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Embedding D^{τ} in Dugundji spaces, with an application to linear topological classification of spaces of continuous functions

by

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Abstract. Pełczyński's class of Dugundji spaces was shown in a previous paper (*On a problem of Pełczyński: Milutin spaces, Dugundji spaces, and AE* (0-dim), Studia Math. 52 (1974)) to coincide with the class of compact absolute extensors for compact zero-dimensional spaces. It was also shown that such spaces allow a particular kind of inverse limit representation. Using this representation, it is now shown that a Dugundji space of weight τ, that satisfies a certain further condition $(*)$, contains a homeomorphic copy of $\{0, 1\}^{\tau}$. (A space S satisfies $(*)$ if we cannot write $S = \bigcup_{n=1}^{\infty} S_n$, where the S_n are closed subspaces with $w(S_n) < w(S)$.) This answers a problem posed by Pełczyński and allows us to conclude that for each such space S the Banach space $\mathscr{C}(S)$ is linearly homeomorphic to $\mathscr{C}(\{0, 1\}^{\tau})$. Some small improvements are also offered to some related results of Efimov on subspaces of dyadic spaces.

1. Notations. All the topological spaces considered, S, T, $X, \ldots,$ will be compact (and Hausdorff). $\boldsymbol{D}$ will denote the two-point space $\{0, 1\}$ and $\boldsymbol{I}$ the unit interval $[0, 1]$. If A is any set, $\boldsymbol{D}^{A}$ will denote the product space $\prod_{a \in A} \boldsymbol{D}_a$ of a family of copies of $\boldsymbol{D}$, indexed by A. The space $\boldsymbol{I}^{A}$ will be defined similarly. In particular, when γ is an ordinal, $\boldsymbol{D}^{\gamma}$ will mean $\prod_{a<\gamma} \boldsymbol{D}_a$, and $\boldsymbol{D}^0$ is, of course, just $\{\emptyset\}$. The *topological weight* $w(S)$ of S is the smallest cardinal τ such that there is a base $\mathscr{B}$ for the open sets of S with card $\mathscr{B} = \tau$. When $w(S)$ is infinite, it is also the smallest cardinal τ such that S can be homeomorphically embedded in $\boldsymbol{I}^{\tau}$.

In the above, and in all that follows, we follow the convention that a cardinal is identified with the corresponding initial ordinal. If γ is a limit ordinal, the cofinality $\mathrm{cf}(\gamma)$ of γ is the smallest cardinal $\varkappa$ such that there is a family $\big(\gamma(\xi)\big)_{\xi<\varkappa}$ of ordinals $\gamma(\xi) < \gamma$ with $\sup\{\gamma(\xi) : \xi < \varkappa\} = \gamma$. We say that an infinite cardinal τ is *regular* is $\mathrm{cf}(\tau) = \tau$. Otherwise, τ is *singular*. For every τ, the successor cardinal τ^{+} is regular, and for every limit ordinal γ, $\mathrm{cf}(\gamma)$ is regular.

When S is a topological space, and $x \in S$, the *neighbourhood character* $\chi(x, S)$ of x in S is the smallest cardinal of a base of neighbourhoods of x

in S. If S is compact Hausdorff, then $\chi(x, S)$ is also the smallest cardinal $\varkappa$ for which there is a set $\mathscr{G}$ of neighbourhoods of x with $\operatorname{card}\mathscr{G} = \varkappa$ and $\bigcap \mathscr{G} = \{x\}$.

The Banach space of all continuous real-valued functions on S will, of course, be denoted by $\mathscr{C}(S)$. When $\varphi\colon S \to T$ is a continuous mapping, an operator $\varphi^0\colon \mathscr{C}(T) \to \mathscr{C}(S)$ is defined by $\varphi^0(g) = g\circ\varphi$. If $\varphi\colon S \hookrightarrow T$ is a continuous injection, a *regular extension operator* for φ is a positive linear operator $u\colon \mathscr{C}(S) \to \mathscr{C}(T)$ satisfying $u(1_S) = 1_T$ and $\varphi^0\big(u(f)\big) = f$ for all $f \in \mathscr{C}(S)$. The space S is said to be a *Dugundji space* if, whenever $\varphi\colon S \hookrightarrow T$ is a continuous injection, there is a regular extension operator for φ.

As a last piece of terminology, we recall that a continuous mapping $\varphi\colon S \to T$ was said in [4] to have a *metrizable kernel* if there exist a compact metrizable space K and a continuous injection $k\colon S \hookrightarrow T\times K$ such that $\varphi = \Pi_1 \circ k$, Π_1 being the projection $T \times K \to T$.

2. Preliminary results concerning dyadic spaces. The results quoted in this paragraph are due, except for one modification, to Efimov ([2] and [3]). We recall that a compact space S is said to be *dyadic* if, for some τ, there is a continuous surjection $\boldsymbol{D}^\tau \to S$. It follows from the results of [4] that every Dugundji space is dyadic.

2.1. Proposition (Corollary 3 of [2]). *If $\mathscr{U}$ is a set of non-empty open subsets of $\mathscr{D}^\tau$ and $\operatorname{card}\mathscr{U}$ is an uncountable regular cardinal, then there is a subset $\mathscr{V}$ of $\mathscr{U}$ with $\operatorname{card}\mathscr{V} = \operatorname{card}\mathscr{U}$ and $\bigcap\mathscr{V} \neq \emptyset$.*

2.2. Theorem (Theorem 14 of [2] and Theorem 1 of [3]). *Let S be a dyadic space, a be an infinite cardinal and M be a subset of S with $\chi(x, S) \leqslant a$ for all $x \in M$. Then $w(\overline{M}) \leqslant a$.*

2.3. Theorem (Stronger version of Theorem 16 of [2]). *If S is a dyadic space and $w(S)$ is an uncountable regular cardinal, then there is a non-empty open subset U of S such that $\chi(x, S) = w(S)$ for every $x \in U$.*

Proof. Efimov proved this result under the additional assumption that every weakly inaccessible cardinal is strongly inaccessible (which will be the case in particular if we assume the Generalized Continuum Hypothesis). But with a small modification in the argument we can do without that assumption.

