ldots, e_n$ are linearly independent vectors in H, there exists a bounded linear operator $V: H \to K$ defined by $Ve_i = k_i$, $1 \leqslant i \leqslant n$, and $V = 0$ on the orthogonal complement of the linear subspace spanned by $e_1, \ldots, e_n$ in H.

We consider $\mu_V : B \to L(K)$ defined by

$$\mu_V(b) = \frac{1}{\|V\|^2} V^* b V \qquad (b \in B \subset L(K)).$$

The map $\mu_V \in \Omega$. Indeed $\|\mu_V(e)\| = 1$, and for each positive matrix $(b_{ij}) \in B \otimes M_m$ and each $h_1, \ldots, h_m \in H$ we have

$$\begin{aligned} \sum_{i,j=1}^{m} \big(\mu_V(b_{ij}) h_j, h_i\big) &= \frac{1}{\|V\|^2} \sum_{i,j=1}^{m} (V^* b_{ij} V h_j, h_i) \\ &= \frac{1}{\|V\|^2} \sum_{i,j=1}^{m} (b_{ij} V h_j, V h_i) \geqslant 0. \end{aligned}$$

We have:

$$\begin{aligned} \operatorname{Re} \sum_{i,j} (u_{ij} k_j, k_i) &= \operatorname{Re} \sum_{i,j} (u_{ij} V e_j, V e_i) = \operatorname{Re} \sum_{i,j} (V^* u_{ij} V e_j, e_i) \\ &= \operatorname{Re} \|V\|^2 \sum_{i,j} \big(\mu_V(u_{ij}) e_j, e_i\big) = \|V\|^2 w(\mu_V) \geqslant 0. \end{aligned}$$

Hence the operator matrix $\operatorname{Re}(u_{ij})$ is positive. It follows that $\operatorname{Re}(u_{ij})$ is a positive matrix over B.

Conversely, if $\operatorname{Re}(u_{ij})$ is a positive matrix then

$$w(\mu) = \operatorname{Re} \sum_{i,j} \big(\mu(u_{ij}) e_j, e_i\big) \geqslant 0,$$

by the complete positivity of $\mu \in \Omega$.

Theorem 1 (Arveson [1], [2]). *For each completely positive mapping $\varphi: S \to L(H)$ there exists a completely positive mapping $\mu: B \to L(H)$ such that $\mu|_S = \varphi$.*

Proof. Let $\varphi_0: S \to L(H)$ be completely positive. For w in $\mathcal{M}$ of the form

$$w(\mu) = \operatorname{Re} \sum_{i,j=1}^{n} \big(\mu(u_{ij}) h_j, h_i\big), \qquad \mu \in \Omega,$$

let us consider the sum $\operatorname{Re} \sum_{i,j=1}^{n} \big(\varphi_0(u_{ij}) h_j, h_i\big)$. If w is positive, then this sum is positive. To see this, choose an orthonormal base $e_1, \ldots, e_m$ $(m \leqslant n)$ for the linear space spanned by $h_1, \ldots, h_n$ in H. If $h_i = \sum_{k=1}^{m} a_{ki} e_k$, $i = 1, 2, \ldots, n$, then we have

$$\begin{aligned} \operatorname{Re} \sum_{i,j} \big(\mu(u_{ij}) h_j, h_i\big) &= \operatorname{Re} \sum_{k,p} \Big(\mu\Big(\sum_{i,j} \bar{a}_{ki} a_{pj} u_{ij}\Big) e_p, e_k\Big) \\ &= \operatorname{Re} \sum_{k,p} \big(\mu(\tilde{u}_{kp}) e_p, e_k\big) \end{aligned}$$

and

$$\operatorname{Re} \sum_{i,j} \big(\varphi_0(u_{ij}) h_j, h_i\big) = \ldots = \operatorname{Re} \sum_{k,p} \big(\varphi_0(\tilde{u}_{kp}) e_p, e_k\big).$$

