ERRATA

| Page, line | For | Read |
|---|---|---|
| $117_4$ | $\Delta_{np}$ | $\Delta_{np}$ |
| $145_{11}$ | $(2C_r)^{-1/w}$ | $(2C_r)^{-1/r}$ |
| $160^{12}$ | $K^\circ = \mathrm{co}\,(K^\circ B_r \setminus (x_1, \ldots, x_n))$ | $K^\circ = \mathrm{co}\,(K^\circ \setminus B_r(x_1, \ldots, x_n))$ |
| $164^2$ | $x^{(n,\sigma)})$ | $x^{(n,\sigma)}$ |
| $180_1$ | $\sum\limits_{n=1}^{\infty}$ | $\sum\limits_{n=0}^{\infty}$ |

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# On the non-existence of linear isomorphisms between Banach spaces of analytic functions of one and several complex variables

by

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**Abstract.** Let $\Phi$ be a closed bounded domain of holomorphy in $\mathbb{C}^n$ $(n \geqslant 2)$ and let the boundary of $\Phi$ be a $C^2$-smooth surface in the $2n$-dimensional real vector space. Then the Banach space $A(\Phi)$ of all functions continuous on $\Phi$ and holomorphic in the interior of $\Phi$ is not isomorphic to the disc algebra $A(D)$, where $D = \{z \in \mathbb{C}: |z| \leqslant 1\}$. In particular, for $n \geqslant 2$, the spaces $A(B_n)$ and $A(D)$ are not isomorphic, where $B_n = \{\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n: \sum_{j=1}^{n} |z_j|^2 \leqslant 1\}$.

Let $\Phi$ be a bounded closed domain of holomorphy in the $n$-dimensional complex vector space $\mathbb{C}^n$ and let $\partial\Phi$ denote the boundary of $\Phi$. By $A(\Phi)$ we denote the Banach space of all continuous complex-valued functions on $\Phi$ which are holomorphic on $\Phi \setminus \partial\Phi$. In the present paper we prove that under some general conditions on $\Phi$ with $\Phi \subset \mathbb{C}^n$ and $n \geqslant 2$, $A(\Phi)$ as a Banach space is not isomorphic to the disc algebra $A(D)$, where $D = \{z \in \mathbb{C}: |z| \leqslant 1\}$; for instance it is enough to assume that the boundary $\partial\Phi$ is a $C^2$-smooth surface (in the $2n$-dimensional real vector space). In particular, if

$$B_n = \left\{\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n: \sum_{j=1}^{n} |z_j|^2 \leqslant 1\right\}$$

then, for $n \geqslant 2$, the Banach space $A(B_n)$ is not isomorphic to $A(D)$. This solves a problem of Henkin. Let us mention that Henkin ([5], [6]) has proved that if $n = 1, 2, \ldots$ and $m = 2, 3, \ldots$, then the spaces $A(B_n)$ and $A(D^m)$ are not isomorphic. (Here $D^m$ denotes the polydisc of dimension $m$.) In particular, the spaces $A(B_n)$ and $A(D^n)$ are not isomorphic for $n \geqslant 2$. It is still unknown whether, for $n > m \geqslant 2$, the spaces $A(B_n)$ and $A(B_m)$ (and similarly, $A(D^n)$ and $A(D^m)$) are or are not isomorphic (cf. [8], Problems 55 and 56).

The present paper consists of three sections. In the first, an isomorphic invariant of a subspace of a $C(S)$-space with a small annihilator (for instance, with the norm separable annihilator) is discussed. The invariant involves the concept of absolutely summing operator. Note that, by the classical F. and M. Riesz Theorem, the disc algebra $A(D)$ regarded as a subspace of $C(\partial D)$ has a small annihilator. The second section contains a proof of the fact that if a domain $\Phi$ in $\mathfrak{C}^n$ contains, roughly speaking, uncountably many analytic discs in transversal positions and the intersections of the discs with $\partial\Phi$ are mutually disjoint, then $A(\Phi)$ is not isomorphic to any subspace of a $C(S)$-space with a small annihilator. In the third section we prove that $A(D)$ is not isomorphic to $A(\Phi)$ for various $\Phi$ by showing that certain either geometric or differential conditions on $\Phi$ imply the existence in $\Phi$ of a family of analytic discs with the properties discussed in Section 2.

**1. Banach spaces with small annihilators.** Throughout this paper $S$ stands for a compact Hausdorff space. We identify (via the Riesz Representation Theorem (cf. [2], IV. 6.3)) the dual $[C(S)]^*$ with the Banach space of all complex Borel measures on $S$ with the norm $\|\mu\|$ = the total variation of $\mu$. We put $[C(S)]^*_+ = \{\mu\in[C(S)]^*\colon \mu \text{ non-negative}\}$. Given a $\mu\in[C(S)]^*_+$ and a $\nu\in[C(S)]^*$, we write $\nu\ll\mu$ if $\nu$ is absolutely continuous with respect to $\mu$, and $\nu\perp\mu$ if $\nu$ is singular with respect to $\mu$. We identify (via the Radon–Nikodym Theorem (cf. [2], III.10.2)) the space $L^1(\mu) = L^1(\mu, S)$ with the subspace $\{\nu\in[C(S)]^*\colon \nu\ll\mu\}$. For a $\mu\in[C(S)]^*_+$ we denote by $i_\mu\colon C(S)\to L^1(\mu)$ the natural map which assigns to a continuous function in $C(S)$, its $\mu$-equivalence class regarded as an element of $L^1(\mu)$. If $X$ is a closed linear subspace of $C(S)$, then the annihilator of $X$ is the subspace $X^\perp = \{\nu\in[C(S)]^*\colon \int_S x(s)\,d\nu = 0 \text{ for all } x\in X\}$. For a $\mu\in[C(S)]^*_+$ we denote by $X_\mu$ the closure of $i_\mu(X)$ in $L^1(\mu)$.

