Let $K_{\mathfrak{w}} = R_{\mathfrak{w}} I_{\mathfrak{w}} - Q_{\mathrm{even}}$ for $\mathfrak{w} \in \Gamma$. Then

$$K_{\mathfrak{w}}(f) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} f^{(2k)}(0) \cdot \big(R_{\mathfrak{w}} I_{\mathfrak{w}}(\chi_{2k}) - \chi_{2k}\big) \quad \text{for} \quad f \in A(D).$$

Thus $K_{\mathfrak{w}}$ is compact (even nuclear). Hence the family $(R_{\mathfrak{w}}, I_{\mathfrak{w}})_{\mathfrak{w} \in \Gamma}$ satisfies condition (2.8) of Proposition 2(a). This is an easy consequence of the following facts.

(i) For every $\mathfrak{w} \in \Gamma$, if $z \in B_2 \setminus (F_{\mathfrak{w}} \cup F_{\mathfrak{w}}^*)$, then there exists a $q(z, \mathfrak{w})$ such that $0 \leqslant q(z, \mathfrak{w}) < 1$ and $|R_{\mathfrak{w}} I_{\mathfrak{w}}(\chi_{2k})(z)| \leqslant 2q(z, \mathfrak{w})^{2k}$ for $k = 1, 2, \ldots$;

(ii) If $\mathfrak{w}_1 \neq \mathfrak{w}_2$, then $(F_{\mathfrak{w}_1} \cup F_{\mathfrak{w}_1}^*) \cap (F_{\mathfrak{w}_2} \cup F_{\mathfrak{w}_2}^*) \cap \partial B_2 = \emptyset$ $(\mathfrak{w}_1 \in \Gamma, \mathfrak{w}_2 \in \Gamma)$. The desired conclusion follows from Proposition 2a.

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The invariance principle for group-valued random variables

by

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Abstract. Let G be a complete, separable metric group. We extend the invariance principle to certain triangular arrays of G-valued random variables. As an application we examine the invariance principle for triangular arrays of random variables with values in a Fréchet space or in a locally compact abelian group.

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables defined on a probability space (Ω, σ, P). For each integer $n \geqslant 1$, let $\xi_n(t)$, $0 \leqslant t \leqslant 1$, denote the stochastic process obtained by the linear interpolation of the normalized sums $S_k/n^{1/2} = (X_1 + X_2 + \ldots + X_k)/n^{1/2}$, $k = 1, \ldots, n$; $S_0 = 0$. The invariance principle asserts that if X_n have mean 0 and variance 1, then the sequence of probability measures on $C[0, 1]$, induced by the stochastic processes $\xi_n(t)$, converges weakly to the Wiener measure (see e.g. [1]). This result has been generalized by J. Kuelbs to triangular arrays of Banach space valued random variables [8]. In this note we prove that the invariance principle has a more general scope. Let G be a Polish (complete separable metric) group. We discuss the invariance principle for certain triangular arrays of G-valued random variables. Instead of $C[0, 1]$ we use the space $D_G \subset G^{[0,1]}$ of all left-continuous functions having the right-hand limits (at every point of $[0, 1]$). As an application of our result we examine the invariance principle for Fréchet space valued random variables and for random variables taking values in any LCA group.

Preliminaries. All statements which we quote here are proved in [1] under the assumption that G is the real line. However, the arguments used there either apply to our general situation without any change or need a slight modification. We shall not repeat those proofs.

Throughout this paper we shall assume that G is a complete separable metric group having a left-invariant and complete metric ϱ. It is well known that every two-sided invariant metric is complete whenever G is topologically complete.

Let $\|\cdot\|$ denote the distance from the identity e of G: $\|x\| = \varrho(x, e)$.

Let $D_G = D_G[0,1]$ be the space of functions f on $[0,1]$ into G that are right-continuous and have left-hand limits:

(i) for $0 \leqslant t < 1$, $f(t+) = \lim\limits_{s \searrow t} f(s)$ exists and $f(t+) = f(t)$,

(ii) for $0 < t \leqslant 1$, $f(t-) = \lim\limits_{s \nearrow t} f(s)$ exists.

Let Λ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself taking 0 onto 0. For f and g in D_G, define $d(f,g)$ to be the infimum of those positive ε for which there exists in Λ a λ such that

$$\sup_t |\lambda t - t| \leqslant \varepsilon$$

and

$$\sup_t \left\| \big(f(t)\big)^{-1} g(\lambda t) \right\| \leqslant \varepsilon.$$

The space D_G is the separable, topologically complete metric space in the topology generated by the metric d (the so-called *Skorohod topology*).