Since $w \geqslant 0$, using Lemma 2, we conclude that the matrix $\operatorname{Re}(\tilde{u}_{kp})$ is positive and since φ_0 is completely positive, we obtain

$$\operatorname{Re} \sum \big(\varphi_0(u_{ij}) h_j, h_i\big) \geqslant 0.$$

Further, if $w_1, w_2 \in \mathcal{M}$,

$$w_1(\mu) = \operatorname{Re} \sum_{i,j=1}^{m} \big(\mu(u'_{ij}) h_j, h_i\big), \qquad w_2(\mu) = \operatorname{Re} \sum_{i,j=1}^{p} \big(\mu(u''_{ij}) h''_j, h''_i\big),$$

and $\alpha,\beta\in\boldsymbol{R}$, and if we write $w_3 = \alpha w_1+\beta w_2$ as in (1.2), then we have

$$\begin{aligned}\operatorname{Re}\sum_{i,j=1}^{m+p}&(\varphi_0(u_{ij})h_j, h_i)\\ &= \operatorname{Re}\sum_{i,j=1}^{m}(\varphi_0(\alpha u'_{ij})h'_j, h'_i)+\operatorname{Re}\sum_{i,j=m+1}^{m+p}(\varphi_0(\beta u_{i-m,j-m})h''_{j-m}, h''_{i-m})\\ &= \alpha\operatorname{Re}\sum_{i,j=1}^{m}(\varphi_0(u'_{ij})h'_j, h'_i)+\beta\operatorname{Re}\sum_{i,j=1}^{p}(\varphi_0(u''_{ij})h''_j, h''_i).\end{aligned}$$

It is clear now that the map

$$w\to\operatorname{Re}\sum_{i,j=1}^{m}(\varphi_0(u_{ij})h_j, h_i)$$

is a well defined positive linear functional on $\mathscr{M}$. Let us write

$$L_{\varphi_0}(w) = \operatorname{Re}\sum_{i,j=1}^{m}(\varphi_0(u_{ij})h_j, h_i).$$

Since, by Lemma 1 (ii), $\mathscr{M}$ contains a strictly positive function, it results that $\mathscr{M}$ contains an interior point of the positive cone of $C_R(\Omega)$. A familiar theorem of M. Krein (cf. [8], Ch. II, § 3, Prop. 6) implies that L_{φ_0} has a positive linear extension to $C_R(\Omega)$ and, consequently, we can extend L_{φ_0} to a positive linear functional $\tilde{L}_{\varphi_0}$ on $C(\Omega)$.

For $b\in B$ and $h, k\in H$ let $w_{b;h,k}$ be the function in $C(\Omega)$ defined by

$$w_{b;h,k}(\mu) = (\mu(b)h, k).$$

It is easy to verify that for a fixed b in B, $(h,k)\to\tilde{L}_{\varphi_0}(w_{b;h,k})$ is a bounded bilinear form. Then there exists an operator $\mu_0(b)$ in $L(H)$ such that

$$(\mu_0(b)h, k) = \tilde{L}_{\varphi_0}(w_{b;h,k})\qquad (h, k\in H).$$

Now, let μ_0 be the map from B into $L(H)$ defined as $\mu_0\colon b\to\mu_0(b)$. A simple calculation shows that μ_0 is a linear extension of φ_0 to B.

It remains to prove that μ_0 is completely positive. For this, let $(b_{ij})\in B\otimes M_n$ be a positive matrix and $h_1,\dots,h_n$ in H. Then:

$$\sum_{i,j}(\mu_0(b_{ij})h_j, h_i) = \sum_{i,j}\tilde{L}_{\varphi_0}(w_{b_{ij};h_j,h_i}) = \tilde{L}_{\varphi_0}\Big(\sum_{i,j}w_{b_{ij};h_j,h_i}\Big)\geqslant 0$$

because $\tilde{L}_{\varphi_0}$ is positive and

$$\sum_{i,j}w_{b_{ij};h_j,h_i}(\mu) = \sum_{i,j}(\mu(b_{ij})h_j, h_i)\geqslant 0\qquad\text{for every } \mu\in\Omega.$$

The proof of the theorem is complete.