We shall need also the concepts of absolutely summing operators and integral ones. A linear operator $U\colon Y\to E$ ($Y$, $E$ Banach spaces) is called *absolutely summing* if for some (equivalently for every) isometric embedding $j$ of $Y$ into a $C(S)$-space there exists a $\mu\in[C(S)]^*_+$ such that

$$\|U(y)\|\leqslant \int_S |j(y)(s)|\,d\mu \quad \text{for every } y\in Y. \tag{1.1}$$

We put $\pi_1(U) = \inf\|\mu\|$, where the infimum is extended over all $\mu$ satisfying (1.1) for a fixed isometric embedding of $Y$ into a $C(S)$-space. In fact, the quantity $\pi_1(U)$ is independent of a particular choice of an isometric embedding of $Y$ into a $C(S)$-space, and for every fixed isometric embedding there exists a $\mu$ satisfying (1.1) with $\|\mu\| = \pi_1(U)$. An operator $V\colon Y\to E$ is called *integral* if there exist a compact Hausdorff space $S$, a $\nu\in[C(S)]^*_+$ and operators $A\colon Y\to C(S)$ and $B\colon L^1(\nu)\to E^{**}$ such that

$$Bi_\nu A = \varkappa V, \qquad \|A\|\,\|B\|\leqslant 1, \tag{1.2}$$

where $\varkappa\colon E\to E^{**}$ denotes the canonical embedding.



We put $i(V) = \inf\|\nu\|$, where the infimum is extended over all possible $A$, $B$, $\nu$ and $S$ satisfying (1.2). The reader is referred to [4] and [10] for details concerning absolutely summing and integral operators.

The following concept plays an important role in the present paper:

Definition 1. A Banach space is said to have a *small annihilator* if it is isometric to a closed linear subspace, say $X$, of a space $C(S)$ such that $X^\perp\subset L^1(\lambda)$ for some non-negative finite Borel measure $\lambda$ on $S$.

By the F. and M. Riesz Theorem (cf. [3], II.7.10) the disc algebra $A(D)$ has a small annihilator: regarding $A(D)$ as the subspace of $C(\partial D)$ (obtained by the restriction operator), we have $(A(D))^\perp\subset L^1(m)$, where $m$ denotes the Haar measure on the circle $\partial D$. Next observe that an $X \subset C(S)$ has a small annihilator whenever $X^\perp$ is norm-separable.

Now we are ready to state the main result of the present section which provides an isomorphic invariant for a Banach space to be isomorphic to a space with a small annihilator.

Proposition 1. *Let $X$ be a closed linear subspace of a $C(S)$ with a small annihilator in $C(S)$, say $X^\perp\subset L^1(\lambda)$, for some $\lambda\in[C(S)]^*_+$. Then for every Banach space $E$ and every absolutely summing operator $U\colon X\to E$ there exist a $\sigma\in[C(S)]^*_+$ and an integral operator $V\colon X\to E$ such that*

$$\pi_1(U)\geqslant\|\sigma\|+i(V), \qquad \sigma\ll\lambda, \tag{1.3}$$

$$\|U(x)-V(x)\|\leqslant\int_S|x(s)|\,d\sigma \quad \textit{for every } x\in X. \tag{1.4}$$

Proof. Pick $\mu\in[C(S)]^*_+$ satisfying (1.1) and such that $\|\mu\| = \pi_1(U)$. It follows from (1.1) that there exists a unique operator $B\colon X_\mu\to E$ with $\|B\|\leqslant 1$ such that

$$B(h) = U(x) \quad \text{whenever} \quad h = i_\mu(x) \quad (x\in X).$$

By the Lebesgue Decomposition Theorem (cf. [2], III, 4.14), there exist measures $\sigma$ and $\nu$ in $[C(S)]^*_+$ such that $\mu = \sigma+\nu$, $\sigma\ll\lambda$, $\nu\perp\lambda$. Clearly, $\|\mu\| = \|\sigma\|+\|\nu\|$. We can also identify the space $L^1(\mu)$ with the direct sum $L^1(\sigma)\oplus L^1(\nu)$ equipped with the norm $\|(a,b)\| = \int_S|a(s)|\,d\sigma+ +\int_S|b(s)|\,d\nu$ for $a\in L^1(\sigma)$ and $b\in L^1(\nu)$. Let us observe that to complete the proof of the proposition it suffices to show that

$$X_\mu = X_\sigma\oplus L^1(\nu). \tag{1.5}$$

Indeed, assuming that (1.5) has been established, we define $V\colon X\to E$ by $V = Bj_\nu Q_\nu i_\mu$, where $Q_\nu\colon L^1(\sigma)\oplus L^1(\nu)\to L^1(\nu)$ is the natural projection and $j_\nu\colon L^1(\nu)\to L^1(\sigma)\oplus L^1(\nu)$ is the natural embedding defined by $j_\nu(b) = (0, b)$. By (1.5), $j_\nu Q_\nu i_\mu(X)\subset X_\mu$; thus $V$ is well defined. Clearly, we have $Q_\nu i_\mu = i_\nu$ and $\|Bj_\nu\|\leqslant\|B\|\,\|j_\nu\|\leqslant 1$. Thus $V$ is an integral operator with $i(V) \leqslant \|\nu\|$. Moreover, for every $x \in X$,

$$\|(U-V)(x)\| = \|Bi_\mu(x) - Bj_\nu Q_\nu i_\mu(x)\| \leqslant \|i_\mu(x) - j_\nu Q_\nu i_\mu(x)\|. \tag{1.6}$$

For every $h \in L^1(\mu) = L^1(\sigma) \oplus L^1(\nu)$ we have $h = j_\sigma Q_\sigma(h) + j_\nu Q_\nu(h)$, where $Q_\sigma$: $L^1(\sigma) \oplus L^1(\nu) \to L^1(\sigma)$ and $j_\sigma$: $L^1(\sigma) \to L^1(\sigma) \oplus L^1(\nu)$ denote the natural projection and the natural embedding, respectively. Thus, for $h = i_\mu(x)$, the above identity combined with (1.6) yields

$$\|(U-V)(x)\| \leqslant \|j_\sigma Q_\sigma i_\mu(x)\| = \|i_\sigma(x)\| = \int_S |x(s)|\, d\sigma.$$

This proves (1.4). Clearly, $\pi_1(U) = \|\mu\| = \|\sigma\| + \|\nu\| \geqslant \|\sigma\| + i(V)$.