The original argument in the case where $w(S)$ is a successor cardinal does not require any additional hypothesis. So let us suppose that $w(S) = \tau$, a regular limit cardinal (i.e., weakly inaccessible). Put $M = \{x \in S\colon \chi(x, S) < \tau\}$. It is required to prove $\overline{M} \neq S$. We suppose to the contrary that $\overline{M} = S$ and show first that every dense subset of M has cardinal at least τ. If not, there is $N \subset M$ with $\operatorname{card} N < \tau$ and $\overline{N} = S$. Since τ is regular,

$$\sup\{\chi(x, S)\colon x \in N\} < \tau$$

and

$$w(S) = w(\overline{M}) = w(\overline{N}) < \tau$$

by 2.2. This is a contradiction. The remainder of the proof (the construction of a 'strong sequence' of order type τ) proceeds exactly as in [2].

2.4. Remark. The same modification to Efimov's method of proof allows us to remove assumptions of GCH type from Lemma 3 and Theorem 7 of [3].

3. Restricted inverse systems. The techniques employed in what follows will make much use of a certain type of inverse system. Similar systems have been studied by Cohen from the point of view of homotopy theory, and from [1] I borrow the term *restricted inverse system* (of compact spaces) for a system $(X_\alpha, p_{\alpha,\beta})_\tau$, satisfying:

(i) τ is an ordinal;

(ii) whenever $\alpha \leqslant \beta < \tau$, $p_{\alpha,\beta}$ is a continuous surjection $X_\beta \to X_\alpha$;

(iii) whenever $\alpha \leqslant \beta \leqslant \gamma < \tau$, $p_{\alpha,\beta}\circ p_{\beta,\gamma} = p_{\alpha,\gamma}$;

(iv) whenever γ is a limit ordinal and $\gamma < \tau$, the continuous surjection

$$X_\gamma \to \varprojlim (X_\alpha, p_{\alpha,\beta})_\gamma$$

determined by the consistent family $(p_{\alpha,\gamma})_{\alpha<\gamma}$ is injective (and thus a homeomorphism).

It will usually be convenient to write X_τ for the inverse limit of a system of the above type, and to write p_α for the canonical surjection $X_\tau \to X_\alpha$.

An elementary observation is the following:

3.1. Proposition. *Let $(X_\alpha, p_{\alpha,\beta})_\tau$ be a restricted inverse system and let F be a closed subset of X_τ. Then $(p_\alpha[F], p_{\alpha,\beta}|p_\beta[F])_\tau$ is a restricted inverse system with inverse limit homeomorphic to F.*

The restricted inverse systems in which we shall be particularly interested will satisfy one or more of:

(a) for every $\alpha < \tau$, $p_{\alpha,\alpha+1}$ has a metrizable kernel;

(b) for every $\alpha < \tau$, $p_{\alpha,\alpha+1}$ is an open mapping;

(c) X_0 is trivial (i.e., $\operatorname{card} X_0 = 1$).

We shall need to make some observations about the weights of the spaces X_α in such systems.

3.2. Proposition. *If $(X_\alpha, p_{\alpha,\beta})_\tau$ is a restricted inverse system satisfying (a), then, for each $\alpha \leqslant \tau$, $w(X_\alpha) \leqslant \max\{\omega, w(X_0), \operatorname{card}\alpha\}$.*

Proof. For each $\beta < \alpha$, there is a compact metrizable space K_β and a continuous mapping $\theta_\beta\colon X_{\beta+1} \to K_\beta$ such that $(p_{\beta,\beta+1}\times\theta_\beta)\colon X_{\beta+1} \to X_\beta \times K_\beta$ is an injection. We note that the continuous mapping $X_\alpha \to X_0 \times \prod_{\beta<\alpha} K_\beta$; $x \mapsto \big(p_{0,\alpha}(x), \big(\theta_\beta\circ p_{\beta,\alpha}(x)\big)_{\beta<\alpha}\big)$ is injective, and that the weight of $X_0 \times \prod_{\beta<\alpha} K_\beta$ is at most $w(X_0) + \omega\cdot\operatorname{card}\alpha$.

3.3. PROPOSITION. *Let $(X_\alpha, p_{\alpha,\beta})_\tau$ be a restricted inverse system. If $\gamma \leqslant \tau$ is an uncountable regular cardinal and*

$$\operatorname{card}\{\beta:\ \beta < \gamma \text{ and } p_{\beta,\beta+1} \text{ is not injective}\} = \gamma,$$

then $w(X_\gamma) \geqslant \gamma$.

Proof. Suppose, if possible, that $w(X_\gamma) < \gamma$, and choose a set $\mathscr{C}_0 \subset \mathscr{C}(X_\gamma)$ with $\operatorname{card}\mathscr{C}_0 = w(X_\gamma)$ and such that $\mathscr{C}_0$ separates the points of X_γ. For each $f \in \mathscr{C}_0$, there is, by a Stone–Weierstrass argument, an ordinal $\alpha(f) < \gamma$ such that f factors through $p_{\alpha(f),\gamma}$. We find that $\alpha = \sup\{\alpha(f): f \in \mathscr{C}_0\} < \gamma$ and that $p_{\alpha,\gamma}$ is injective. Thus $p_{\beta,\beta+1}$ can fail to be injective only if $\beta < \alpha$ and, since $\operatorname{card}\alpha < \gamma$, we have a contradiction.

3.4. PROPOSITION. *Let $(X_\alpha, p_{\alpha,\beta})_\tau$ be a restricted inverse system and let $\gamma \leqslant \tau$ be a limit ordinal. Then $w(X_\gamma) \leqslant \sup\{w(X_\alpha)^+:\ \alpha < \gamma\}$.*

Proof. If not, there is a cardinal $\varkappa$ such that $\varkappa < w(X_\gamma)$ and $\varkappa > w(X_\alpha)$ when $\alpha < \gamma$.

If we put $\lambda = \operatorname{cf}(\gamma)$, it must be that $\lambda > \varkappa$. For if not, we could find $(\gamma(\xi))_{\xi<\varkappa}$ with $\gamma(\xi) < \gamma$ and

$$\sup\{\gamma(\xi):\ \xi < \varkappa\} = \gamma.$$

In this case there would be a continuous injection of X_γ into $\prod_{\xi<\varkappa} X_{\gamma(\xi)}$, which has weight at most

$$\varkappa \cdot \sup\{w(X_{\gamma(\xi)}):\ \xi < \varkappa\} = \varkappa < w(X_\gamma).$$

Let us suppose first that $\varkappa$ can be chosen to be a regular cardinal. There is an increasing family $(\beta(\xi))_{\xi<\lambda}$ of ordinals with $\beta(\xi) < \gamma$ and $\sup\{\beta(\xi):\ \xi < \lambda\} = \gamma$, and such that $p_{\beta(\xi),\,\beta(\xi)+1}$ fails to be injective for each $\xi < \lambda$. (If not, then for some $\beta < \gamma$, $p_{\beta,\gamma}$ would be injective, giving us $w(X_\gamma) = w(X_\beta)$.) If we put $\alpha = \sup\{\beta(\xi):\ \xi < \varkappa\}$, we see that $\alpha < \gamma$, because $\varkappa < \operatorname{cf}(\gamma)$, while by 3.3 applied to the system $(X_{\beta(\xi)}, p_{\beta(\xi),\beta(\gamma)})_\varkappa$, $w(X_\alpha)$ is at least $\varkappa$, a contradiction.

