Note added in proof. J. Hagler (Trans. Amer. Math. Soc. 214 (1975), pp. 415–428) has shown that if the weight of a dyadic space S is an uncountable regular cardinal τ then S contains a subspace homeomorphic to D^{τ}.

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On absolute retracts, $P(S)$, and complemented subspaces of $\mathscr{C}(D^{\omega_1})$

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Abstract. It is shown that, if S is a compact Hausdorff space, then the space $P(S)$ of probability measures on S is an absolute retract (in the category of compact spaces and continuous mappings) if and only if S is a Dugundji space (in Pełczyński's terminology) and the weight of S is at most ω_1. The crucial point of the proof consists of showing that $P(\{0,1\}^A)$ is an absolute retract if and only if the cardinality of A is at most ω_1. As a corollary, it follows that if S is a compact Hausdorff space of weight ω_1, then S is a Dugundji space if and only if the Banach space $\mathscr{C}(S)$ is linearly isometric to the range of a contractive projection on $\mathscr{C}(\{0,1\}^{\omega_1})$.

1. When S is a compact (Hausdorff) space, the Banach space of all continuous real valued functions on S will be denoted by $\mathscr{C}(S)$. The dual space of $\mathscr{C}(S)$ will as usual be identified with $M(S)$, the space of all Radon measures on S. We shall be particularly interested in

$$P(S) = \{\mu \in M(S):\ \|\mu\| = \langle \mu, 1_s\rangle = 1\},$$

the set of all probability measures on S, which is itself a compact space under the weak topology $\sigma(M(S), \mathscr{C}(S))$. We write δ or δ_s for the canonical embedding $S \hookrightarrow P(S)$.

If $\varphi: S \hookrightarrow T$ is a continuous injection, and $\varrho: T \to S$ is a continuous mapping satisfying $\varrho \circ \varphi = \iota_s$, where ι_s is the identity mapping on S, we say that ϱ is a *retraction* for φ, and that S is a *retract* of T. A compact space S is an *absolute retract* (AR) if every continuous injection $S \hookrightarrow T$ allows a retraction. The question dealt with in this paper, namely that of characterizing those S for which $P(S)$ is an AR, arose out of some problems posed by Pełczyński in [8], concerning extension operators and averaging operators on spaces of continuous functions.

A linear operator $u: \mathscr{C}(S) \to \mathscr{C}(T)$ is called *regular* if u is continuous with $\|u\| = 1$ and $u(1_s) = 1_T$, where 1_s denotes, of course, the function that is identically 1 on the space S. Equivalently, u is regular if and only if the transpose u' takes $P(T)$ into $P(S)$. If $\varphi: S \to T$ is a continuous mapping, a regular operator $\varphi^0: \mathscr{C}(T) \to \mathscr{C}(S)$ is defined by $\varphi^0(g) = g \circ \varphi$ $(g \in \mathscr{C}(T))$. Restricting the transpose $(\varphi^0)'$ to the set $P(S)$ gives us a continuous map $\tilde{\varphi}$: $P(S) \to P(T)$. In the particular case where φ is a continuous injection,

φ^0 is a surjection. A regular right inverse for φ^0 (i.e. a regular operator u: $\mathscr{C}(S)\to\mathscr{C}(T)$ with $u(f)\circ\varphi = f\ (f\in\mathscr{C}(S))$), if one exists, is called a *regular extension operator* (r.e.o.). If, on the other hand, φ is a surjection, φ^0 is an injection and a regular left inverse for φ^0, if one exists, is a *regular averaging operator* (r.a.o.). Pełczyński [8] defined a *Dugundji space* to be a compact space S such that every continuous injection φ: $S\hookrightarrow T$ allows an r.e.o. It is enough that some continuous injection $S\hookrightarrow \boldsymbol{I}^A$ should allow an r.e.o., where $\boldsymbol{I}$ is the unit interval $[0,1]$ and A is some index set. If, for some A, there is a continuous surjection $\boldsymbol{D}^A\twoheadrightarrow S$ that allows an r.a.o., where $\boldsymbol{D}$ is the two-point space $\{0,1\}$, S is said to be a *Milutin space*.