To prove (1.5) observe first that the inclusion $X_\mu \subset X_\sigma \oplus L^1(\nu)$ is trivial. To prove the reverse inclusion, fix functions $a \in X_\sigma$ and $b \in L^1(\nu)$ so that

$$\int_S |(a+b)(s)|\, d(\sigma+\nu) = \int_S \big(|a(s)| + |b(s)|\big)\, d(\sigma+\nu) = \int_S |a(s)|\, d\sigma + \int_S |b(s)|\, d\nu. \tag{1.7}$$

Now fix $\varepsilon > 0$ and pick an $x \in X$ so that $\int_S |a(s) - x(s)|\, d\sigma < \varepsilon$. Next pick a positive $M < +\infty$ so that, if $Z = \{s \in S\colon |b(s)| \leqslant M\}$, then

$$\int_{S\setminus Z} |b(s)|\, d\nu < \varepsilon \quad \text{and} \quad \nu(S\setminus Z) < \varepsilon(M + \|x\| + 1)^{-1}. \tag{1.8}$$

(If the required $M$ did not exist one would have

$$\nu(S\setminus Z) > (M + \|x\| + 1)^{-1} \int_{S\setminus Z} |b(s)|\, d\nu \geqslant 2^{-1} M^{-1} \int_{S\setminus Z} |b(s)|\, d\nu$$

for all values of $M$ big enough. Remembering that $S\setminus Z = \{s\colon |b(s)| > M\}$, we would get a contradiction with the fact that $\int_S |b(s)|\, d\nu < \infty$.)

By (1.7) and by the relations $\nu \perp \lambda$ and $\sigma \ll \lambda$ and by Lusin's Theorem, there exist a compact set $F \subset Z$ and an open set $G \supset F$ such that

(1.9) *the restriction* $b|F$ *is continuous*, $\lambda(F) = 0$,

$$\max\big(\nu(S\setminus F),\ \sigma(G)\big) < \varepsilon(M + \|x\| + 1)^{-1}.$$

Since $\lambda(F) = 0$ and $X^\perp \subset L^1(\lambda)$, we have $\tau(H) = 0$ for every $H \subset F$ and $\tau \in X^\perp$. Thus, by Bishop's General Rudin–Carleson Theorem (cf. [1], [3], II.12.4), there exists a $y \in X$ such that

(1.10) $y(s) = b(s) - x(s)$ for $s \in F$, $|y(s)| < \varepsilon$ for $s \in S\setminus G$, $\|y\| \leqslant M + \|x\| + 1$.



It follows from (1.7)–(1.10) that

$$\int_S |(a+b)(s) - (x+y)(s)|\, d(\sigma+\nu)$$
$$= \int_S |(a-x-y)(s)|\, d\sigma + \int_{S\setminus F} |(b-x-y)(s)|\, d\nu$$
$$\leqslant \int_S |(a-x)(s)|\, d\sigma + \int_G |y(s)|\, d\sigma + \int_{S\setminus G} |y(s)|\, d\sigma + \int_{S\setminus F} |b(s)|\, d\nu + \int_{S\setminus F} |(x+y)(s)|\, d\nu$$
$$\leqslant \varepsilon + \sigma(G)\|y\| + \sigma(S)\cdot\varepsilon + \int_{S\setminus Z} |b(s)|\, d\nu + M\cdot\nu(Z\setminus F) + (\|x\| + \|y\|)\nu(S\setminus F)$$
$$\leqslant \varepsilon + \varepsilon + \sigma(S)\varepsilon + \varepsilon + (2\|x\| + M + 1)\varepsilon\cdot(\|x\| + M + 1)^{-1} \leqslant \big(5 + \sigma(S)\big)\varepsilon.$$

Letting $\varepsilon \to 0$, we infer that the pair $(a, b)$ belongs to $X_\mu$. This proves the inclusion $X_\sigma \oplus L^1(\nu) \subset X_\mu$ and completes the proof of Proposition 1.

Corollary 1. *Let $E$ be a reflexive Banach space; alternatively, le $E$ be a separable dual. Then, under the assumption of Proposition* 1, *for every absolutely summing operator $U$: $X \to E$, there exist a $\sigma \in [C(S)]^*_+$ and a nuclear operator $V$: $X \to E$ satisfying* (1.4) *and* (1.3) *with $i(V)$ replaced by $n(V)$, where $n(V)$ denotes the nuclear norm of $V$* (cf. [4] and [10] for the definitions).

Proof. Combine Proposition 1 with the fact that if $E$ is reflexive or if $E$ is a separable dual, then every integral operator $V$ into $E$ is nuclear and $i(V) \leqslant n(V)$ (cf. [4], [9], [10]).

We end this section by specifying Corollary 1 in the form in which we shall apply it in the next section.

Corollary 2. *If $X$ is a closed linear subspace of a $C(S)$-space with $X^\perp \subset L^1(\lambda)$ for some $\lambda \in [C(S)]^*_+$, then, for every absolutely summing operator $U$: $X \to l^2$, there exist a compact operator $V$: $X \to l^2$ and a positive function $a \in L^1(\lambda)$ such that*

$$\|U(x) - V(x)\| \leqslant \int_S |x(s)|\, a(s)\, d\lambda \quad \textit{for} \quad x \in X. \tag{1.11}$$

Proof. Put $a = d\sigma/d\lambda$, the Radon–Nikodym derivative of $\sigma$, and observe that every nuclear operator is compact.