We say that a mapping ξ from a probability space (Ω, σ, P) into D_G is a *random element* iff it is measurable with respect to the Borel σ-field of D_G. Let us define, for $t_1, t_2, \ldots, t_k$; $t_i \in [0,1]$, the natural projection $\pi_{t_1 \ldots t_k}$ from D_G to G^k:

$$\pi_{t_1 \ldots t_k}(f) = \big(f(t_1), f(t_2), \ldots, f(t_k)\big).$$

By *finite-dimensional sets* we shall mean sets of the form $\pi^{-1}_{t_1 \ldots t_k}(H)$, where $H \in \mathscr{B}_{G^k}$ (the Borel σ-field in G^k).

For the random element ξ, by $\xi(t)$ we shall denote the composition of ξ and π_t:

$$\xi(t) = \pi_t(\xi).$$

From the usual arguments regarding this subject, it follows that ξ is a random element of D_G iff $\xi(t)$ is a random variable of G for each $t \in [0,1]$ (i.e., $\xi(t)$ is the Borel measurable mapping from the probability space into G).

Finally, we say that a random element ξ has *independent increments* iff, for each $t_1, t_2, \ldots, t_k$; $t_i \in [0,1]$ such that $t_1 < t_2 < \ldots < t_k$, the random variables

$$\xi(t_1)^{-1}\xi(t_2),\ \xi(t_2)^{-1}\xi(t_3), \ldots, \xi(t_{k-1})^{-1}\xi(t_k)$$

are independent.

The invariance principle. First of all we state some lemmas needed in the sequel. A proof of the first one may be found e.g. in [5].

LEMMA 1. *Let $X_1, X_2, \ldots, X_n$ be n independent, G-valued random variables and $S_k = X_1 X_2 \ldots X_k$, $k = 1, 2, \ldots, n$, $S_0 = e$. Then*

$$P(\{\max_{1 \leqslant k \leqslant n} \|S_k\| > 2\varepsilon\}) \leqslant a^{-1} P(\{\|S_n\| > \varepsilon\})$$

whenever

$$P(\{\|S_k^{-1} S_n\| \leqslant \varepsilon\}) \geqslant a > 0 \quad \textit{for} \quad k = 0, 1, \ldots, n-1.$$

The next lemma, which may be found in [4], gives some property of random elements with independent increments and continuous paths.

LEMMA 2. *Let ξ be a random element of D_G with independent increments. If ξ has continuous paths with probability one, then*

$$\sum_{k=1}^{m_n} P\left(\{\|\xi(t_{k-1}^{(n)})^{-1}\xi(t_k^{(n)})\| \geqslant \varepsilon\}\right) \to 0$$

whenever $n \to \infty$, for every $\varepsilon > 0$ and every sequence $\{t_k^{(n)}\}_{k=1}^{m_n}$ of partitions of $[0,1]$ such that $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{m_n}^{(n)} = 1$ and $\lambda_n = \max\limits_{1 \leqslant k \leqslant m_n} (t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ if $n \to \infty$.

The arguments needed to prove the following lemma do not differ essentially from those used in the proofs of Theorems 14.3, 15.2 and 15.5, Chapt. 3 of [1], so we shall not repeat them.

LEMMA 3. *Let $\{X_j^{(n)}: j = 1, \ldots, n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued, independent and identically distributed random variables. Let $S_k^{(n)} = X_1^{(n)} X_2^{(n)} \ldots X_k^{(n)}$. Suppose that for each $\varepsilon > 0$ and every $\eta > 0$ there exist: δ, $0 < \delta < 1$, a positive integer n_0, and a family $\{K_t\}_{t \in S}$ of compact subsets of G, where S is a dense subset of $[0,1]$ containing 1, such that*

(a) $\dfrac{1}{\delta} P(\{\max\limits_{1 \leqslant i \leqslant n\delta} \|S_i^{(n)}\| > \varepsilon\}) \leqslant \eta$ *for* $n \geqslant n_0$,

(b) $P(\{S_{[nt]}^{(n)} \notin K_t \textit{ for } t \in S\}) \leqslant \eta$.

Let $\xi_n(t) = S_{[nt]}^{(n)}$, $\xi_n(0) = e$.

Then the sequence of random elements ξ_n is tight, and if μ is the weak limit of a certain subsequence $\xi_{n'}$, then $\mu(C_G) = 1$, where C_G denotes the space of all continuous functions $\subseteq G^{[0,1]}$.

The following lemma will also be useful in the sequel.