In what follows, we shall always identify a cardinal with the corresponding initial ordinal; ω and ω_1 will denote, respectively, the first infinite cardinal and the first uncountable cardinal. The weight $w(S)$ of a space S is the smallest cardinality of a base for the topology of S. When S is compact, we can always embed S in $\boldsymbol{I}^{w(S)}$.

2. THEOREM 1. *The compact space $P(\boldsymbol{D}^{\omega_1})$ is an absolute retract.*

Proof. We can represent $P(\boldsymbol{D}^{\omega_1})$ as an inverse limit

$$\varprojlim\big(P(\boldsymbol{D}^{\alpha}),\ \tilde\pi_{\alpha,\beta}\big)_{\alpha<\beta<\omega_1},$$

where $\tilde\pi_{\alpha,\beta}$: $P(\boldsymbol{D}^\beta)\to P(\boldsymbol{D}^\alpha)$ $(\alpha<\beta)$ is the mapping induced by the projection

$$\pi_{\alpha,\beta}:\ \boldsymbol{D}^\beta\to\boldsymbol{D}^\alpha.$$

(We are, of course, identifying the ordinal α as a subset of the ordinal β.)

To prove $P(\boldsymbol{D}^{\omega_1})$ is AR we shall show that, whenever φ: $S\hookrightarrow T$ is a continuous injection, and ψ: $S\to P(\boldsymbol{D}^{\omega_1})$ is a continuous mapping, there is a continuous θ: $T\to P(\boldsymbol{D}^{\omega_1})$ with $\theta\circ\varphi=\psi$. Such a map θ will be determined if we obtain a family $(\theta_\alpha)_{\alpha<\omega_1}$ of continuous mappings

$$\theta_\alpha:\ T\to P(\boldsymbol{D}^\alpha)\quad\text{ with }\quad \theta_\alpha=\tilde\pi_{\alpha,\beta}\circ\theta_\beta\quad(\alpha<\beta<\omega_1)$$

and

$$\theta_\alpha\circ\varphi=\tilde\pi_\alpha\circ\psi,$$

where $\tilde\pi_\alpha$: $P(\boldsymbol{D}^{\omega_1})\to P(\boldsymbol{D}^\alpha)$ is defined in the same way as $\tilde\pi_{\alpha,\beta}$.

We construct the family (θ_α) by transfinite induction. With $\alpha=0$, the situation is trivial, $P(\boldsymbol{D}^0)$ being a one-point set, so that θ_0 is determined. Suppose now that θ_α has been defined for all $\alpha\leqslant\beta$. We consider the set-valued mapping

$$\Theta_\beta:\ T\to\mathrm{p}P(\boldsymbol{D}^{\beta+1})$$

defined by

$$\Theta_\beta(t)=\begin{cases}\{\tilde\pi_{\beta+1}\psi(s)\} & \text{if}\quad t=\varphi(s)\in\varphi[S],\\ \tilde\pi^{-1}_{\beta,\beta+1}\big(\theta_\beta(t)\big) & \text{if}\quad t\notin\varphi[S].\end{cases}$$

Using the fact (Section 4 of (2)) that $\tilde\pi_{\beta,\beta+1}$ is an open mapping, we can check that Θ_β is, in Michael's terminology, a lower semicontinuous carrier. Moreover, $P(\boldsymbol{D}^{\beta+1})$ is compact, convex and metrizable (β is countable), and each $\Theta_\beta(t)$ is a compact convex subset. So by Theorem 1.2 of [7] there is a continuous mapping

$$\theta_{\beta+1}:\ T\to P(\boldsymbol{D}^{\beta+1})$$

with

$$\theta_{\beta+1}(t)\in\Theta_\beta(t)\qquad(t\in T).$$

Finally, if γ is a limit ordinal, and a consistent family $(\theta_\alpha)_{\alpha<\gamma}$ has been defined already, we note that we can identify $P(\boldsymbol{D}^\gamma)$ with $\varprojlim\big(P(\boldsymbol{D}^\alpha),\ \tilde\pi_{\alpha,\beta}\big)_{\alpha<\beta<\gamma}$. So the maps θ_α do already determine the required map

$$\theta_\gamma:\ T\to P(\boldsymbol{D}^\gamma).$$

Note. Alternatively, we could notice that, by the inverse limit representation used above, $P(\boldsymbol{D}^{\omega_1})$ satisfies the conditions of Theorem 2 of [4]. Thus, as a *convex Dugundji space*, it is AR (cf. the discussion in Section 2 of [4]).