In the remaining case, $\varkappa = \sup\{w(X_\alpha):\ \alpha < \gamma\}$ and $\varkappa$ is a singular cardinal. For each cardinal $\mu < \varkappa$, there is $\alpha(\mu) < \gamma$ with $w(X_{\alpha(\mu)}) \geqslant \mu$. This time if we put $\alpha = \sup\{\alpha(\mu): \mu < \varkappa\}$, we see again that $\alpha < \gamma$, while $w(X_\alpha) \geqslant \varkappa$. This contradiction ends the proof.

3.5. COROLLARY. *Let $(X_\alpha, p_{\alpha,\beta})_\tau$ be a restricted inverse system satisfying* (a) *and* (c), *and suppose further that none of the mappings $p_{\alpha,\alpha+1}$ is injective. Then $w(X_\alpha) = \operatorname{card}\alpha$ for all $\omega \leqslant \alpha \leqslant \tau$.*

Proof. In the light of 3.2, it is enough to prove that $w(X_\alpha) \geqslant \operatorname{card}\alpha$. Since, by 3.3, $w(X_\alpha) \geqslant w(X_\gamma)$ whenever $\gamma \leqslant \alpha$ and γ is a regular uncountable cardinal, we see that this is indeed the case when α is uncountable. It remains only to note that $w(X_\omega)$ is not finite. (If it were, then $p_{n,\omega}$ would be injective for some $n < \omega$).

4. Construction of the regular inverse system. It was shown in [4] that a space S is Dugundji if and only if S is homeomorphic to the limit X_τ of a restricted inverse system $(X_\alpha, p_{\alpha,\beta})_\tau$ that satisfies (a), (b) and (c). In this section, we sketch the construction of this system, starting from a space S that is assumed to be Dugundji, and isolating as lemmas certain arguments that we shall need again later on.

Write τ for the weight of S (which, to avoid trivialities, we can assume to be uncountable). Choose a set A with $\operatorname{card}A = \tau$, and an embedding of S in $\boldsymbol{I}^A$. Then, by hypothesis, there is a regular extension operator u: $\mathscr{C}(S) \to \mathscr{C}(\boldsymbol{I}^A)$.

Let us fix some notation: when $B \subseteq C \subseteq A$, write $\pi_{B,C}$ for the projection $\boldsymbol{I}^C \to \boldsymbol{I}^B$ (for future reference, let us agree to use the same notation π_{BC} for the projection $\boldsymbol{D}^C \to \boldsymbol{D}^B$); denote by $S(B)$ the subspace $\pi_{B,A}[S]$ of $\boldsymbol{I}^B$ and by $P_{B,C}$ the restriction of $\pi_{B,C}$ to $S(C)$. For simplicity, write π_B and P_B for $\pi_{B,A}$ and $P_{B,A}$, respectively.

Let us say that $B \subseteq A$ is an *(F) set* (for 'factorization') if $u(f \circ P_B)$ factors through π_B for every $f \in \mathscr{C}(S(B))$. Let us say that B is a *(WF) set* if the restriction of $u(f \circ P_B)$ to the subspace $S(B) \times \boldsymbol{I}^{A\setminus B}$ of $\boldsymbol{I}^A$ factors through π_B for every $f \in \mathscr{C}(S(B))$. The essential lemmas on (F) and (WF) sets are the following:

4.1. LEMMA. *If B is (WF), then P_B: $S \to S(B)$ is an open mapping.*

Proof. We consider the diagram

$$\begin{tikzcd} S \arrow[rr, hook] \arrow[dr, "P_B"'] & & S(B) \times \boldsymbol{I}^{A\setminus B} \arrow[dl, "\pi_B"] \\ & S(B) & \end{tikzcd}$$

If we define $w(f) = u(f) \mid (S(B) \times \boldsymbol{I}^{A\setminus B})$, we see that w is a regular extension operator $\mathscr{C}(S) \to \mathscr{C}(S(B) \times \boldsymbol{I}^{A\setminus B})$. Moreover, $w(f \circ P_B) = f \circ \pi_B$ for all $f \in \mathscr{C}(S(B))$, and π_B is an open mapping, so that P_B is open by Lemma 2 of [4].

4.2. LEMMA. *If C and D are (WF) sets, then $B = C \cup D$ is also (WF).*

Proof. $S(B)$ can be identified with a subspace of $S(C) \times S(D)$, so every $f \in \mathscr{C}(S(B))$ can be approximated uniformly by sums of functions $(g \otimes h)$ with $g \in \mathscr{C}(S(C))$ and $h \in \mathscr{C}(S(D))$. We consider the regular extension operator

$$v:\ \mathscr{C}(S) \to \mathscr{C}(S(C) \times \boldsymbol{I}^{A\setminus C})$$

defined by $v(f) = u(f) \mid (S(C) \times \boldsymbol{I}^{A\setminus C})$. Since $v(g \circ P_C) = g \circ \pi_C$ for all

$g \in \mathscr{C}(S(C))$, it follows from the module property of Corollary 2 of [4] that

$$v\big((g\otimes h)\circ P_B\big) = v\big((g\circ P_C)\cdot(h\circ P_D)\big) = (g\circ\pi_C)\cdot v(h\circ P_B).$$

Now on the intersection $(S(C)\times I^{A\setminus C})\cap(S(D)\times I^{A\setminus B})$ we know that $v(h\circ P_B)$ coincides with $(h\circ\pi_D)$. But $S(B)\times I^{A\setminus B}$ is contained in this intersection, so on this subspace it is clearly true that $u\big((g\otimes h)\circ P_B\big)$ coincides with $(g\otimes h)\circ\pi_B$. The required result now follows by the uniform approximation of an arbitrary $f\in\mathscr{C}(S(B))$ by sums of functions $g\otimes h$.

4.3. LEMMA. *If H is an infinite subset of A, then there is an (F) set B with $H\subseteq B$ and* $\operatorname{card} B = \operatorname{card} H$.