LEMMA 4. *Let $\{X_j^{(n)}: j = 1, \ldots, n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued, independent and identically distributed random variables. Let $S_k^{(n)} = X_1^{(n)} X_2^{(n)} \ldots X_k^{(n)}$. Suppose that*

(1) *$S_{[nt]}^{(n)}$ converges weakly for $t \in S$, where S is a dense subset of $[0,1]$ containing 1,*

(2) *for each* $\varepsilon > 0$ *and a certain sequence* δ_n, $0 < \delta_n < 1$, $\delta_n \to 0$

$$\frac{1}{\delta_k} \limsup_n P(\{\|S^{(n)}_{[n\delta_k]}\| > \varepsilon\}) \to 0 \quad \text{if} \quad k \to \infty,$$

(3) *for each* $\varepsilon > 0$ *there exists* $h > 0$ *such that*

$$\limsup_n \max_{1 \leqslant r \leqslant nh} P(\{\|S^{(n)}_r\| > \varepsilon\}) < 1.$$

Let $\xi_n(t) = S^{(n)}_{[nt]}$. *Then the sequence of random elements* ξ_n *converges weakly to a certain element of* D_G *with continuous paths.*

Proof. First we shall show that conditions (a) and (b) of Lemma 3 are satisfied.

Condition (b) follows directly from assumption (1) and from Prohorov Theorem (see [1]).

Now, let ε, η be given real, positive numbers. By (3), there exist real positive numbers h_1, α such that $\alpha < 1$ and a positive integer n_1 such that

$$\max_{1 \leqslant r \leqslant nh_1} P(\{\|S^{(n)}_r\| > \varepsilon\}) \leqslant \alpha$$

for every $n \geqslant n_1$. Lemma 1 implies that

$$(*) \qquad P(\{\max_{1 \leqslant i \leqslant n\delta_k} \|S^{(n)}_i\| > \varepsilon\}) \leqslant \frac{1}{1-\alpha} P\left(\left\{\|S^{(n)}_{[n\delta_k]}\| > \frac{\varepsilon}{2}\right\}\right)$$

for $n \geqslant n_1$ and $\delta_k \leqslant h_1$. By (2), there exists a positive integer N such that if $k \geqslant N$ then

$$\limsup_n P\left(\left\{\|S^{(n)}_{[n\delta_k]}\| > \frac{\varepsilon}{2}\right\}\right) < \delta_k \eta (1-\alpha) \quad \text{and} \quad \delta_k \leqslant h_1.$$

Let $n_2 = \max(n_1, N)$ and $\delta = \delta_{n_2}$. There exists an $n_0 \geqslant n_2$ such that

$$(**) \qquad P\left(\left\{\|S^{(n)}_{[n\delta]}\| > \frac{\varepsilon}{2}\right\}\right) < \delta\eta(1-\alpha)$$

for $n \geqslant n_0$. From $(*)$ and $(**)$, we have

$$\frac{1}{\delta} P(\{\max_{1 \leqslant i \leqslant n\delta} \|S^{(n)}_i\| > \varepsilon\}) \leqslant \eta$$

for $n \geqslant n_0$ and thus condition (a) of Lemma 3 is satisfied.

Now, let a certain subsequence $\xi_{n'}$ converge weakly to μ. Let $\xi_{n''}$ be another subsequence converging to a distribution ν. Since $\mu(C_G) = \nu(C_G) = 1$, it follows (see [1], Chapt. 3, § 15) that, for every $t \in [0, 1]$, $\xi_{n'}(t)$ and $\xi_{n''}(t)$ converge weakly to one-dimensional distributions of μ and ν, respectively. Since, for $t \in S$, $\xi_n(t)$ also converges weakly, one-dimensional distributions of μ and ν based on points in S are equal. Since the limits of $\xi_{n'}$, $\xi_{n''}$ are random elements with independent increments, the corresponding finite-dimensional distributions of μ and ν based on points in S are also equal. Since S is dense in $[0, 1]$ and contains 1, we have $\mu = \nu$ (see Th. 14.5, [1]). This completes the proof.

PROPOSITION. *Suppose that there exists a homogeneous random element* W *on* D_G *with independent increments and with continuous paths (with probability one) such that* $W(0) = e$. *Let* $\{X^{(n)}_k; k = 1, 2, \ldots, n; n = 1, 2, \ldots\}$ *be a triangular array of independent, identically distributed random variables with values in* G. *Let* $S^{(n)}_k = X^{(n)}_1 X^{(n)}_2 \ldots X^{(n)}_k$, $S^{(n)}_0 = e$. *Suppose that*

(i) $S^{(n)}_{[nt]}$ *converges weakly to* $W(t)$ *for every* $t \in [0, 1]$,

(ii) *for each* $\varepsilon > 0$ *there exists* $h > 0$ *such that*

$$\limsup_n \max_{1 \leqslant r \leqslant nh} P(\{\|S^{(n)}_r\| > \varepsilon\}) < 1.$$

Let $\xi_n(t) = S^{(n)}_{[nt]}$. *Then* ξ_n *converges weakly to* W.