PROPOSITION 1. *Let S be a compact space and suppose $P(S)$ is an AR. Then S is a Dugundji space.*

Proof. Let φ: $S\hookrightarrow T$ be a continuous injection. Then there exists a continuous map θ: $T\to P(S)$ with $\theta\circ\varphi=\delta_S$. Define u: $\mathscr{C}(S)\to\mathscr{C}(T)$ by $\big(u(f)\big)(t)=\langle\theta(t),f\rangle$ $(f\in\mathscr{C}(S),\ t\in T)$. Then u is a regular extension operator for φ.

The next proposition, which we shall need in order to prove the second theorem of this section, uses methods similar to some employed by Amir and Arbel [1] and McDonald [6]. For convenience, we shall use some category theory terminology from [9]. The Banach spaces $\mathscr{C}(S)$ may be regarded as objects in either the category $\mathbf{Ban}_1$, where the morphisms are linear contractions, or $\mathbf{Bcfd}$, where the morphisms are regular operators. Thus, to say that $\mathscr{C}(S)$ is a $\mathbf{Ban}_1$-retract (resp. a $\mathbf{Bcfd}$-retract) of $\mathscr{C}(T)$ is just to say that there are linear contractions (resp. regular operators)

$$j:\ \mathscr{C}(S)\to\mathscr{C}(T),\qquad p:\ \mathscr{C}(T)\to\mathscr{C}(S),$$

with $p\circ j=\iota_{\mathscr{C}(S)}$. If S is a Milutin space, then $\mathscr{C}(S)$ is a $\mathbf{Bcfd}$-retract of $\mathscr{C}(\boldsymbol{D}^A)$ for some A.

PROPOSITION 2. *Let S, T be compact spaces and suppose that $\mathscr{C}(S)$ is a $\mathbf{Ban}_1$-retract of $\mathscr{C}(T)$. Then there is a closed G_δ subset T_0 of T such that $\mathscr{C}(S)$ is a $\mathbf{Bcfd}$-retract of $\mathscr{C}(T_0)$.*

Proof. There exist, by hypothesis, linear contractions $j\colon \mathscr{C}(S)\to\mathscr{C}(T)$, $p\colon \mathscr{C}(T)\to\mathscr{C}(S)$ with $p\circ j = \iota_{\mathscr{C}(S)}$. We start by defining

$$T_+ = \{t\in T\colon \big(j(1_s)\big)(t) = +1\},$$
$$T_- = \{t\in T\colon \big(j(1_s)\big)(t) = -1\},$$
$$T_0 = T_+\cup T_-.$$

So T_0 is certainly a closed G_δ subset of T.

Since j is evidently isometric, the transpose j' takes the unit ball ball $M(T)$ onto ball $M(S)$. By the Krein–Mil'man theorem, for each extreme point μ of ball $M(S)$, there is an extreme point ν of ball $M(T)$ with $j'(\nu) = \mu$. Recalling that the extreme points of ball $M(T)$ are exactly the measures $\pm\delta(t)$, we see that for each $s\in S$, there is $t\in T$ with $j'\delta(t) = \pm\delta(s)$. Such a point t is necessarily in T_0, so we see that j' takes ball $M(T_0)$ (identified as a subset of $M(T)$) onto ball $M(S)$. Thus

$$j_0\colon \mathscr{C}(S)\to\mathscr{C}(T_0),$$

defined by $j_0(f) = j(f)|T_0$ is isometric and

$$|j_0(1_S)| = 1_{T_0}.$$

Let us define $k\colon \mathscr{C}(S)\to\mathscr{C}(T_0)$ by

$$\big(k(f)\big)(t) = \begin{cases}\big(j_0(f)\big)(t) & (t\in T_+),\\ -\big(j_0(f)\big)(t) & (t\in T_-).\end{cases}$$

Then k is a regular isometric embedding.