Proof. We put $B(0) = H$ and define $B(n)$ inductively as follows. If $B(n)$ has been defined already and $\operatorname{card} B(n) = \operatorname{card} H = \varkappa$, say, we note that $w\big(S(B(n))\big)$ is at most $\varkappa$, so that there is a subset $\mathscr{C}_n$ of $\mathscr{C}\big(S(B(n))\big)$ with $\operatorname{card}\mathscr{C}_n = \varkappa$, which separates the points of $S(B(n))$. For each $f\in\mathscr{C}_n$, there exists, by the Stone–Weierstrass theorem, a countable subset $E(f)$ of A such that $u(f\circ P_{B(n)})$ factors through $\pi_{E(f)}$. If we put $B(n+1) = B(n)\cup \bigcup\{E(f)\colon f\in\mathscr{C}_n\}$, we find that $u(g\circ P_{B(n)})$ factors through $\pi_{B(n+1)}$ for all $g\in\mathscr{C}\big(S(B(n))\big)$. Finally, we set $B = \bigcup_{n=0}^{\infty} B(n)$. For each $f\in\mathscr{C}(S(B))$, we can approximate f uniformly with functions $h = g\circ P_{B(n),B}$ with $g\in\mathscr{C}\big(S(B(n))\big)$. For each such h, $u(h\circ P_B) = u(g\circ P_{B(n)})$ factors through π_B, and so $u(f\circ P_B)$ does the same.

4.4. LEMMA. *The union of an increasing family of (WF) sets is again* (WF)

Proof. We apply the same uniform approximation argument as was used at the end of the proof of 4.3.

We see now that to obtain the system $(X_\alpha, p_{\alpha,\beta})_\tau$ it is enough to obtain an increasing family of (WF) sets $\big(A(\alpha)\big)_{\alpha<\tau}$ satisfying

(α) $A(0) = \emptyset$,

(β) $A(\gamma) = \bigcup\{A(\alpha)\colon \alpha<\gamma\}$ when γ is a limit ordinal,

(γ) $A(\alpha+1)\setminus A(\alpha)$ is countable for all α, and such that the mappings $P_{A(\alpha)}$ separate the points of S.

We put $X_\alpha = S\big(A(\alpha)\big)$ and $p_{\alpha,\beta} = P_{A(\alpha),A(\beta)}$, and check the required properties as follows:

condition (iv) of the definition of a restricted inverse system is satisfied because of (β);

condition (a) is satisfied because of (γ);

condition (c) is satisfied because of (α);

condition (b) is satisfied because $P_{A(\alpha)}$ is an open mapping, by 4.1, so that $p_{\alpha,\alpha+1} = P_{A(\alpha),A(\alpha+1)}$ is also open.

If the mappings $P_{A(\alpha)}$ separate the points of S, then the surjection $S\to X_\tau$ determined by the family $(P_{A(\alpha)})$ is a homeomorphism.

The family $\big(A(\alpha)\big)$ is constructed by transfinite induction. We choose, once and for all, a family $(f_\xi)_{\xi<\tau}$ in $\mathscr{C}(S)$ that separates the points of S. We define $A(0) = \emptyset$. To perform the inductive step of obtaining $A(\alpha+1)$ from $A(\alpha)$, we take ξ to be the smallest ordinal such that f_ξ does not factor through $P_{A(\alpha)}$. (We note that $w\big(S(A(\alpha))\big)\leqslant\omega\cdot\operatorname{card}\alpha<\tau$ by 3.2, so that such a ξ will always exist.) We then let B be a countable (F) set such that f_ξ does factor through P_B. We put $A(\alpha+1) = A(\alpha)\cup B$, which is a (WF) set by 4.3. When we reach a limit ordinal γ, we define

$$A(\gamma) = \bigcup\{A(\alpha)\colon \alpha<\gamma\}$$

which is a (WF) set by 4.4.

In the proof of the main theorem it will be necessary to consider what happens when an (F) set B is adjoined simultaneously to all the sets $A(\alpha)$ we have just constructed.

We formulate this as a proposition.

4.5. PROPOSITION. *Retaining the notation of the preceding discussion, let B be a (WF) set with $\operatorname{card} B = \varkappa<\tau$. For each $\alpha<\tau$, define $B(\alpha) = A(\alpha)\cup B$, $Y_\alpha = S\big(B(\alpha)\big)$, $q_\alpha = P_{B(\alpha)}$, $q_{\alpha,\beta} = P_{B(\alpha),B(\beta)}$ and $r_\alpha = P_{A(\alpha),B(\alpha)}$.*

Then $(Y_\alpha, q_{\alpha,\beta})_\tau$ is a restricted inverse system, with inverse limit S, which satisfies (a) *and* (b). *The maps r_α are open. Moreover, $w(Y_\alpha) = \operatorname{card}\alpha$ for all $\alpha\geqslant\max\{\omega,\varkappa\}$.*

Proof. All the assertions are immediate consequences of what we know already, except perhaps the last. For this, we note that, by construction, the system $(X_\alpha, p_{\alpha,\beta})_\tau$ satisfies the hypotheses of 3.5, so that $w(Y_\alpha)\geqslant w(X_\alpha) = \operatorname{card}\alpha$ for all α, while by 3.2, $w(Y_\alpha)\leqslant\max\{\omega, w(Y_0), \operatorname{card}\alpha\}$. We now only have to note that $w(Y_0)\leqslant\operatorname{card} B = \varkappa$.

5. The embedding theorem.

5.1. PROPOSITION. *Let $p\colon X\to Z$ be a continuous, open surjection with a metrizable kernel. Let us suppose that in the following diagram T is zero-dimensional, j is a continuous injection, that φ and θ are continuous mappings, and that $p\circ\theta = \varphi\circ j$,*

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[d, hook, "j"'] & X \arrow[d, "p"] \\ T \arrow[r, "\varphi"'] \arrow[ur, dashed, "\psi"] & Z \end{tikzcd}$$

Then there is a continuous mapping $\psi\colon T\to X$ such that $\psi\circ j = \varphi$ and $p\circ\psi = \varphi$.

Proof. The method is that used in the course of the proof of Theorem 2 of [4]. We consider the mapping Ψ from T into the non-empty closed subsets of X defined by

$$\Psi(t) = \{\varphi(r)\} \quad \text{if} \quad t = j(r) \in j[R],$$
$$\Psi(t) = p^{-1}(\varphi(t)) \quad \text{if} \quad t \notin j[R].$$

If V is an open subset of X, then

$$\{t \in T: \Psi(t) \cap V \neq \emptyset\} = j\varphi^{-1}[V] \cup (\varphi^{-1}p[V] \setminus j[R]) = (\varphi^{-1}p[V]) \setminus (j[R \setminus \varphi^{-1}[V]]),$$

which is an open subset of T. Thus, in the terminology of [5], Ψ is lower semicontinuous.