Remark. The compactness of the operator $V$ of Corollary 2 can be proved more directly. To this end, note that, by Proposition 1, there exists an integral operator $V$: $X \to l^2$ satisfying (1.11). Hence $V = Bi_\nu A$, where $A$, $B$ and $\nu$ are as in (1.2). Now note that the natural map $i_\nu$: $C(S) \to L^1(\nu)$ is weakly compact while $B$: $L^1(\nu) \to l^2$, being weakly compact, takes weak Cauchy sequences in $L^1(\nu)$ into convergent sequences in $l^2$ (cf. [2], VI.8.12). Hence $Bi_\nu$ is compact and so is $V$.

**2. Spaces of holomorphic functions in domains with sufficiently many supports of the disc algebra.**

DEFINITION 2. A subset $F$ of a closed domain of holomorphy $\Phi \subset \mathfrak{C}^n$ is called a *strict support of the disc algebra* if there exist a map $\varphi_F\colon D\to\Phi$, with $\varphi_F(D) = F$, a positive constant $C_F$ and functions $g_{F,n}\in A(\Phi)$ for $n = 1, 2, \ldots$ such that

(2.1) $g_{F,n}\circ\varphi_F(z) = z^n$ for every $z\in A(D)$ and for $n = 1, 2, \ldots$,

(2.2) $\|g_{F,n}\|\leqslant C_F$ for $n = 1, 2, \ldots$, and for every open $G\supset F\cap\partial\Phi$ and $\varepsilon>0$ there exists an $n(G,\varepsilon)$ such that $|g_{F,n}(\mathfrak{w})|<\varepsilon$ for $\mathfrak{w}\in\Phi\setminus G$ and for every $n>n(G,\varepsilon)$;

(2.3) $I_F(f) = f\circ\varphi_F\in A(D)$ for every $f\in A(\Phi)$.

Let us observe that $I_F\colon A(\Phi)\to A(D)$ is an algebraic homomorphism with $\|I_F\|\leqslant 1$. Next observe that the following properties imply that a closed subset $F$ of $\Phi$ is a strict support of the disc algebra:

There exist a map $\varphi_F\colon D\to F$ satisfying (2.3) and a function $g_F\in A(\Phi)$ such that

(2.1a) $g_F\circ\varphi_F(z) = z$ for every $z\in D$,

(2.2a) $|g_F(\mathfrak{w})|<1$ for every $\mathfrak{w}\in\Phi\setminus(F\cap\partial\Phi)$.

(We put $C_F = 1$ and $g_{F,n} = (g_F)^n$ for $n = 1, 2, \ldots$)

Our next result provides a useful criterion for the non-isomorphism of the Banach spaces $A(D)$ and $A(\Phi)$ for some domains $\Phi$.

PROPOSITION 2. *Let $\Phi$ be a closed bounded domain of holomorphy in $\mathfrak{C}^n$ such that*

(*) *there exists in $\Phi$ an uncountable family $(F_\gamma)_{\gamma\in\Gamma}$ of strict supports of the disc algebra such that $F_\alpha\cap F_\beta\cap\partial\Phi = \emptyset$ whenever $\alpha\neq\beta$ $(\alpha,\beta\in\Gamma)$.*

*Then $A(\Phi)$ is not isomorphic to any complemented subspace of a Banach space with a small annihilator.*

Proof. Let us set $I_\gamma = I_{F_\gamma}$, $g_{\gamma,n} = g_{F_\gamma,n}$ and $C_\gamma = C_{F_\gamma}$ for $\gamma\in\Gamma$, where the homomorphism $I_{F_\gamma}$, the constants $C_{F_\gamma}$ and the functions $g_{F_\gamma,n}$ are the $I_F$, $C_F$ and $g_{F,n}$ of Definition 2 for $F = F_\gamma$. Next define the Paley operator $P\colon A(D)\to l^2$ by

$$P(h) = \Big(\int_0^1 h(e^{it2\pi})e^{-2^nti\pi}\,dt\Big)_{0\leqslant n<+\infty}\quad\text{for}\quad h\in A(D).$$

By a result of Paley (cf. [13], Chap. XII, Theorem 7.8), there exists a $C>0$ such that

$$\|P(h)\| = \Big(\sum_{n=1}^{\infty}\Big|\int_0^1 h(e^{it2\pi})e^{-i2^nt\pi}\,dt\Big|^2\Big)^{1/2}\leqslant C\int_0^1|h(e^{it2\pi})|\,dt\quad\text{for}\quad h\in A(D).$$



Hence $P$ is an absolutely summing operator. Thus, for every $\gamma\in\Gamma$, the operator $P_\gamma = PI_\gamma$ is absolutely summing with $\pi_1(P_\gamma)\leqslant\pi_1(P)\|I_\gamma\|\leqslant C$.

Now assume to the contrary that there exists a subspace $X$ of a $C(S)$-space such that $X^\perp\subset L^1(\lambda)$ for some $0\neq\lambda\in[C(S)]^*_+$ and there exist bounded linear operators $R\colon A(\Phi)\to X$ and $Q\colon X\to A(\Phi)$ such that $QR$ is the identity on $A(\Phi)$. Then, by Corollary 2, for every $\gamma\in\Gamma$, there exist a positive function $a_\gamma$ in $L^1(\lambda)$ and a compact linear operator $V_\gamma\colon X\to l^2$ such that

(2.4) $$\|P_\gamma Q(x)-V_\gamma(x)\|\leqslant\int_S|x(s)|\,a_\gamma(s)\,d\lambda\quad\text{for}\quad x\in X.$$

Next fix $\varepsilon$ with $0<\varepsilon<(\sqrt{2}-1)/(2\|R\|+1)$ and, for every $\gamma\in\Gamma$, pick a positive function $b_\gamma\in L^\infty(\lambda)$ so that