Proof. From (i) it follows that

$$\limsup_n P(\{\|S^{(n)}_{[nt]}\| \geqslant \varepsilon\}) \leqslant P\big(\{\|W(t)\| \geqslant \varepsilon\}\big),$$

which, together with Lemma 2, establishes condition (2) of Lemma 4 and ends the proof.

We now present some applications of these statements.

Let G be a separable Fréchet space, i.e., a real vector space which is a complete, separable metric group, and is such that the mappings $a \to ax$ of R onto G and $x \to ax$ of G onto G are continuous.

We say that the random variable X with values in G has a *symmetric Gaussian distribution* iff for any pair Y_1, Y_2 of independent random variables having the distributions of X, and for every pair of real numbers (s, t) such that $s^2 + t^2 = 1$, the random variables $sY_1 + tY_2$ and $tY_1 - sY_2$ are independent and have the distribution of X. This definition was introduced by Fernique in [4]. If the continuous linear functionals of G generate the Borel σ-field of G, then this definition is equivalent to the following one: X *has a symmetric Gaussian distribution* iff $f(X)$ is a symmetric real Gaussian random variable, for every continuous linear functional f.

THEOREM 1. *Let* X *be a symmetric Gaussian random variable on* G. *The following conditions are equivalent*:

(i) *for each* $\varepsilon > 0$, $nP(\{\|X/n^{1/2}\| \geqslant \varepsilon\}) \to 0$ *if* $n \to \infty$,

(ii) *there exists a homogeneous Gaussian random element* W *on* D_G, *with independent increments, having continuous paths with probability one and such that* $W(0) = 0$ *and* $W(1)$ *has the distribution of* X,

(iii) *if $\{X_n\}$ is a sequence of independent, identically distributed random variables such that $(X_1+X_2+\ldots+X_n)/n^{1/2}$ converges weakly to the distribution of X, then the sequence ξ_n, $\xi_n(t) = (X_1+X_2+\ldots+X_{[nt]})/n^{1/2}$ converges weakly to a certain random element on D_G with continuous paths with probability one.*

Proof. First we shall prove that (iii) implies (ii). Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables such that X_i has the distribution of X. Then $(X_1+X_2+\ldots+X_n)/n^{1/2}$ also has the distribution of X, and if (iii) holds, then ξ_n converges weakly to a certain random element with the properties as in (ii). Thus (ii) holds.

That (ii) implies (i) follows directly from Lemma 2 and the definition of Gaussian distribution.

Now, suppose that condition (i) holds. Let $\{X_n\}$ be a sequence of independent, identically distributed random variables such that $(X_1+ +X_2+\ldots+X_n)/n^{1/2}$ converges weakly to the distribution of X. In order to conclude that ξ_n converges weakly to a random element with properties as described in (ii), it suffices to show that the assumptions of Lemma 4 are satisfied.

For this purpose let us observe that if

$$\xi_n(t) = \left(\frac{[nt]}{n}\right)^{1/2} \frac{X_1+\ldots+X_{[nt]}}{[nt]^{1/2}}$$

and

$$\xi_n'(t) = t^{1/2}\frac{X_1+\ldots+X_{[nt]}}{[nt]^{1/2}},$$

then $\xi_n(t)-\xi_n'(t)$ converges to 0 with probability 1, and $\xi_n'(t)$ converges weakly to the distribution of $t^{1/2}X$. Therefore, by [1], Theorem 4.1, $\xi_n(t)$ also converges weakly to the distribution of $t^{1/2}X$, and condition (1) of Lemma 4 holds.