By hypothesis, there is a metrizable space K and a continuous injection $k: X \to Z \times K$ such that $p = \Pi_1 \circ k$. The mapping Γ defined by $\Gamma(t) = \Pi_2 k[\Psi(t)]$ is lower semicontinuous from T into the closed non-empty subsets of K. By Theorem 2 of [5], there is a continous mapping $\gamma: T \to K$ with $\gamma(t) \in \Gamma(t)$ for all $t \in T$. To finish the proof, we note that ψ, with the required properties, is well defined by

$$k\psi(t) = (\varphi(t), \gamma(t)).$$

5.2. Lemma. *Let $(X_a, p_{a,\beta})_\tau$ be a restricted inverse system satisfying* (a) *and* (b). *Suppose that $\gamma < \tau$ is a limit ordinal and that $\alpha < \gamma$.*

Suppose further that, for some ξ, φ is a continuous injection $D^\xi \subset\!\to X_a$ which has the property that, for each $z \in D^\xi$, $p_{a,\gamma}^{-1}(\varphi(z))$ contains more than one point.

Then there is an ordinal β with $\alpha < \beta < \gamma$ and a continuous injection ψ: $D^{\xi+1} \subset\!\to X_\beta$ such that $p_{a,\beta} \circ \psi = \varphi \circ \pi_{\xi,\xi+1}$.

Proof. For each $z \in D^\xi$, because $p_{a,\gamma}^{-1}(\varphi(z))$ contains more than one point, there is an ordinal $\beta(z)$ with $\alpha < \beta(z) < \gamma$ such that $p_{a,\beta(z)}^{-1}(\varphi(z))$ contains distinct points, $y(z, 0)$ and $y(z, 1)$, say. By repeated applications of 5.1 (with $T = D^\xi$ and $R = \{z\}$) we can find, for each z, continuous mappings

$$\psi_{z,i}: D^\xi \to X_{\beta(z)},$$

such that $p_{a,\beta(z)} \circ \psi_{z,i} = \varphi$ and

$$\psi_{z,i}(z) = y(z, i) \quad (i = 0, 1).$$

For each z, there is an open and closed neighbourhood $U(z)$ of z in D^ξ such that $\psi_{z,0}(z') \neq \psi_{z,1}(z')$ for $z' \in U(z)$. Let us cover D^ξ with finitely many such sets $U(z_k)$ $(1 \leqslant k \leqslant n)$ and obtain an open and closed partition (V_k) of D^ξ with $V_k \subseteq U(z_k)$ $(1 \leqslant k \leqslant n)$.

Define $\beta = \sup\{\beta(z_k): 1 \leqslant k \leqslant n\}$, and obtain, again by 5.1, continuous mappings $\theta_{k,i}: D^\xi \to X_\beta$ such that $p_{\beta(z_k),\beta} \circ \theta_{k,i} = \psi_{z_k,i}$ $(i = 0, 1;$

$1 \leqslant k \leqslant n)$. Finally, note that we can regard $D^{\xi+1}$ as $D^\xi \times D$ and so write a typical element as (z, i) with $z \in D^\xi$ and $i \in \{0, 1\}$. We define

$$\theta: D^{\xi+1} \to X_\beta$$

by $\theta(z, i) = \theta_{k,i}(z)$ if $z \in V_k$.

It is easy to check that θ is injective.

The above lemma provides us with the essential technique required to build up, by transfinite induction, a continuous injection $D^\tau \subset\!\to S$. However, the (obviously necessary) hypothesis that each $p_{a,\gamma}^{-1}(\varphi(z))$ should contain more than one point means that we have to make a few final preparations.

5.3. Lemma. *Let p: $S \to X$ be a continuous mapping, let $x \in X$ and suppose that $p^{-1}(x)$ is a singleton, $\{s\}$, say.*

Then $\chi(s, S) \leqslant \chi(x, X)$.

Proof. Let $\mathcal{G}$ be a set of neighbourhoods of x in X with card $\mathcal{G} = \chi(x, X)$ and $\bigcap \mathcal{G} = \{x\}$. Then each $p^{-1}[G]$ is a neighbourhood of s in S and

$$\{p^{-1}[G]: G \in \mathcal{G}\} = \{s\}.$$

This lemma tells us that the hypothesis placed on the continuous injection φ in 5.2 will be satisfied in particular if

$$\chi(x, X_\gamma) > \chi(p_{a,\gamma}(x), X_a)$$

whenever $p_{a,\gamma}(x) \in \varphi[D^\xi]$.

5.4. Proposition. *Let $(X_a, p_{a,\beta})_\tau$ be a restricted inverse system satisfying* (a) *and* (b). *Let $\varkappa$ be a regular uncountable cardinal, α, γ, ξ be ordinals with $\alpha < \gamma < \tau$ and $\xi < \varkappa$. Suppose Z is a closed subset of X_a such that*

$$\chi(x, X_\gamma) = \varkappa \quad \textit{when} \quad x \in p_{a,\gamma}^{-1}[Z]$$

and

$$\chi(x, X_\beta) < \varkappa \quad \textit{when} \quad x \in p_{a,\beta}^{-1}[Z]$$

with $\alpha \leqslant \beta < \gamma$. Then if φ: $D^\xi \subset\!\to Z$ is a continuous injection, there is a continuous injection θ: $D^\varkappa \subset\!\to X_\gamma$ with

$$p_{a,\gamma} \circ \theta = \varphi \circ \pi_{\xi,\varkappa}.$$

Proof. We define, by transfinite induction, an increasing family $(\alpha(\eta))_{\xi \leqslant \eta < \varkappa}$ of ordinals $\alpha(\eta) < \gamma$, together with continuous injections θ_η: $D^\eta \to X_{\alpha(\eta)}$ in such a way that

$$\theta_\eta \circ \pi_{\eta,\zeta} = p_{\alpha(\eta),\alpha(\zeta)} \circ \theta_\zeta$$

when $\xi \leqslant \eta < \zeta < \varkappa$. We start, of course, by putting $\alpha(\xi) = \alpha$ and $\theta_\xi = \varphi$.