(2.5) $$\int_S|b_\gamma(s)-a_\gamma(s)|\,d\lambda\leqslant\frac{\varepsilon}{C_\gamma+1}.$$

Let

$$\Gamma_m = \{\gamma\in\Gamma\colon \|b_\gamma\|_\infty+C_\gamma+1\leqslant m\}\quad(m = 1, 2, \ldots).$$

Since $\bigcup_{m=1}^{\infty}\Gamma_m = \Gamma$ and since $\Gamma$ is uncountable, at least one of the sets $\Gamma_m$, say $\Gamma_{m_0}$, is infinite. Let $M>2\|Q\|\,\|\lambda\|m_0^2$ and let $\gamma_1,\gamma_2,\ldots,\gamma_M$ be fixed indices in $\Gamma_{m_0}$. For simplicity we shall write in the sequel instead of the index $\gamma_j$ the index $j$; for instance, $P_j$ instead of $P_{\gamma_j}$, $b_j$ instead of $b_{\gamma_j}$, etc. Let us set

$$y_{j,r} = R(g_{j,2^r-1})\quad\text{for}\quad j = 1, 2, \ldots, M;\ r = 1, 2, \ldots$$

Clearly, $Q(y_{j,r}) = g_{j,2^r-1}$ and, by (2.1), $\|y_{j,r}\|\leqslant\|R\|\|g_{j,2^r-1}\|\leqslant\|R\|C_j$. Thus, using the fact that the operators $V_j$ are compact, we extract an infinite increasing subsequence $(r(k))$ of the indices such that

$$\|V_j(x_{j,k})\|<\varepsilon\quad\text{for}\quad j = 1, 2, \ldots, M;\ k = 1, 2, \ldots,$$

where $x_{j,k} = y_{j,r(2k-1)}-y_{j,r(2k)}$.

Thus, for $j = 1, 2, \ldots, M$ and for $k = 1, 2, \ldots$, by (2.4) and (2.5), we have

(2.6) $$\begin{aligned}\|P_jQ(x_{j,k})\| &\leqslant\int_S|x_{j,k}(s)|\,a_j(s)\,d\lambda+\varepsilon\leqslant\int_S|x_{j,k}(s)|\,b_j(s)\,d\lambda+\|x_{j,k}\|\frac{\varepsilon}{C_j+1}+\varepsilon\\ &\leqslant\int_S|x_{j,k}(s)|\,b_j(s)\,d\lambda+(2\|R\|+1)\varepsilon,\end{aligned}$$

because

$$\|x_{j,k}\| \leqslant \|y_{j,r(2k-1)}\| + \|y_{j,r(2k)}\| \leqslant 2\,\|R\|\,C_j.$$

On the other hand, the definitions of the Paley operator and of the functions $y_{j,r}$ yield

$$P_jQ(y_{j,r}) = P_j(g_{j,2^r-1}) = P(\chi_{2^r-1}) = e_r \quad (j = 1, 2, \ldots, M),$$

where $\chi_{2^r-1} \in A(D)$ is defined by $\chi_{2^r-1}(z) = z^{2^r-1}$ and $e_r$ is the $r$th unit vector of $l^2$ $(r = 1, 2, \ldots)$.

Hence $P_jQ(x_{j,k})$ is the difference of two orthogonal vectors, each of norm one. Thus $\|P_jQ(x_{j,k})\| \geqslant \sqrt{2}$. Hence, by (2.6) and by the choice of $\varepsilon$, for $j = 1, 2, \ldots, M$ and for $k = 1, 2, \ldots,$ we have

$$\int_S |x_{j,k}(s)|\, b_j(s)\, d\lambda \geqslant \sqrt{2} - (2\,\|R\|+1)\,\varepsilon \geqslant 1. \tag{2.7}$$

Now, using the fact that the closed sets $F_1 \cap \partial\Phi, F_2 \cap \partial\Phi, \ldots, F_M \cap \partial\Phi$ are mutually disjoint and conditions (2.1) and (2.2), we may pick an index $r_0$ such that, for $r > r_0$,

$$\sum_{j=1}^{M} |g_{j,2^r-1}(\mathfrak{w})| \leqslant 1 + \max_{1\leqslant j\leqslant M} C_j \quad \text{for all} \quad \mathfrak{w} \in \Phi.$$

Thus there exists an index $k$ such that for arbitrary complex numbers $c_1, c_2, \ldots, c_M$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^{M} c_j x_{j,k}\Big\| &\leqslant \Big\|\sum_{j=1}^{M} c_j y_{j,r(2k-1)}\Big\| + \Big\|\sum_{j=1}^{M} c_j y_{j,r(2k)}\Big\| \\ &\leqslant \|Q\|\,\Big\|\sum_{j=1}^{M} c_j g_{j,2^{r(2k-1)}-1}\Big\| + \|Q\|\,\Big\|\sum_{j=1}^{M} c_j g_{j,2^{r(2k)}-1}\Big\| \\ &\leqslant \|Q\| \max_{1\leqslant j\leqslant M} |c_j| \Big(\sup_{\mathfrak{w}\in\Phi} \sum_{j=1}^{M} |g_{j,2^{r(2k-1)}-1}(\mathfrak{w})| + \sup_{\mathfrak{w}\in\Phi} \sum_{j=1}^{M} |g_{j,2^{r(2k)}-1}(\mathfrak{w})|\Big) \\ &\leqslant 2\,\|Q\| \max_{1\leqslant j\leqslant M} |c_j| \cdot m_0,\end{aligned}$$

because, for $j = 1, 2, \ldots, M$, $\gamma_j \in \Gamma_{m_0}$ and therefore $1 + \max_{1\leqslant j\leqslant M} C_j \leqslant m_0$. For the same reason $\max_{1\leqslant j\leqslant M} \|b_j\|_\infty \leqslant m_0$. Hence

$$\sup_{s\in S} \sum_{j=1}^{M} |x_{j,k}(s)|\, b_j(s) \leqslant 2\,\|Q\|\, m_0^2.$$



Thus, by (2.7),

$$M \leqslant \sum_{j=1}^{M} \int_S |x_{j,k}(s)|\, b_j(s)\, d\lambda = \int_S \sum_{j=1}^{M} |x_{j,k}(s)|\, b_j(s)\, d\lambda \leqslant 2\,\|Q\|\, m_0^2\, \|\lambda\|,$$

which contradicts our choice of $M$. This completes the proof.