Next, we shall show that for each $\varepsilon, \eta > 0$ there exists an $h > 0$ such that

$$(*) \qquad \limsup_n \max_{1\leqslant r\leqslant nh} P\left(\left\{\left\|\frac{1}{n^{1/2}}\sum_{j=1}^{r} X_j\right\| \geqslant \varepsilon\right\}\right) \leqslant \eta.$$

Let

$$Y_n = \frac{X_1+\ldots+X_n}{n^{1/2}}, \qquad U = \{x;\ \|x\| < \varepsilon\}.$$

Since Y_n converges weakly, there exists a compact subset K of G (by the Prohorov Theorem) such that $P(\{Y_n \notin K\}) < \eta$. Since U is

an open neighbourhood of 0, there exists an $h > 0$ such that for $t \leqslant h^{1/2}$ we have $tK \subset U$ (see [3], Chapt. II). Therefore, for $r \leqslant nh$ we have

$$\left\{\left\|\frac{1}{n^{1/2}}\sum_{j=1}^{r} X_j\right\| \geqslant \varepsilon\right\} = \left\{\left\|\left(\frac{r}{n}\right)^{1/2}\frac{X_1+\ldots+X_r}{r^{1/2}}\right\| \geqslant \varepsilon\right\}$$

$$= \left\{\left(\frac{r}{n}\right)^{1/2}\frac{X_1+\ldots+X_r}{r^{1/2}} \notin U\right\} = \left\{\frac{X_1+\ldots+X_r}{r^{1/2}} \notin \left(\frac{n}{r}\right)^{1/2} U\right\} \subset \{Y_r \notin K\},$$

which proves (*). Thus, condition (3) of Lemma 4 holds.

Finally, since $(X_1+X_2+\ldots+X_{[k/n]})/k^{1/2}$ converges to the distribution of $X/n^{1/2}$ as $k\to\infty$, by condition (i) of this theorem it follows that condition (2) of Lemma 4 is also satisfied. This ends the proof of the theorem.

Remark. It is interesting to know which separable Fréchet spaces G have the property that every symmetric Gaussian random element taking values in G satisfies condition (i) of Theorem 1. By Theorem 2 in [7] it follows that every separable locally pseudoconvex Fréchet space has this property. The corresponding result for Banach spaces can be derived from an earlier paper of Fernique [4] and also from Landau and Shepp [10]; for locally convex spaces — from Kuelbs [9]. The forthcoming example shows that every symmetric Gaussian random element with values in the space $S(T, \mathscr{A}, m)$ (all $\mathscr{A}$-measurable real functions on T with the convergence in measure m) also satisfies condition (i). The idea of this example is due to Professor C. Ryll-Nardzewski.

However, the author does not know either any characterization of Fréchet spaces having the above property or any example of a symmetric Gaussian random element which does not satisfy (i).

EXAMPLE. Let $(T, \mathscr{A}, m)$ be a finite measure space. Let S denote the space of all real-valued $\mathscr{A}$-measurable functions defined on T. It is well known that S with the norm

$$\|f\| = \int_T \frac{|f(t)|}{1+|f(t)|}\, dm(t),$$

which induces the topology of convergence in measure m, is a real Fréchet space. If m is nonatomic, then S admits no nonzero continuous linear functionals. Suppose that S is separable.

Let $\{\xi(t);\ t\in T\}$ be a stochastic process defined on a probability space (Ω, σ, P); it is said to be *measurable* if the map ξ, $\xi\colon T\times\Omega\to R$ defined by $(t,\omega)\to\xi(t,\omega)$ is measurable relative to the σ-algebras $\mathscr{B}_R$ and $\mathscr{A}\times\sigma$. From the measurability of the process and the separability of S it follows that the mapping $\omega\to\xi(\cdot,\omega)$ is measurable relative to $\mathscr{B}_S$ and σ. The probability measure on $(S, \mathscr{B}_S)$ induced by this mapping will be denoted by μ_ξ.

A stochastic process $\{\xi(t);\ t\in T\}$ is said to be *Gaussian* if, for every $t_1, t_2, \ldots, t_k \in T$, $\langle \xi(t_1), \xi(t_2), \ldots, \xi(t_k)\rangle$ is a Gaussian random vector with values in R^k.

The proof that every symmetric Gaussian random element with values in $(S, \mathscr{B}_S)$ satisfies condition (i) of Theorem 1 is divided into two parts; each of these parts may be of independent interest.

In the first part we shall prove that every symmetric Gaussian measure on $(S, \mathscr{B}_S)$ is induced by a Gaussian measurable stochastic process. In the second part we shall show that every measurable symmetric Gaussian process is induced by a continuous linear mapping Φ, $\Phi\colon L_2\to S$, from a measurable Gaussian process with samples in L_2; so the induced random element with values in S is symmetric Gaussian and satisfies condition (i).

I. Let μ be a probability measure on $(S, \mathscr{B}_S)$. We construct a measurable stochastic process $\{\xi(t);\ t\in T\}$ such that the induced measure μ_ξ equals μ.