In view of 5.3 and the remark above, obtaining $\alpha(\eta+1)$ and $\theta_{\eta+1}$

from $\alpha(\eta)$ and θ_η is exactly 5.2. If $\zeta < \varkappa$ is a limit ordinal, we put

$$\alpha(\zeta) = \sup\{\alpha(\eta)\colon \eta < \zeta\}.$$

We can note that, by an argument similar to one used early in the proof of 3.4, we must have $\mathrm{cf}(\gamma) \geqslant \varkappa$, so that $\alpha(\zeta) < \gamma$. A continuous injection $\theta_\zeta\colon \boldsymbol{D}^\zeta \to X_{\alpha(\zeta)}$ is defined by the family $(\theta_\eta)_{\eta<\zeta}$ when we identify $\boldsymbol{D}^\zeta$ with $\varprojlim(\boldsymbol{D}^\eta)_{\eta<\zeta}$ and $X_{\alpha(\zeta)}$ with $\varprojlim(X_{\alpha(\eta)})_{\eta<\zeta}$.

In the same way, when the induction is complete, $\theta\colon \boldsymbol{D}^\varkappa \to X_\beta$ is determined, where $\beta = \sup\{\alpha(\eta)\colon \eta < \varkappa\}$. Certainly $\beta \leqslant \gamma$, but since $p_{\alpha,\beta}^{-1}[Z]$ contains points x with $\chi(x, X_\beta) \geqslant w(\boldsymbol{D}^\varkappa) = \varkappa$, we see that $\beta = \gamma$ (so that, incidentally, we must have the equality $\mathrm{cf}(\gamma) = \varkappa$).

Finally, a last lemma, no doubt well known, on the behaviour of the neighbourhood character under an open mapping.

5.5. Lemma. *Let $r\colon Y \to X$ be an open mapping. Then, for every $y \in Y$, $\chi(y, Y) \geqslant \chi(r(y), X)$.*

Proof. Let $\mathscr{H}$ be a base of neighbourhoods of y in Y with $\mathrm{card}\,\mathscr{H} = \chi(y, Y)$. If $x \in \bigcap\{r[H]\colon H \in \mathscr{H}\}$, then the closed set $r^{-1}(x)$ meets each H and so contains y. Thus $\mathscr{G} = \{r[H]\colon H \in \mathscr{H}\}$ is a set of neighbourhoods of $r(y)$ in X with $\mathrm{card}\,\mathscr{G} = \chi(y, Y)$ and $\bigcap\mathscr{G} = \{r(y)\}$.

Let us say, following the notation of p. 71 of [6], that the *space S satisfies condition* (*) if we cannot represent S as a union $\bigcup_{n=0}^{\infty} S_n$ of closed subspaces S_n satisfying $w(S_n) < w(S)$.

5.6. Theorem. *Let S be a Dugundji space that satisfies* (*). *Then S contains a subspace homeomorphic to $\boldsymbol{D}^\tau$, where $\tau = w(s)$.*

Proof. We may certainly assume that τ is uncountable, since a compact metrizable space that satisfies (*) (and is thus not countable) certainly contains a copy of $\boldsymbol{D}^\omega$. We consider, then, three cases, taking the easiest first.

(I) τ is an uncountable regular cardinal.

There is, by 2.3, a non-empty open subset U of S with $\chi(x, S) = \tau$ for all $x \in U$. Choose a function $f_0 \in \mathscr{C}(S)$ such that $f_0(x_0) = 1$ for some $x_0 \in U$ and $f_0|(S \setminus U) = 0$. Construct the inverse system $(X_\alpha, p_{\alpha,\beta})_\tau$ as in Section 4, ensuring that f_0 factors through $p_1\colon S \to X_1$. In this case, we see that $p_1^{-1}(p_1(x_0)) \subseteq U$. Define

$$\varphi\colon \boldsymbol{D}^0 \hookrightarrow X_1$$

by $\varphi(\emptyset) = p_1(x_0)$ and apply 5.4 to get $\theta\colon \boldsymbol{D}^\tau \hookrightarrow X_\tau = S$.

(II) τ is singular and $\mathrm{cf}(\tau) = \varkappa > \omega$.

Let $(\beta(\xi))_{\xi<\varkappa}$ be an increasing family of regular cardinals with $\beta(\xi) < \tau$ and $\sup\{\beta(\xi)\colon \xi < \varkappa\} = \tau$. Construct the inverse system $(X_\alpha, p_{\alpha,\beta})_\tau$ as in

Section 4, and note that for each ξ we have $w(X_{\beta(\xi)}) = \beta(\xi)$ by 3.5. Hence, by 3.2, there is a non-empty open subset V_ξ of $X_{\beta(\xi)}$ with $\chi(x, X_{\beta(\xi)}) = \beta(\xi)$ for all $x \in V_\xi$. Applying 2.1 to

$$\mathscr{U} = \{p_{\beta(\xi)}^{-1}[V_\xi]\colon \xi < \varkappa\},$$

we see that there is a subset J of $\varkappa$ with $\mathrm{card}\,J = \varkappa$ and

$$\bigcap\{p_{\beta(\xi)}^{-1}[V_\xi]\colon \xi \in J\} \neq \emptyset.$$

Let us label J as an increasing family $(\xi(\eta))_{\eta<\varkappa}$ and put $\alpha(\eta) = \beta(\xi(\eta))$, $U_\eta = V_{\xi(\eta)}$. We may suppose $\alpha(0) \geqslant \varkappa$.

Now let x_0 be a point in the non-empty intersection $\bigcap\{p_{\alpha(\xi)}^{-1}[U_\xi]\colon \xi < \varkappa\}$. For each $\xi < \varkappa$, let g_ξ be a function in $\mathscr{C}(S)$ with $g_\xi(x_0) = 1$ and $g_\xi|(S \setminus p_{\alpha(\xi)}^{-1}[U_\xi]) = 0$. Then there is, by 4.3, an (F) set $B \subset A$ with $\mathrm{card}\,B = \varkappa$ and with the property that each g_ξ factors through P_B. We form $(Y_\alpha, p_{\alpha,\beta})_\tau$ and define r_α as in 4.5.