2. Let φ be a completely positive linear map of S into $L(H)$. Arvenson's theorem says that φ has a completely positive linear extension $\mu\colon B\to L(H)$. Stinespring's theorem implies that there exists a Hilbert space K, a bounded linear operator $V\colon H\to K$ and a representation π of B on K such that

$$\mu(b) = V^*\pi(b)V\qquad (b\in B).$$

Such a triplet $[K, V, \pi]$ we shall call a *dilation of* φ. The dilation $[K, V, \pi]$ of φ is called *minimal* if K is the closed linear span of all vectors $\pi(b)Vh$, $b\in B$, $h\in H$. It is easy to see that if φ has a dilation then it has a minimal one. In what follows all considered dilation will be supposed to be minimal.

For φ_1,φ_2 completely positive linear maps, let us write $\varphi_1\leqslant\varphi_2$ if $\varphi_2-\varphi_1$ is completely positive.

The following theorem was proved in [14] in the case $B = C(X)$, with X a compact Hausdorff space. Since the proof is exactly the same in the noncommutative case we shall omit it.

THEOREM 2. *Let φ_1,φ_2 be completely positive linear maps from S into $L(H)$. The following assertions are equivalent:*

(i) *There exists a constant c $(0<c<1)$ such that*

$$c\varphi_2\leqslant\varphi_1\leqslant c^{-1}\varphi_2.$$

(ii) *There exist a constant c $(0<c<1)$, an extension μ_1 for φ_1 and an extension μ_2 for φ_2 such that*

$$c\mu_2\leqslant\mu_1\leqslant c^{-1}\mu_2.$$

(iii) *There exist dilations $[K_1, V_1, \pi_1]$ and $[K_2, V_2, \pi_2]$ of φ_1 and φ_2 respectively, and a bounded linear operator $S\colon K_2\to K_1$ with bounded inverse, such that*

$$SV_2 = V_1\quad\text{and}\quad \pi_2(b) = S^{-1}\pi_1(b)S\qquad (b\in B).$$

Let us now consider the set:

$$\Omega(S;H) = \{\varphi\colon S\to L(H);\ \varphi\text{ completely positive linear map, } \|\varphi(e)\| = 1\}.$$

If we endow $\mathscr{B}(S;H)$ with the BW-topology, in a similar way as $\mathscr{B}(B;H)$ was endowed, then it is easy to see that $\Omega(S;H)$ is a compact convex subset of $\mathscr{B}(S;H)$.

We say that $\varphi_1,\varphi_2\in\Omega(S;H)$ are *Harnack equivalent* if they satisfy one of the (equivalent) assertions of Theorem 2. The equivalence classes induced by this equivalence relation will be called the *Harnack parts of* $\Omega(S;H)$. The concept is analogous in noncommutative case to that of Gleason parts defined in the context of function spaces (see [10], [7], [4], [5]).

Moreover we will show that in an adequate "Bear schema" attached to S and B, Harnack equivalence turns into Gleason equivalence.

Let us recall "Bear schema" of Gleason parts in a slightly more general case. Let Ω be a compact Hausdorff space and $\mathcal{M} \subset C_R(\Omega)$ a linear subspace which separates the points of Ω and contains a strictly positive function ($\mathcal{M}$ cannot contain non-zero constants). Let $\mathcal{M}'$ be the dual of $\mathcal{M}$ and denote by

$$T_{\mathcal{M}} = \{L \in \mathcal{M}';\ L \geqslant 0,\ \|L\| = 1\}.$$

The space Ω can be embedded in $\mathcal{M}'$ as a point-evaluation functionals, and then $T_{\mathcal{M}}$ is the compact convex closure (in weak*-topology) of Ω. Two elements $L_1, L_2 \in T_{\mathcal{M}}$ are called *Gleason equivalent* if there exists a constant c, $0 < c \leqslant 1$, such that for any positive function $u \in \mathcal{M}$ we have

$$cL_1(u) \leqslant L_2(u) \leqslant \frac{1}{c} L_1(u).$$

A consistent study of the Gleason parts induced by this equivalence on $T_{\mathcal{M}}$ was done by H.S. Bear in [5].