We now assert that

$$p\colon \mathscr{C}(T)\to\mathscr{C}(S)$$

factors through the restriction mapping $\mathscr{C}(T)\to\mathscr{C}(T_0)$. It will be enough to prove that each measure $p'\delta(s)\in M(T)$ is supported by T_0. If $\mu = p'\delta(s)$, then

$$\langle\mu, j(1_S)\rangle = \big(pj(1_S)\big)(s) = 1.$$

That is to say,

$$\int\big(j(1_S)\big)d\mu = 1, \quad \text{while} \quad \|\mu\|\leqslant 1$$

and $|j(1_S)|\leqslant 1$. Thus

$$|\mu|(T_0) = |\mu|\{t\in T\colon \big|\big(j(1_S)\big)(t)\big| = 1\} = 1,$$

as required. So $p_0\colon \mathscr{C}(T_0)\to\mathscr{C}(S)$ is well defined by $p_0(f) = p(g)$ if $g\in\mathscr{C}(T)$ and $g|T_0 = f$. Let us finally define $q\colon \mathscr{C}(T_0)\to\mathscr{C}(S)$ by

$$q(f) = p_0\big(f\cdot\big(j(1_S)\big)\big).$$

Then q is regular and $q\circ k = \iota_{\mathscr{C}(S)}$.

COROLLARY 1. *Let S be a compact space and suppose that $\mathscr{C}(S)$ is a* $\mathbf{Ban}_1$*-retract of $\mathscr{C}(\boldsymbol{D}^A)$. Then $\mathscr{C}(S)$ is a* $\mathbf{Bcfd}$*-retract of $\mathscr{C}(\boldsymbol{D}^A)$.*

Proof. If A is countable, S is necessarily metrizable, hence a Milutin space ([8], Theorem 5.6), so that $\mathscr{C}(S)$ is a $\mathbf{Bcfd}$-retract of $\mathscr{C}(\boldsymbol{D}^A)$. If A is uncountable, the result follows immediately from Proposition 2 when we recall that every closed G_δ subset of $\boldsymbol{D}^A$ is homeomorphic to $\boldsymbol{D}^A$ ([3], Theorem 6).

THEOREM 2. *Let S be a compact space and suppose that $\mathscr{C}(S)$ is a* $\mathbf{Ban}_1$*-retract of $\mathscr{C}(\boldsymbol{D}^{\omega_1})$. Then $P(S)$ is an absolute retract and S is a Dugundji space.*

Proof. By Corollary 1, there exist regular operators

$$k\colon \mathscr{C}(S)\to\mathscr{C}(\boldsymbol{D}^{\omega_1}),$$
$$q\colon \mathscr{C}(\boldsymbol{D}^{\omega_1})\to\mathscr{C}(S),$$

with $q\circ k = \iota_{\mathscr{C}(S)}$. Passing to the adjoints, we have

$$q'\colon P(S)\to P(\boldsymbol{D}^{\omega_1}),$$
$$k'\colon P(\boldsymbol{D}^{\omega_1})\to P(S),$$

with $k'\circ q' = \iota_{P(S)}$. Thus $P(S)$ is a retract of $P(\boldsymbol{D}^{\omega_1})$, and so, by Theorem 1, an absolute retract. Proposition 2 tells us that S is a Dugundji space.

COROLLARY 2. *Let S be a compact space with $w(S)\leqslant\omega_1$. Then the following are equivalent:*

(a) *S is a Dugundji space;*

(b) *S is a Milutin space;*

(c) *$P(S)$ is an absolute retract.*

Proof. (a) $\Rightarrow$ (b) follows from the results of [4], (b) $\Rightarrow$ (c) $\Rightarrow$ (a) from Theorem 2.

COROLLARY 3. *Let S be a non-metrizable compact space, and suppose that $\mathscr{C}(S)$ is a* $\mathbf{Ban}_1$*-retract of $\mathscr{C}(\boldsymbol{D}^{\omega_1})$. Then $\mathscr{C}(S)$ is linearly homeomorphic to $\mathscr{C}(\boldsymbol{D}^{\omega_1})$.*

Proof. This follows from Corollary 5.7 of [5], when we note that, by Theorem 2, S is a Dugundji space of weight ω_1.