Let us define $\theta_0\colon \boldsymbol{D}^0 \to Y_{\alpha(0)}$ by $\theta_0(\emptyset) = q_{\alpha(0)}(x_0)$. I assert that $\chi(y, Y_{\alpha(\xi)}) = \alpha(\xi)$ whenever $y \in q_{\alpha(0),\alpha(\xi)}^{-1}(q_{\alpha(0)}(x_0))$. For any such y, $q_{\alpha(\xi)}^{-1}(y) \subseteq q_{\alpha(0)}^{-1}(q_{\alpha(0)}(X_0)) \subseteq \bigcap\{p_{\alpha(\eta)}^{-1}[U_\eta]\colon \eta < \varkappa\}$. So $r_{\alpha(\xi)}(y) \in U_\xi$ and $\chi(r_{\alpha(\xi)}(y), X_{\alpha(\xi)}) = \alpha(\xi)$. The assertion now follows from 5.5 when we recall that $r_{\alpha(\xi)}$ is an open mapping.

It is certainly true that $w(Y_\alpha) = \mathrm{card}\,\alpha$ when $\alpha \geqslant \varkappa$, so, since we are assuming $\alpha(0) \geqslant \varkappa$, we see that we can apply 5.4 repeatedly to obtain a family $(\theta_\xi)_{\xi<\varkappa}$ of continuous injections

$$\theta_\xi\colon \boldsymbol{D}^{\alpha(\xi)} \hookrightarrow Y_{\alpha(\xi)}$$

satisfying $q_{\alpha(\xi),\alpha(\eta)} \circ \theta_\eta = \theta_\xi \circ \pi_{\alpha(\xi),\alpha(\eta)}$ $(\xi < \eta < \varkappa)$. This, of course, gives us the required mapping $\theta\colon \boldsymbol{D}^\tau \hookrightarrow S$.

(III) τ is cofinal with ω.

As before, let $(X_\alpha, p_{\alpha,\beta})_\tau$ be the inverse system of Section 4. Let $(\beta(n))$ be a sequence of regular cardinals that increases to τ. Write $M_n = \{x \in S\colon \chi(x, S) \leqslant \beta(n)\}$ and $S_n = \overline{M}_n$. Then $w(S_n) \leqslant \beta(n)$, by 2.2, and by our assumption (*) $F = S \setminus (\bigcup_{n=0}^{\infty} S_n) \neq \emptyset$. The set F is a non-empty G_δ subset of S and so contains a non-empty closed G_δ subset, Z, say.

We shall now construct by induction a decreasing sequence (Z_n) of non-empty closed G_δ subsets of S and a sequence $(\alpha(n))$ of ordinals that increases to τ. These will have the property that, for $n \geqslant 1$,

$$\chi(x, X_{\alpha(n)}) \geqslant \beta(n)$$

for every $x \in p_{\alpha(n)}[Z_n]$, while

$$\chi(x, X_\alpha) < \beta(n)$$

when $\alpha < \alpha(n)$ and $x \in p_\alpha[Z_n]$. We start by putting $Z_0 = Z$.

If we have defined Z_n already, and $Z_n \subseteq Z$, we note that $w(Z_n) = \tau$, so that, by 3.4, applied to the system $(p_\alpha[Z_n], p_{\alpha,\beta}|p_\beta[Z_n])_\tau$ (which is a restricted inverse system by 3.1), there is some $\alpha < \tau$ with $w(p_\alpha[Z_n]) \geqslant \beta(n+1)$. Let $\alpha(n+1)$ be the smallest such α. Noting that Z_n is dyadic, as a closed G_δ subspace of a dyadic space (Theorem 6 of [2]), and that $Z' = p_{\alpha(n+1)}[Z_n]$ is thus also dyadic, we know that there is a non-empty open subset U of Z' with $X(x, Z') \geqslant \beta(n+1)$ for each $x \in U$. Let Z'' be a compact G_δ contained in U and put $Z_{n+1} = p^{-1}_{\alpha(n+1)}[Z'']\cap Z_n$.

Finally, we put $W = \bigcap_{n=0}^{\infty} Z_n$, which is still a closed G_δ, and choose $f \in \mathscr{C}(S)$ such that $W = f^{-1}(0)$. Let B be a countable (F) set such that f factors through P_B and form the system $(Y_\alpha, q_{\alpha,\beta})_\tau$ as in 4.5.

We note that $\chi(y, Y_{\alpha(n)}) \geqslant \beta(n)$ whenever $y \in q_{\alpha(n)}[W] = q^{-1}_{0,\alpha(n)} q_0[W]$, and that $\chi(y, Y_\alpha) < \beta(n)$ whenever $y \in q_\alpha[W]$ and $\alpha < \alpha(n)$. These facts assure us that we can proceed as in case (II) to obtain a family (θ_n) of continuous injections $\theta_n: \boldsymbol{D}^{\beta(n)} \to Y_{\alpha(n)}$ with $\theta_n \circ \pi_{\beta(n),\beta(n+1)} = q_{\alpha(n),\alpha(n+1)} \circ \theta_{n+1}$, which is exactly what we require.

5.7. Corollary. *Let S satisfy the conditions of 5.6. Then $\mathscr{C}(S)$ is linearly homeomorphic to $\mathscr{C}(\boldsymbol{D}^\tau)$.*

Proof. This is now exactly Theorem 8.4 of [6].

5.8. Remark. It will be seen that the basic technique used in our embedding theorem is much the same as that used by Pełczyński in 8.10 of [6] to show that an infinite compact group G of weight τ contains a homeomorphic copy of $\boldsymbol{D}^\tau$. There, a result of Pontryagin was used to represent G as the limit of a restricted inverse system $(G_\alpha, \varphi_\alpha^\beta)_\tau$, where each $\varphi_\alpha^{\alpha+1}$ is a continuous group epimorphism (hence an open mapping), with a metrizable kernel in the usual (algebraic) sense. We can note, in passing, that this representation, together with Theorems 1 and 2 of [4], assures us that every compact group is a Dugundji space (cf. Problem 21 of [6]).

6. On subspaces of dyadic spaces. It does not seem to be known whether a *dyadic* space S of weight τ which satisfies $(*)$ need contain a copy of $\boldsymbol{D}^\tau$. However, Efimov's basic theorem of [3] states that if either

(1) τ is a successor cardinal, or

(2) $\omega_1 \leqslant \mathrm{cf}(\tau) < \tau$,

then S contains a closed subspace X that can be mapped continuously onto $\boldsymbol{D}^\tau$. The vital tool in proving this is the notion of the ν-character $\nu(x, S)$, a cardinal function that shares many of the properties of the neighbourhood character $\chi(x, S)$. Corollary 3 of [3] tells us that if F is a G_δ subset of a dyadic space S and $\nu(x, S) \geqslant \varkappa \geqslant \omega$ for every $x \in F$, then F contains a compact subset X that can be mapped continuously onto $\boldsymbol{D}^\varkappa$. Lemma 3 of [3] (which we have already remarked does not require special assumptions of GCH type) states that if S is dyadic and $w(S)$ is an uncountable regular cardinal, then there is a non-empty open $U \subset S$ with $\nu(x, S) = w(S)$ for all $x \in U$. Thus we can extend the validity of Efimov's basic theorem to the case where

$$\omega_1 \leqslant \mathrm{cf}(\tau) = \tau.$$

In fact, we can, without using any new ideas, extend the result one step further still.