Let now $\Omega = \Omega(B; H)$ and $\mathcal{M} \subset C_R(\Omega)$ be as in the first section. If $\varphi \in \Omega(S; H)$, we have already seen how the positive functional L_φ on $\mathcal{M}$ can be defined. In fact $L_\varphi \in T_{\mathcal{M}}$ and $\varphi \to L_\varphi$ is a one-to-one mapping from $\Omega(S; H)$ onto $T_{\mathcal{M}}$. Indeed, if $\varphi \in \Omega(S; H)$ and μ is a completely positive extension of φ to B then $\|\mu\| = \|\varphi\| = 1$, i.e. $\mu \in \Omega(B; H)$. For $w \in \mathcal{M}$ with $\|w\| \leqslant 1$ we have

$$|L_\varphi(w)| = |w(\mu)| \leqslant 1.$$

If we take now $h \in H$ such that $|(\varphi(e)h, h)| = 1$ then

$$\|w_{e;h,h}\| \leqslant 1 \quad \text{and} \quad |L_\varphi(w_{e;h,h})| = |(\varphi(e)h, h)| = 1.$$

It results $\|L_\varphi\| = 1$ hence $L_\varphi \in T_{\mathcal{M}}$. Clearly $\varphi \to L_\varphi$ is a one-to-one mapping from $\Omega(S; H)$ into $T_{\mathcal{M}}$. If $L \in T_{\mathcal{M}}$ then, using similar arguments as in the last part of the proof of Theorem 1, we can construct $\mu \in \Omega(B; H)$ such that $L(w) = w(\mu)$, $w \in \mathcal{M}$, and taking $\varphi = \mu|_S$ we have $\varphi \in \Omega(S; H)$ and $L = L_\varphi$.

THEOREM 3. *$\varphi_1, \varphi_2 \in \Omega(S; H)$ are Harnack equivalent if and only if $L_{\varphi_1}, L_{\varphi_2}$ are Gleason equivalent.*

Proof. Let φ_1, φ_2 be Harnack equivalent. This happens if and only if there exists c, $0 < c < 1$ such that $c\varphi_2 \leqslant \varphi_1 \leqslant c^{-1}\varphi_2$ or, equivalently, for each matrix $(u_{ij}) \in S \otimes M_n$ so that $\operatorname{Re}(u_{ij}) \geqslant 0$ and each n-tuple $h_1, \ldots, h_n$ of elements of H we have:

$$c\operatorname{Re}\sum_{i,j}(\varphi_2(u_{ij})h_j, h_i) \leqslant \operatorname{Re}\sum_{i,j}(\varphi_1(u_{ij})h_j, h_i) \leqslant c^{-1}\operatorname{Re}\sum_{i,j}(\varphi_2(u_{ij})h_j, h_i).$$

By Lemma 2 and definition of L_φ, this is equivalent to:

$$cL_{\varphi_2}(w) \leqslant L_{\varphi_1}(w) \leqslant \frac{1}{c} L_{\varphi_2}(w)$$

for every positive w in $\mathcal{M}$.

The proof of the theorem is complete.

3. As in the scalar case we can define on a Harnack part of $\Omega(S; H)$ the *hyperbolic metric*, setting for φ_1, φ_2 in the same Harnack part

$$d(\varphi_1, \varphi_2) = \inf\left\{\log\frac{1}{c};\ 0 < c < 1,\ c\varphi_2 \leqslant \varphi_1 \leqslant \frac{1}{c}\varphi_2\right\}.$$

Because of Theorem 3 we have

$$d(\varphi_1, \varphi_2) = d(L_{\varphi_1}, L_{\varphi_2})$$

where the hyperbolic metric on Gleason part is defined as in [5].