3. THEOREM 3. *If $\operatorname{card} A$ exceeds ω_1, $P(\boldsymbol{D}^A)$ is not an absolute retract.*

Proof. Let us suppose, if possible, that $\operatorname{card} A > \omega_1$ and that $P(\boldsymbol{D}^A)$ is an absolute retract. The weight of $P(\boldsymbol{D}^A)$ is $\operatorname{card} A$, so there is a continuous injection

$$\varphi\colon P(\boldsymbol{D}^A)\to \boldsymbol{I}^A.$$

Let ϱ be a retraction for φ.

When $B \subseteq C \subseteq A$, let us agree to write $\pi_{B,C}$ (resp. $\tilde{\pi}_{B,C}$) for the canonical mappings $\boldsymbol{I}^C \to \boldsymbol{I}^B$ (resp. $P(\boldsymbol{D}^C) \to P(\boldsymbol{D}^B)$). Let us write π_B, $\tilde{\pi}_B$ for $\pi_{B,A}$, $\tilde{\pi}_{B,A}$, respectively.

We shall say that $B \subseteq A$ is a *factorization set* for φ and ϱ if

(i) $\pi_B \circ \varphi\colon P(\boldsymbol{D}^A) \to \boldsymbol{I}^B$ factors through $\tilde{\pi}$, and

(ii) $\tilde{\pi}_B \circ \varrho\colon \boldsymbol{I}^A \to P(\boldsymbol{D}^B)$ factors through π_B.

If B is a factorization set, there are continuous mappings $\varphi_B\colon P(\boldsymbol{D}^B) \to \boldsymbol{I}^B$, $\varrho_B\colon \boldsymbol{I}^B \to P(\boldsymbol{D}^B)$ with $\varrho_B \circ \varphi_B = \iota_{P(\boldsymbol{D}^B)}$ and such that the diagram below commutes.

$$\begin{tikzcd}
P(\boldsymbol{D}^A) \arrow[r] \arrow[d, "\tilde{\pi}_B"] & \boldsymbol{I}^A \arrow[r] \arrow[d, "\pi_B"] & P(\boldsymbol{D}^A) \arrow[d, "\tilde{\pi}_B"] \\
P(\boldsymbol{D}^B) \arrow[r] & \boldsymbol{I}^B \arrow[r] & P(\boldsymbol{D}^B)
\end{tikzcd}$$

We need the following lemmas to guarantee the existence of such sets.

LEMMA 1. *Let θ be a continuous map from $\boldsymbol{I}^A$ (resp. $P(\boldsymbol{D}^A)$) into X. Then there is a subset E of A with* $\operatorname{card} E \leqslant w(X)$ *such that θ factors through π_E (resp. $\tilde{\pi}_E$).*

Proof. The assertion is evidently trivial if the image of θ contains only one point. Otherwise, since the image of θ is connected and non-trivial, the weight of X is infinite. It is easy to see by the Stone–Weiestrass theorem (cf. [9], 7.3.13) that every $f \in \mathscr{C}(\boldsymbol{I}^A)$ (resp. $f \in \mathscr{C}\big(P(\boldsymbol{D}^A)\big)$) factors through π_C (resp. $\tilde{\pi}_C$) for some countable $C \subseteq A$. Let $F \subset \mathscr{C}(X)$ be a set which separates the points of X and which has cardinality $w(X)$. Then there is a subset E of A with cardinality $\omega \cdot w(X) = w(X)$ and such that, for each $f \in F$, $f \circ \theta$ factors through π_E (resp. $\tilde{\pi}_E$).

LEMMA 2. *With the notation of Theorem 3, let H be an infinite subset of A. Then there is a factorization set B with $H \subseteq B$ and* $\operatorname{card} B = \operatorname{card} H$.