We shall also need the following variation of Proposition 2.

PROPOSITION 2a. *Let $Y$ be a Banach space such that there exist uncountable families of bounded linear operators $(R_\gamma\colon Y \to A(D))_{\gamma\in\Gamma}$ and $(I_\gamma\colon A(D) \to Y)_{\gamma\in\Gamma}$ and a constant $C > 0$ such that*

(2.8) $\qquad R_\gamma I_\gamma = Q_{\text{even}} + K_\gamma$ *with $K_\gamma$ compact* $(\gamma \in \Gamma)$,

*(here $Q_{\text{even}}\colon A(D) \to A(D)$ is the projection defined by $Q_{\text{even}}(f)(z) = \frac{1}{2}\big(f(z) + f(-z)\big)$ for $f \in A(D)$),*

(2.9) *for every uncountable subset $\Gamma'$ of $\Gamma$ and every positive integer $M$ there exist indices $\gamma_1, \gamma_2, \ldots, \gamma_M$ in $\Gamma'$ and a positive integer $r$ such that*

$$\Big\|\sum_{j=1}^{M} I_{\gamma_j}(f_j)\Big\| \leqslant C \cdot \max_{1\leqslant j\leqslant M} \sup_{z\in D} |f_j(z)| \quad \textit{for every } f_1, f_2, \ldots, f_M \textit{ in } A_r,$$

*where $A_r = \{f \in A(D)\colon f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, r\}$.*

*Then $Y$ is not isomorphic to a complemented subspace of any Banach space with a small annihilator.*

We omit the proof of Proposition 2a; it is essentially the same as the proof of Proposition 2.

**3. Spaces $A(\Phi)$ non-isomorphic to $A(D)$.** Since $A(D)$ has a small annihilator (cf. the paragraph following Definition 1), to show that, for some bounded closed domain of holomorphy $\Phi$, the space $A(\Phi)$ is not isomorphic to any complemented subspace of $A(D)$ it suffices to check that $\Phi$ satisfies condition $(*)$ of Proposition 2. This is very simple in the case of the unit ball $B_n$ of $\mathbb{C}^n$ and in the case of the $n$-polydisc $D^n$ $(n \geqslant 2)$.

PROPOSITION 3. *If $n \geqslant 2$, then the spaces:*

(a) $A(B_n)$,

(b) $A(D^n)$

*are not isomorphic to any complemented subspace of $A(D)$.*

Proof. (a) Let $\Gamma = \{\mathfrak{w} = (w_1, w_2, \ldots, w_n) \in \partial B_n\colon w_1 = \operatorname{Re} w_1 > 0\}$. For $\mathfrak{w} \in \Gamma$ define $\varphi_{\mathfrak{w}}\colon D \to B_n$ by $\varphi_{\mathfrak{w}}(z) = z\cdot\mathfrak{w}$ for $z \in D$, $g_{\mathfrak{w}} \in A(B_n)$, by $g_{\mathfrak{w}}(\mathfrak{z}) = \sum_{k=1}^{n} z_k \overline{w}_k$ for $\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in B_n$, and put $F_{\mathfrak{w}} = \varphi_{\mathfrak{w}}(D)$. Clearly, each $F_{\mathfrak{w}}$ satisfies the conditions (2.1a), (2.2a), (2.3), and $B_n$ together with the family $(F_{\mathfrak{w}})_{\mathfrak{w}\in\Gamma}$ satisfies condition $(*)$ of Proposition 2.

(b) Let $\Gamma = \{\mathfrak{w} = (w_1, w_2, \ldots, w_{n-1}) \in C^{n-1}\colon |w_1| = |w_2| = \ldots = |w_{n-1}| = 1\}$. For $\mathfrak{w} \in \Gamma$ define $\varphi_{\mathfrak{w}}\colon D \to D^n$ by $\varphi_{\mathfrak{w}}(z) = (w_1, w_2, \ldots, w_{n-1}, z)$ for $z \in D$,

$g_{\mathfrak{w}} \in A(D^n)$ by $g_{\mathfrak{w}}(\mathfrak{z}) = z_n \prod_{k=1}^{n-1} \frac{1}{2}(z_k + w_k)$ for $\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in D^n$, and put $F_{\mathfrak{w}} = \varphi_{\mathfrak{w}}(D)$. Clearly, each $(F_{\mathfrak{w}})_{\mathfrak{w} \in \Gamma}$ satisfies conditions (2.1a), (2.2a), (2.3), and $D^n$ together with the family $(F_{\mathfrak{w}})_{\mathfrak{w} \in \Gamma}$ satisfies condition $(*)$ of Proposition 2.