6.1. Theorem. *If S is a dyadic space of weight τ which satisfies $(*)$, then S contains a closed subspace X that can be mapped continuously onto $\boldsymbol{D}^\tau$.*

Proof. The only case that remains to be considered is where $\mathrm{cf}(\tau) = \omega$. We can again, of course, assume $\tau > \omega$. Let $\alpha(n)$ be a sequence of cardinals with $\alpha(n) < \tau$ and $\sup\{\alpha(n): n < \omega\} = \tau$. For each n put

$$M_n = \{x \in S: \nu(x, S) \leqslant \alpha(n)\}$$

and $S_n = \overline{M}_n$. Then by Theorem 1 of [3] $w(S_n) \leqslant \alpha(n)$ and $S \neq \bigcup_{n=0}^{\infty} S_n$ because of $(*)$. Thus $F = S \setminus \left(\bigcup_{n=0}^{\infty} S_n\right)$ is a non-empty G_δ and $\nu(x, S) = \tau$ for each $x \in F$. Our desired result now follows from the already quoted Corollary 3 of [3].

6.2. Corollary. *If S is a dyadic space of weight τ which satisfies $(*)$, then there is a positive, linear, isometric embedding $\mathscr{C}(\boldsymbol{D}^\tau) \to \mathscr{C}(S)$.*

Proof. Let ψ be a continuous mapping of a closed subspace X of S onto $\boldsymbol{D}^\tau$. Let φ be an embedding of $\boldsymbol{D}^\tau$ in $\boldsymbol{I}^\tau$ and let u be a regular extension operator for φ. Since $\boldsymbol{I}^\tau$ is an absolute retract, there is a continuous mapping $\theta: S \to \boldsymbol{I}^\tau$ such that $\theta | X = \varphi \circ \psi$. Let us define $v: \mathscr{C}(\boldsymbol{D}^\tau) \to \mathscr{C}(S)$ by $v = \theta^0 \circ u$. Then v is a positive, linear, isometric embedding.

Pełczyński showed on p. 71 of [6] that if $w(S) = \tau$ and S does not satisfy $(*)$, then there is not even a linear *homeomorphic* embedding of $\mathscr{C}(\boldsymbol{D}^\tau)$ into $\mathscr{C}(S)$. We can, however, formulate the following proposition without $(*)$.

6.3. Proposition. *If S is a dyadic space of weight τ, then for every $\varkappa < \tau$ there is a closed subset X of S that can be mapped continuously onto $\boldsymbol{D}^\varkappa$. If, further, we assume S to be a Dugundji space, then S contains a homeomorphic copy of $\boldsymbol{D}^\varkappa$.*

Proof. There is an open subset U of S with $\chi(x, S) \geqslant \nu(x, S) \geqslant \varkappa$ for every $x \in U$. Corollary 3 of [3] gives us the required conclusion when we assume only that S is dyadic. When S is Dugundji, we can imitate the proof of 5.6, case (I) to embed $\boldsymbol{D}^\varkappa$ in S.

Note added in proof. J. Hagler (Trans. Amer. Math. Soc. 214 (1975), pp. 415–428) has shown that if the weight of a dyadic space S is an uncountable regular cardinal τ then S contains a subspace homeomorphic to D^{τ}.

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On absolute retracts, $P(S)$, and complemented subspaces of $\mathscr{C}(D^{\omega_1})$

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Abstract. It is shown that, if S is a compact Hausdorff space, then the space $P(S)$ of probability measures on S is an absolute retract (in the category of compact spaces and continuous mappings) if and only if S is a Dugundji space (in Pełczyński's terminology) and the weight of S is at most ω_1. The crucial point of the proof consists of showing that $P(\{0,1\}^A)$ is an absolute retract if and only if the cardinality of A is at most ω_1. As a corollary, it follows that if S is a compact Hausdorff space of weight ω_1, then S is a Dugundji space if and only if the Banach space $\mathscr{C}(S)$ is linearly isometric to the range of a contractive projection on $\mathscr{C}(\{0,1\}^{\omega_1})$.

1. When S is a compact (Hausdorff) space, the Banach space of all continuous real valued functions on S will be denoted by $\mathscr{C}(S)$. The dual space of $\mathscr{C}(S)$ will as usual be identified with $M(S)$, the space of all Radon measures on S. We shall be particularly interested in

$$P(S) = \{\mu \in M(S) : \|\mu\| = \langle \mu, 1_s\rangle = 1\},$$

the set of all probability measures on S, which is itself a compact space under the weak topology $\sigma(M(S), \mathscr{C}(S))$. We write δ or δ_s for the canonical embedding $S \hookrightarrow P(S)$.

If $\varphi\colon S \hookrightarrow T$ is a continuous injection, and $\varrho\colon T \to S$ is a continuous mapping satisfying $\varrho\circ\varphi = \iota_s$, where ι_s is the identity mapping on S, we say that ϱ is a *retraction* for φ, and that S is a *retract* of T. A compact space S is an *absolute retract* (AR) if every continuous injection $S \hookrightarrow T$ allows a retraction. The question dealt with in this paper, namely that of characterizing those S for which $P(S)$ is an AR, arose out of some problems posed by Pełczyński in [8], concerning extension operators and averaging operators on spaces of continuous functions.

A linear operator $u\colon \mathscr{C}(S) \to \mathscr{C}(T)$ is called *regular* if u is continuous with $\|u\| = 1$ and $u(1_s) = 1_T$, where 1_s denotes, of course, the function that is identically 1 on the space S. Equivalently, u is regular if and only if the transpose u' takes $P(T)$ into $P(S)$. If $\varphi\colon S \to T$ is a continuous mapping, a regular operator $\varphi^0\colon \mathscr{C}(T) \to \mathscr{C}(S)$ is defined by $\varphi^0(g) = g\circ\varphi$ $(g \in \mathscr{C}(T))$. Restricting the transpose $(\varphi^0)'$ to the set $P(S)$ gives us a continuous map $\tilde{\varphi}\colon P(S) \to P(T)$. In the particular case where φ is a continuous injection,