Many facts about hyperbolic metric on Harnack parts can be obtained by a simple reformulation in this context of the known results relative to hyperbolic metric on Gleason parts.

We point out only the following

COROLLARY. *The hyperbolic metric is complete on Harnack part of $\Omega(S; H)$.*

On the other hand, one of the most important results from Bear's theory of hyperbolic metric on Gleason parts, which establish the equivalence between convergence on Gleason part and the convergence in L^∞-norm of corresponding Radon-Nikodym derivatives, has not an immediate reformulation in the context of Harnack parts. This is what we intend to do in the remainder of this section.

Let $\Omega(S; H)$ and $\Omega(B; H)$ be as above. We can define the Harnack parts on $\Omega(B; H)$ by using the Harnack inequalities for elements in $\Omega(B; H)$. According to Theorem 2 (ii), if $\varphi_1, \varphi_2 \in \Omega(S; H)$ belong to the same Harnack part of $\Omega(S; H)$ then we can extend φ_1, φ_2 to $\mu_1, \mu_2 \in \Omega(B; H)$ which lie in the same Harnack part of $\Omega(B; H)$.

But it is known ([5]) that (even in the scalar case) it is not generally true that for any Harnack part Δ of $\Omega(S; H)$ there is a Harnack part Δ of $\Omega(B; H)$ which contains extentions to B for every element of Δ. We first give a description of metric convergence on the Harnack part of $\Omega(B; H)$ and finally we shall return with some comments relative to this question.

Let $\mu_1, \mu_2 \in \Omega(B; H)$ be Harnack equivalent. According to Theorem 2 (iii), there exist the dilations $[K_1, V_1, \pi_1]$ and $[K_2, V_2, \pi_2]$ of μ_1 and μ_2 respectively, and a bounded invertible operator $S\colon K_2 \to K_1$ such that $SV_2 = V_1$ and $S\pi_2(b) = \pi_1(b)S$, $b \in B$.

For any $b\in B$ we have

$$\mu_2(b) = V_2^*\pi_2(b)V_2 = V_2^*S^{-1}\pi_1(b)SV_2 = V_1^*(S^{-1})^*S^{-1}\pi_1(b)V_1.$$

Write $D = (S^{-1})^*S^{-1}$; then D is a positive operator on K_1 belonging to $\pi_1(B)'$, the commutant of $\pi_1(B)$ in $L(K_1)$, and we have

$$\mu_2(b) = V_1^*D\pi_1(b)V_1 \qquad (b\in B).$$

Let us call D the *Radon–Nikodym derivative* of μ_2 with respect to μ_1.

It is easy to show (see the proof of Theorem 2 in [14]) that if D is a positive operator in $\pi_1(B)'$ then, setting $\mu(b) = V_1^*D\pi_1(b)V_1$, $b\in B$, we obtain an element $\mu\in\Omega(B;H)$ Harnack equivalent to μ_1.

Let now Δ be a Harnack part of $\Omega(B;H)$ and let us fix an element μ_0 in Δ and denote by $[K_0, V_0, \pi_0]$ its (minimal, necessarily unique) dilation. For μ in Δ let us denote by D_μ its Radon–Nikodym derivative with respect to μ_0.

THEOREM 4. *A sequence $\{\mu_n\}$ in Δ is convergent in the hyperbolic metric on Δ if and only if the sequence $\{D_{\mu_n}\}$ of corresponding Radon–Nikodym derivatives is convergent in the norm metric on $L(K_0)$.*

Proof. Consider $\Omega = \Omega(B;H)$ and $\mathcal{M}\subset C_R(\Omega)$ constructed as in Section 1 with B instead of S. According to Theorem 3, if $\mu,\nu\in\Delta$, we have:

$$d(\mu,\nu) = \inf\left\{\log\frac{1}{c};\ c\mu\leqslant\nu\leqslant\frac{1}{c}\mu\right\}$$
$$= \sup\{|\log w(\mu)-\log w(\nu)|,\ w\in\mathcal{M},\ w>0\}.$$

Thus if $\mu_n,\mu\in\Delta$ then $d(\mu_n,\mu)\to 0$ if and only if $w(\mu_n)\to w(\mu)$ uniformly for $w\in\mathcal{M}$, $w>0$, $w(\mu_0)\leqslant 1$.