**Remark.** Part (b) of Proposition 3 was first proved by Henkin [5], who used a different argument.

Using a more refined analytic argument, we generalize part (a) of Proposition 3 as follows:

THEOREM 1. *Let $n \geq 2$. Let $\Phi \subset \mathbb{C}^n$ be a bounded closed domain of holomorphy such that $\partial\Phi$ as a real surface is $C^2$-smooth. Then $A(\Phi)$ is not isomorphic to any complemented subspace of $A(D)$.*

**Proof.** By the compactness and $C^2$-smoothness of $\partial\Phi$, we can choose (cf. [12], Theorem 3.1) a point $\mathfrak{z}_0 \in \partial\Phi$ and its neighbourhood $U$ so that $U \cap \partial\Phi$ is a strictly convex surface. Without loss of generality we may assume that $\mathfrak{z}_0 = 0$ and that the inner normal direction to $\partial\Phi$ at $\mathfrak{z}_0$ is the positive direction of the imaginary part of the $z_n$-axis of $\mathbb{C}^n$. Then there exists an $\varepsilon_0 > 0$ (depending on the behaviour of the curvature radius of $\partial\Phi \cap U$) such that, for every $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$, the complex line $\mathbb{C}_\varepsilon = \{\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\colon z_1 = 0, z_2 = 0, \ldots, z_{n-1} = 0, z_n = \varepsilon\}$ intersects $\partial\Phi \cap U$ transversally in $U$ and the curve $\mathbb{C}_\varepsilon \cap (\partial\Phi \cap U)$ is a $C^2$-smooth boundary (relatively to $\mathbb{C}_\varepsilon$) of the topological disc $F_\varepsilon = \mathbb{C}_\varepsilon \cap U \cap \Phi$. Since the curve $\mathbb{C}_\varepsilon \cap (\partial\Phi \cap U)$ is $C^2$-smooth, a strengthening of the Riemann Mapping Theorem (cf. [11], 14. 19) yields the existence of a homeomorphism $\varphi_\varepsilon\colon D \to F_\varepsilon$ which is holomorphic in the interior of $D$. Consequently, $g \circ \varphi_\varepsilon \in A(D)$ for every $g \in A(\Phi)$ and for every $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$. Since $F_\varepsilon$ is a transversal analytic disc of a bounded closed domain of holomorphy with a strictly pseudo-convex boundary, a result of Henkin (cf. [7], (1.1) and Theorem) implies the existence of a linear operator $L_\varepsilon\colon A(D) \to A(\Phi)$ such that if $g_{\varepsilon,n} = L_\varepsilon(\chi_n)$, where $\chi_n(z) = z^n$ for $z \in D$ and for $n = 1, 2, \ldots$, then $g_{\varepsilon,n} \circ \varphi_\varepsilon(z) = z^n$ for $z \in D$ and for $n = 1, 2, \ldots$, and the sequence $(g_{\varepsilon,n})$ tends uniformly to 0 on every compact subset of $\Phi$ which is disjoint with the curve $\partial\Phi \cap F_\varepsilon = \mathbb{C}_\varepsilon \cap (\partial\Phi \cap U)$ (as $n \to +\infty$, $\varepsilon$ being fixed). Hence each $F_\varepsilon$ is a strict support of the disc algebra (we put $C_{F_\varepsilon} = \|L_\varepsilon\|$). Thus $\Phi$ together with the family $(F_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ satisfy condition $(*)$ of Proposition 2.

We are indebted to T. Figiel for the following generalization of part (a) of Proposition 3.

PROPOSITION 4. *If $n \geq 2$ and if $W$ is a strictly convex circled bounded closed domain in $\mathbb{C}^n$ whose boundary does not contain intervals, then $A(W)$ is not isomorphic to any complemented subspace of $A(D)$.*

**Proof.** Applying the Bohnenblust–Sobczyk Theorem (cf. [2], II. 3. 11) to the one-dimensional linear subspaces of the $n$-dimensional complex Banach space whose unit ball is $W$ we construct an uncountable family $(g_\gamma)_{\gamma \in \Gamma}$ of linear functionals on $\mathbb{C}^n$ such that $\sup_{\mathfrak{w} \in W} |g_\gamma(\mathfrak{w})| = 1$ for every $\gamma \in \Gamma$ and such that every two different members of the family are linearly independent. Since the boundary of $W$ does not contain intervals, there exists, for every $\gamma$ in $\Gamma$, a unique point $\mathfrak{w}_\gamma \in W$ such that $g_\gamma(\mathfrak{w}_\gamma) = 1$. Let us define $\varphi_\gamma\colon D \to W$ by $\varphi_\gamma(z) = z \cdot \mathfrak{w}_\gamma$ for $z \in D$ and put $F_\gamma = \varphi_\gamma(D)$ for every $\gamma \in \Gamma$. Then $W$ together with the family $(F_\gamma)_{\gamma \in \Gamma}$ of strict supports of the disc algebra satisfies condition $(*)$ of Proposition 2.



We close this section by discussing the case of the $n$-octohedron

$$\Sigma^n = \Big\{\mathfrak{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\colon \sum_{j=1}^{n} |z_j| \leq 1\Big\}.$$

Let us observe that every point of $\partial\Sigma^n$ with all coordinates different from zero is a point of strict pseudo-convexity and therefore a pick point of the algebra $A(\Sigma^n)$. Therefore the technique of the proof of our Theorem 1 can be carried out to obtain Proposition 5 given below. However we present here a simple elementary argument.

PROPOSITION 5. *If $n \geq 2$, then the space $A(\Sigma^n)$ is not isomorphic to any complemented subspace of $A(D)$.*

**Proof.** Note that it is enough to consider the case of $n = 2$, because $A(\Sigma^2)$ is isometrically isomorphic to a complemented subspace of $A(\Sigma^n)$ for $n \geq 2$ (the desired projection is defined by $Q(f)(z_1, z_2, \ldots, z_n) = f(z_1, z_2, 0, \ldots, 0)$ for $f \in A(\Sigma^n)$). Next observe that $A(\Sigma^2)$ is isometrically isomorphic with a subspace $Y$ of $A(B_2)$ consisting of the functions $g \in A(B_2)$ such that $g(z_1, z_2) = g(c_1 z_1, c_2 z_2)$ for $c_i = \pm 1$ $(i = 1, 2)$ and for every $(z_1, z_2) \in B_2$. The desired isometric isomorphism assigns to a function $f \in A(\Sigma^2)$ the function $g \in Y$ defined by $g(z_1, z_2) = f(z_1^2, z_2^2)$ for $(z_1, z_2) \in B_2$. Let $\Gamma = \{\mathfrak{w} = (w_1, w_2) \in B_2\colon w_1 > 0,\ w_2 > 0,\ w_1^2 + w_2^2 = 1\}$. For every $\mathfrak{w} \in \Gamma$ we define linear operators $I_{\mathfrak{w}}\colon A(D) \to Y$ and $R_{\mathfrak{w}}\colon Y \to A(D)$ by