For any set $A\subseteq S$ we denote its diameter by $\delta(A)$. Since S is separable, we can write, for every positive integer n, $S=\bigcup_{k=1}^{\infty} S_k^{(n)}$, where $S_k^{(n)}$ are non-empty, $\delta(S_k^{(n)})<1/n$, $S_k^{(n)}\in\mathscr{B}_s$ $(k=1,2,\ldots)$ and $S_k^{(n)}\cap S_m^{(n)}=0$ if $k\neq m$. Without loss of generality we may assume that $\{S_k^{(n+1)}\}$ is a refinement of $\{S_k^{(n)}\}$. Let us choose, for each m, an element of $S_m^{(n)}$, namely $h_m^{(n)}$, and let $\bar{h}_m^{(n)}$ be a representative of the equivalence class $h_m^{(n)}$. Now, let us define, for $n=1,2,\ldots$,

$$\xi_n(f,t)=\bar{h}_k^n(t)\quad \text{if}\quad f\in S_k^{(n)}.$$

It is easy to check that $\xi_n\colon S\times T\to R$ is $\mathscr{B}_S\times\mathscr{A}$-measurable. Next, by the construction of ξ_n it follows immediately that for $\varepsilon>0$

$$m\{t;\ |\xi_n(f,t)-\xi_m(f,t)|>\varepsilon\}\to 0$$

if $n, m\to\infty$ uniformly with respect to f. Fubini's Theorem implies that ξ_n is fundamental in the $\mu\times m$ measure. So, there is a function $\xi\colon S\times T\to R$, $\mathscr{B}_S\times\mathscr{A}$-measurable and such that ξ_n converges to ξ in the $\mu\times m$-measure. Let $\tilde{\xi}$ denote the mapping of S into S induced by ξ as follows:

$$\tilde{\xi}(f)(\cdot)=\xi(f,\cdot).$$

Let us observe that $\tilde{\xi}=I$ μ-a.e., where I denotes the identity map. For, let ξ_{n_k} be a subsequence of ξ_n converging to ξ $\mu\times m$-a.e. Then, by Fubini's Theorem, $\xi_{n_k}(f,\cdot)$ converges m-a.e. to $\xi(f,\cdot)$, for μ-almost all f, and hence in the m-measure. So, $\tilde{\xi}_{n_k}\to\tilde{\xi}$ μ-a.e. On the other hand, $\tilde{\xi}_n(f)\to I(f)$, for every $f\in S$, which gives the desired conclusion. Next, for every pair (s,u) of real numbers we have

$$\xi(sf+ug,t)=s\xi(f,t)+u\xi(g,t) \tag{1}$$

for $\mu\times\mu\times m$-almost all (f,g,t). This follows from the construction of ξ_n and from the fact that S is topologized by the norm of convergence in measure m.

Now, let μ be a symmetric Gaussian distribution. Let (s,u) be a fixed pair of real numbers such that $0<s, u<1$, $s^2+u^2=1$. Let us define the mapping $F\colon S\times S\to S\times S$:

$$F(f,g)=(sf+ug,\ uf-sg).$$

Since μ is symmetric Gaussian, we have

$$(\mu\times\mu)(A)=(\mu\times\mu)\big(F^{-1}(A)\big) \tag{2}$$

for every $A\in\mathscr{B}_S$. Let N be a subset of T such that $N\in\mathscr{A}$, $m(N)=0$ and

$$(\mu\times\mu)\big(\{(f,g);\ \xi(sf+ug,t_0)\neq s\xi(f,t_0)+u\xi(g,t_0),\ \xi(uf-sg,t_0)\neq u\xi(f,t_0)-s\xi(g,t_0)\}\big)=0\quad\text{for}\quad t_0\in T\setminus N. \tag{3}$$

Let $t_1, t_2, \ldots, t_k\in T\setminus N$ and let

$$X=\big(\xi(f,t_1), \xi(f,t_2), \ldots, \xi(f,t_k)\big),$$
$$Y=\big(\xi(g,t_1), \xi(g,t_2), \ldots, \xi(g,t_k)\big).$$