Suppose that D_{μ_n} converges to D_μ in the norm metric on $L(K_0)$. Since D_μ is a positive operator in $\pi_0(B)'$, we have that μ defined as $\mu(b) = V_0^*D_\mu(b)V_0$, $b\in B$, belongs to Δ. We have

$$|w(\mu_n)-w(\mu)| \leqslant \Big|\sum_{i,j}(\mu_n(u_{ij})h_j, h_i) - \sum_{i,j}(\mu(u_{ij})h_j, h_i)\Big|$$
$$= \Big|\sum_{i,j}(V_0^*D_{\mu_n}\pi_0(u_{ij})V_0h_j, h_i) - \sum_{i,j}(V_0^*D_\mu\pi_0(u_{ij})V_0h_j, h_i)\Big|$$
$$= \Big|\sum_{i,j}\Big(V_0^*(D_{\mu_n}-D_\mu)\pi_0\Big(\sum_m v_{mi}^*v_{mj}\Big)V_0h_j, h_i\Big)\Big|$$
$$= \Big|\sum_m\Big((D_{\mu_n}-D_\mu)\sum_j\pi_0(v_{mj})V_0h_j, \sum_i\pi_0(v_{mi})V_0h_i\Big)\Big|$$
$$\leqslant \|D_{\mu_n}-D_\mu\|\sum_m\Big\|\sum_j\pi_0(v_{mj})V_0h_j\Big\|^2$$

$$= \|D_{\mu_n}-D_\mu\|\sum_m\Big(\sum_j\pi_0(v_{mj})V_0h_j, \sum_i\pi_0(v_{mi})V_0h_i\Big)$$
$$= \|D_{\mu_n}-D_\mu\|\sum_{i,j}(V_0^*\pi_0(u_{ij})V_0h_j, h_i)$$
$$= \|D_{\mu_n}-D_\mu\|w(\mu_0)\leqslant\|D_{\mu_n}-D_\mu\|.$$

Hence $\|D_{\mu_n}-D_\mu\|\to 0$, $n\to\infty$, implies $d(\mu_n,\mu)\to 0$, $n\to\infty$.

Conversely, if we suppose that $d(\mu_n,\mu)\to 0$, $n\to\infty$, and D_{μ_n} does not converge to D_μ, then there exists $\varepsilon_0>0$ and a subsequence $\{\mu_{n_p}\}$ such that $\|D_{\mu_{n_p}}-D_\mu\|>\varepsilon_0$, hence there is $k_p\in K_0$, $\|k_p\|\leqslant 1$, such that

$$|((D_{\mu_{n_p}}-D_\mu)k_p, k_p)|>\varepsilon_0.$$

Since K is the closed linear span of all vectors $\pi_0(b)Vh$, $b\in B$, $h\in H$, we can consider $k_p = \sum_i\pi_0(b_i^p)V_0h_i^p$, $b_i^p\in B$, $h_i^p\in H$, $i=1,2,\ldots,n$. Let $w_p\in\mathcal{M}$ be defined as

$$w_p(\nu) = \operatorname{Re}\sum_{i,j}\big(\nu((b_i^p)^*b_j^p)h_j^p, h_i^p\big), \qquad (\nu\in\Omega(B;H)).$$

We have $w_p\geqslant 0$ and

$$w_p(\mu_0) = \operatorname{Re}\sum_{i,j}\big(\mu_0((b_i^p)^*b_j^p)h_j^p, h_i^p\big) = \operatorname{Re}\sum_{i,j}\big(V_0^*\pi_0((b_i^p)^*b_j^p)V_0h_j^p, h_i^p\big)$$
$$= \operatorname{Re}\Big(\sum_j\pi_0(b_j^p)V_0h_j^p, \sum_i\pi_0(b_i^p)V_0h_i^p\Big) = \|k_p\|^2\leqslant 1.$$