$$I_{\mathfrak{w}}(\chi_{2k+1}) = 0, \quad I_{\mathfrak{w}}(\chi_{2k})(z_1, z_2) = (z_1 w_1 + z_2 w_2)^{2k} + (-z_1 w_1 + z_2 w_2)^{2k}$$

for $(z_1, z_2) \in B_2$ and for $k = 0, 1, \ldots$ ($\chi_n \in A(D)$ is defined by $\chi_n(z) = z^n$),

$$R_{\mathfrak{w}}(f)(z) = f(z \cdot \mathfrak{w}) \quad \text{for } z \in D \text{ and for } f \in Y.$$

It is easily seen that $\|I_{\mathfrak{w}}\| \leq 2$ and $\|R_{\mathfrak{w}}\| \leq 1$ for every $\mathfrak{w} \in \Gamma$. Let

$$F_{\mathfrak{w}} = \{\mathfrak{z} \in \mathbb{C}^2\colon \mathfrak{z} = z \cdot \mathfrak{w} \text{ for } z \in D\}, \quad F_{\mathfrak{w}}^* = \{\mathfrak{z} \in C^2\colon \mathfrak{z} = (-z w_1, z w_2) \text{ for } z \in D\}.$$

Then, for $\mathfrak{w} = (w_1, w_2) \in \Gamma$ and for $\mathfrak{z} \in F_{\mathfrak{w}}$, we have

$$|-\mathfrak{z} w_1 + \mathfrak{z} w_2| \leq |w_1^2 - w_2^2| = q_{\mathfrak{w}} < 1.$$

Hence

$$\|R_{\mathfrak{w}} I_{\mathfrak{w}}(\chi_{2k}) - \chi_{2k}\| \leq q_{\mathfrak{w}}^{2k} \quad \text{for} \quad k = 1, 2, \ldots$$

Let $K_{\mathfrak{w}} = R_{\mathfrak{w}} I_{\mathfrak{w}} - Q_{\text{even}}$ for $\mathfrak{w} \in \Gamma$. Then

$$K_{\mathfrak{w}}(f) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} f^{(2k)}(0) \cdot \big(R_{\mathfrak{w}} I_{\mathfrak{w}}(\chi_{2k}) - \chi_{2k}\big) \quad \text{for} \quad f \in A(D).$$

Thus $K_{\mathfrak{w}}$ is compact (even nuclear). Hence the family $(R_{\mathfrak{w}}, I_{\mathfrak{w}})_{\mathfrak{w} \in \Gamma}$ satisfies condition (2.8) of Proposition 2(a). This is an easy consequence of the following facts.

(i) For every $\mathfrak{w} \in \Gamma$, if $z \in B_2 \setminus (F_{\mathfrak{w}} \cup F_{\mathfrak{w}}^*)$, then there exists a $q(z, \mathfrak{w})$ such that $0 \leqslant q(z, \mathfrak{w}) < 1$ and $|R_{\mathfrak{w}} I_{\mathfrak{w}}(\chi_{2k})\,(z)| \leqslant 2q(z, \mathfrak{w})^{2k}$ for $k = 1, 2, \ldots$;

(ii) If $\mathfrak{w}_1 \neq \mathfrak{w}_2$, then $(F_{\mathfrak{w}_1} \cup F_{\mathfrak{w}_1}^*) \cap (F_{\mathfrak{w}_2} \cup F_{\mathfrak{w}_2}^*) \cap \partial B_2 = \emptyset$ $(\mathfrak{w}_1 \in \Gamma,\ \mathfrak{w}_2 \in \Gamma)$. The desired conclusion follows from Proposition 2a.

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# The invariance principle for group-valued random variables

by

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**Abstract.** Let $G$ be a complete, separable metric group. We extend the invariance principle to certain triangular arrays of $G$-valued random variables. As an application we examine the invariance principle for triangular arrays of random variables with values in a Fréchet space or in a locally compact abelian group.

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables defined on a probability space $(\Omega, \sigma, P)$. For each integer $n \geqslant 1$, let $\xi_n(t)$, $0 \leqslant t \leqslant 1$, denote the stochastic process obtained by the linear interpolation of the normalized sums $S_k/n^{1/2} = (X_1 + X_2 + \ldots \ldots + X_k)/n^{1/2}$, $k = 1, \ldots, n$; $S_0 = 0$. The invariance principle asserts that if $X_n$ have mean 0 and variance 1, then the sequence of probability measures on $C[0, 1]$, induced by the stochastic processes $\xi_n(t)$, converges weakly to the Wiener measure (see e.g. [1]). This result has been generalized by J. Kuelbs to triangular arrays of Banach space valued random variables [8]. In this note we prove that the invariance principle has a more general scope. Let $G$ be a Polish (complete separable metric) group. We discuss the invariance principle for certain triangular arrays of $G$-valued random variables. Instead of $C[0, 1]$ we use the space $D_G \subset G^{[0,1]}$ of all left-continuous functions having the right-hand limits (at every point of $[0, 1]$). As an application of our result we examine the invariance principle for Fréchet space valued random variables and for random variables taking values in any LCA group.

**Preliminaries.** All statements which we quote here are proved in [1] under the assumption that $G$ is the real line. However, the arguments used there either apply to our general situation without any change or need a slight modification. We shall not repeat those proofs.

Throughout this paper we shall assume that $G$ is a complete separable metric group having a left-invariant and complete metric $\varrho$. It is well known that every two-sided invariant metric is complete whenever $G$ is topologically complete.