Let $A, B\in\mathscr{B}_{R^k}$ (the Borel σ-field in R^k). Let us define

$$A'=\{f;\ X\in A\},\quad B'=\{g;\ Y\in B\}.$$

Clearly, by the measurability of ξ it follows that $A', B'\in\mathscr{B}_S$. By (2), we have

$$(\mu\times\mu)(A'\times B')=(\mu\times\mu)\big(F^{-1}(A'\times B')\big).$$

In view of (3) we have

$$\begin{aligned}
&(\mu\times\mu)\big(F^{-1}(A'\times B')\big)\\
&=(\mu\times\mu)\big(\{(f,g);\ F(f,g)\in A'\times B'\}\big)\\
&=(\mu\times\mu)\Big(\{(f,g);\ \big(\xi(sf+ug,t_1),\ldots,\xi(sf+ug,t_k)\big)\in A;\ \big(\xi(uf-sg,t_1),\ldots\\
&\qquad\qquad \ldots,\xi(uf-sg,t_k)\big)\in B\}\Big)\\
&=(\mu\times\mu)\big(\{(f,g);\ (sX+uY,\ uX-sY)\in A\times B\}\big)\\
&=(\mu\times\mu)\big(\{(f,g);\ (X,Y)\in F_k^{-1}(A\times B)\}\big)=(\mu_X\times\mu_Y)\big(F_k^{-1}(A\times B)\big),
\end{aligned}$$

where μ_X, μ_Y denote the distributions of X, Y, respectively, and $F_k(x_1,x_2)=(sx_1+ux_2,\ ux_1-sx_2)$ for each $x_1, x_2\in R^k$. Thus we have

$$(\mu_X\times\mu_Y)(A\times B)=(\mu_X\times\mu_Y)\big(F_k^{-1}(A\times B)\big).$$

By Theorem 2, § 8. XV, in [5], we infer that X is a symmetric Gaussian random variable with values in R^k. Thus, if we define

$$\bar{\xi}(f,t) = \begin{cases} \xi(f,t) & \text{if} \quad t \in T \setminus N, \\ 0 & \text{if} \quad t \in N, \end{cases}$$

then $\mu_{\bar{\xi}} = \mu_{\xi} = \mu$ and $\bar{\xi}$ is a symmetric measurable Gaussian process.

II. Next, let $\{\xi(t); t \in T\}$ be a symmetric measurable Gaussian process. Let us define $K(s,t) = E(\xi(t)\xi(s))$. Now, let

$$\eta(\omega, t) = \xi(\omega, t)\left(1+K(t,t)\right)^{-1/2}.$$

Then $E(\eta(t)^2) = K(t,t)\left(1+K(t,t)\right)^{-1}$ and so $\eta(\omega,\cdot) \in L_2(T, \mathscr{A}, m)$ with probability one (see Proposition 3.4 in [12]). Thus, the probability measure μ_η induced on L_2 by η is symmetric Gaussian (one uses the fact that L_2 has sufficiently many continuous functionals and Theorem 3.2 in [12]). Now, let us define a linear mapping from L_2 into S as follows:

$$\Phi f = f(1+K)^{1/2}.$$

It is easy to see that Φ is continuous, $\xi(\omega,\cdot) = \Phi\eta(\omega,\cdot)$, and that μ_ξ is symmetric Gaussian. Let us denote by X, Y the random elements induced by the stochastic processes η, ξ on L_2, S, respectively. Let U be any open neighbourhood of 0 in S. Then

$$nP\{Y/n^{1/2} \notin U\} = nP\{\Phi X/n^{1/2} \notin U\} = nP\{X/n^{1/2} \notin \Phi^{-1}U\} \to 0$$

if $n \to \infty$, since by the continuity of Φ, $\Phi^{-1}U$ is an open neighbourhood of 0 in L_2 and X satisfies condition (i) of Theorem 1 (as a symmetric Gaussian random element with values in a Banach space). So, Y satisfies condition (i) of Theorem 1.

Now, let G be a locally compact, second countable abelian group and Γ its character group. The random variable X with values in G has a symmetric Gaussian distribution if its characteristic function has the form

$$\hat{\mu}(\gamma) = \exp\left(-\varphi(\gamma)\right),$$

where $\gamma \in \Gamma$ and φ is a continuous, nonnegative function on Γ satisfying the equality

$$\varphi(\gamma_1+\gamma_2)+\varphi(\gamma_1-\gamma_2) = 2[\varphi(\gamma_1)+\varphi(\gamma_2)]$$

for all γ_1, γ_2 in Γ (see § 6, IV in [11]).

Let X be a G-valued symmetric Gaussian random variable having the characteristic function $\hat{\mu}(\gamma) = \exp\left(-\varphi(\gamma)\right)$, where φ is a certain fixed function on Γ with the properties as described above.