On the other hand

$$|w_p(\mu_{n_p})-w_p(\mu)| = \Big|\sum_{i,j}\big(\mu_{n_p}((b_i^p)^*b_j^p)h_j^p, h_i^p\big) - \sum_{i,j}\big(\mu((b_i^p)^*b_j^p)h_j^p, h_i^p\big)\Big|$$
$$= \Big|\sum_{i,j}\big(V_0^*(D_{\mu_n}-D_\mu)\pi_0((b_i^p)^*b_j^p)V_0h_j^p, h_i^p\big)\Big|$$
$$= \Big|\Big((D_{\mu_{n_p}}-D_\mu)\sum_j\pi_0(b_j^p)V_0h_j^p, \sum_i\pi_0(b_i^p)V_0h_i\Big)\Big|$$
$$= |((D_{\mu_{n_p}}-D_\mu)k_p, k_p)|>\varepsilon_0.$$

If we take $w_p' = (1+\delta)^{-1}(w_p+\delta w_0)$, where $w_0\in\mathcal{M}$, $0<w_0\leqslant 1$, and $\delta>0$ sufficiently small, then we have $w_p'>0$, $w_p'(\mu_0)\leqslant 1$ and

$$|w_p'(\mu_{n_p})-w_p'(\mu)|>\varepsilon_0$$

which contradicts $d(\mu_n,\mu)\to 0$, $n\to\infty$.

The proof of the theorem is complete.

4. As we already remarked, if Δ is a Harnack part in $\Omega(S;B)$, then, in general, it is not possible to find a Harnack part of $\Omega(B;H)$ containing

extensions of any element $\varphi \in A$. An example in this matter was given by Bear in [5] (for the scalar case). Like in the scalar case, in this context, interesting problems relative to the selection of mutually absolutely continuous dilations and integral kernels arise.

From this point of view, the case of functional calculi for contraction is of particular interest.

Let $\boldsymbol{T}$ be the one-dimensional torus in complex plane, $B = C(\boldsymbol{T})$ and $S = A$, the disc algebra. If T is a contraction on a Hilbert space H, then the von Neumann functional calculus with function in A gives up an element $\varphi_T \in \Omega(A; H)$. We say that two contractions T_1, T_2 on H are *Harnack equivalent* if $\varphi_{T_1}, \varphi_{T_2}$ lie in the same Harnack part of $\Omega(A; H)$ (cf. [15]). In [9] C. Foiaş proved that the set of all strict contractions on H forms a Harnack part, the Harnack part of contraction 0. Let us remark that in this case we have a usual formula for hyperbolic distance, namely:

$$d(T, 0) = \log\frac{1+\|T\|}{1-\|T\|}.$$

Since in this case we have the unique dilation, according to Theorem 2 there is a Harnack part in $\Omega\big(C(\boldsymbol{T}); H\big)$ which contains extension to $C(\boldsymbol{T})$ of φ_T for any strict contraction T.

If we take contraction 0 as a center of the Harnack part of strict contractions, and the bilateral shift of multiplicity $\dim H$ as a unitary dilation of 0, then the Radon–Nikodym derivative of a strict contraction will be a positive operator in the commutant of the bilateral shift of $\dim H$.

Corollary. *Let T_n, T be strict contractions on H and D_n, D the corresponding Radon–Nikodym derivatives. Then T_n converge to T in the hyperbolic metric if and only if D_n converge to D in the norm metric.*

The problem of selection of mutually absolutely dilations effectively appear in the case of functional calculus for pairs of commuting contractions [12]. In this case we have unitary dilation (Ando Theorem), but it is no more unique.

In a subsequent paper we shall study in details the topic of selection of mutually absolutely continuous dilations and integral kernels.

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