THEOREM 2. *The following statements are valid:*

(i) *$nP(\{\|X_n\| \geqslant \varepsilon\}) \to 0$ if $n \to \infty$, where X_n is a random variable with the characteristic function $\hat{\nu}_n(\gamma) = \exp\left(-\varphi(\gamma)/n\right)$.*

(ii) *There exists a homogeneous Gaussian random element W with independent increments, having, with probability one, continuous paths and such that $W(1)$ has the distribution of X and $W(0) = 0$.*

(iii) *Let $\{X_j^{(n)};\ j = 1,2,\ldots,n;\ n = 1,2,\ldots\}$ be an infinitesimal triangular array of symmetric, independent and identically distributed random variables such that $X_1^{(n)}+\ldots+X_n^{(n)}$ converges weakly to the distribution of X. Let $\xi_n(t) = X_1^{(n)}+\ldots+X_{[nt]}^{(n)}$. Then ξ_n converges weakly to W.*

Proof. It follows immediately from [2] that condition (ii) of the theorem is satisfied.

In order to prove that conditions (i), (ii) and (iii) are equivalent we need only show that condition (3) of Lemma 4 holds for the array as is described in (iii). The remaining arguments are almost the same as in the proof of Theorem 1.

Let $S_k^{(n)} = X_1^{(n)}+\ldots+X_k^{(n)}$ and let $\hat{\mu}_n$ be the characteristic function of $X_j^{(n)}$. We shall show that for each compact subset $K \subseteq \Gamma$

$$\lim_{h\to 0}\overline{\lim_{n}}\max_{k\leqslant nh}\sup_{\gamma\in K}|\hat{\mu}_n(\gamma)^k-1| = 0.$$

Let us observe that $(\hat{\mu}_n)^{[nh]}$ converges to $\exp\left(-h\varphi(\gamma)\right)$ uniformly on each compact set $K \subseteq \Gamma$ for a fixed h, $0 < h \leqslant 1$. Let ν_h be the distribution defined by the characteristic function $\exp(-h\varphi)$. Notice that if $h \to 0$, then ν_h converges to the measure concentrated at the identity of G. So, given $\varepsilon > 0$ and a compact subset K of Γ, there exists an h, $0 < h < 1$, such that if $0 < t \leqslant h$ then

$$\sup_{\gamma\in K}\left|\exp\left(-t\varphi(\gamma)\right)-1\right| < \varepsilon/2.$$

Now, by the infinitesimality of array, it follows that there exists a positive integer n_0 such that if $n \geqslant n_0$ then $\hat{\mu}_n(\gamma) > 0$ for $\gamma \in K$ and

$$\sup_{\gamma\in K}\left|\hat{\mu}_n(\gamma)^{nh}-\exp\left(-h\varphi(\gamma)\right)\right| < \varepsilon/2.$$

Hence

$$\sup_{\gamma\in K}|\hat{\mu}_n(\gamma)^{nh}-1| < \varepsilon \quad \text{for} \quad n \geqslant n_0$$

and, since $\hat{\mu}_n(\gamma)$ is positive if $\gamma \in K$ and $n \geqslant n_0$, we have

$$\sup_{\gamma\in K}|\hat{\mu}_n(\gamma)^{nt}-1| = \sup_{\gamma\in K}\left|\left(\hat{\mu}_n(\gamma)^{nh}\right)^{t/h}-1\right| < \varepsilon$$

for $t \leqslant h$ and $n \geqslant n_0$.

We now complete the proof by citing the following lemma, resulting from the remark after Definition 5.1, IV in [11].

LEMMA 5. *Let $\{S_k^{(n)}:\ k = 1,2,\ldots,n;\ n = 1,2,\ldots\}$ be a triangular array of G-valued random variables. The following conditions are equivalent:*

(i) $\lim_{h\to 0}\overline{\lim_{n}}\max_{k\leqslant nh} P(\{\|S_k^{(n)}\|\geqslant \varepsilon\}) = 0$ *for each* $\varepsilon > 0$.

(ii) *For each compact subset* K *of* Γ

$$\lim_{h\to 0}\overline{\lim_{n}}\max_{k\leqslant nh}\sup_{\gamma\in K}|\psi_k^{(n)}(\gamma)-1| = 0,$$

where $\psi_k^{(n)}$ *is the characteristic function of* $S_k^{(n)}$.

Remark. It is easy to see that we can formulate and prove, in a similar way, an analogue of Theorem 1 for C_G-valued random elements. Obviously, instead of random elements used in the formulation of Theorem 1, we have to deal with the stochastic process obtained by linear interpolation of the sums $(X_1+\ldots+X_k)/n^{1/2}$, $k = 0, 1, \ldots, n$.

Added in proof. I have noticed that the fact that the condition (i) of Theorem 2 holds in every LCA group follows also from the paper of V. V. Sazonow and V. N. Tutubalin *Probability distributions on topological groups,* Theor. Prob. Appl. (1966) (see Theorem 4.11).

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