Proof. Let us write $\varkappa$ for $\operatorname{card} H$ and define subsets $B(n)$ of A inductively. We put $B(0) = H$. Suppose that we have defined $B(0) \subseteq B(1) \subseteq \ldots \subseteq B(n)$ and that $\operatorname{card} B(n) = \varkappa$. Then $w(\boldsymbol{I}^{B(n)}) = \varkappa$ and $w\big(P(\boldsymbol{D}^{B(n)})\big) = \varkappa$, so that, by Lemma 1, there are subsets E, F of A with $\operatorname{card} E = \operatorname{card} F = \varkappa$, such that

$$\pi_{B(n)} \circ \varphi \text{ factors through } \tilde{\pi}_E$$

and

$$\tilde{\pi}_{B(n)} \circ \varrho \text{ factors through } \pi_F.$$

Let us put $B(n+1) = B(n) \cup E \cup F$.

Finally, we take B to be $\bigcup_{n=0}^{\infty} B(n)$ and assert that B is a factorization set. To show that $\pi_B \circ \varphi$ factors through $\tilde{\pi}_B$, we suppose that $\mu, \nu \in P(\boldsymbol{D}^A)$ and that $\pi_B \varphi(\mu) \neq \pi_B \varphi(\nu)$. Then, for some n, $\pi_{B(n)} \varphi(\mu) \neq \pi_{B(n)} \varphi(\nu)$. So, by construction, $\tilde{\pi}_{B(n+1)}(\mu) \neq \tilde{\pi}_{B(n+1)}(\nu)$, and $\tilde{\pi}_B(\mu) \neq \tilde{\pi}_B(\nu)$. A similar argument shows that $\tilde{\pi}_B \circ \varrho$ factors through π_B.

Proof of Theorem 3 (continued). By Lemma 2, there exists a factorization set $B \subset A$ with $\operatorname{card} B = \omega_1$. Let us choose a point $a \in A \setminus B$ and a countably infinite subset C of B. Then there is a countable factorization set E with $C \cup \{a\} \subseteq E$. We note that $F = B \cap E$ is countably infinite, and it is easy to see that F is also a factorization set. If φ_B, ϱ_B, φ_E, ϱ_E, φ_F are defined as in the discussion at the start of the proof, then the following identities hold:

$$\tilde{\pi}_{F,B} \circ \varrho_B = \varrho_F \circ \pi_{F,B}, \qquad \pi_{F,B} \circ \varphi_B = \varphi_F \circ \tilde{\pi}_{F,B},$$
$$\tilde{\pi}_{F,E} \circ \varrho_E = \varrho_F \circ \pi_{F,E}, \qquad \pi_{F,E} \circ \varphi_E = \varphi_F \circ \tilde{\pi}_{F,E}.$$

Let Q denote the set

$$\{(\mu, \nu) \in P(\boldsymbol{D}^B) \times P(\boldsymbol{D}^E)\colon \tilde{\pi}_{F,B}(\mu) = \tilde{\pi}_{F,E}(\nu)\}.$$

If $(\mu, \nu) \in Q$,

$$\pi_{F,B} \varphi_B(\mu) = \varphi_F \tilde{\pi}_{F,B}(\mu) = \varphi_F \tilde{\pi}_{F,E}(\nu) = \pi_{F,E} \varphi_E(\nu).$$

So there is a (uniquely determined) element z of $\boldsymbol{I}^{B \cup E}$ with

$$\pi_{B,B\cup E}(z) = \varphi_B(\mu) \quad \text{and} \quad \pi_{E,B\cup E}(z) = \varphi_E(\nu).$$

We can define a continuous map $\psi\colon Q \to \boldsymbol{I}^{B\cup E}$ by $\psi(\mu, \nu) = z$, as above. On the other hand, if $z \in \boldsymbol{I}^{B\cup E}$, we have

$$\begin{aligned}
\tilde{\pi}_{F,B} \varrho_B \pi_{B,B\cup E}(z) &= \varrho_F \pi_{F,B} \pi_{B,B\cup E}(z) \\
&= \varrho_F \pi_{F,B\cup E}(z) = \tilde{\pi}_{F,E} \varrho_E \pi_{E,B\cup E}(z),
\end{aligned}$$

so that $\sigma(z)$, defined to be $\big(\varrho_B \pi_{B,B\cup E}(z), \varrho_E \pi_{E,B\cup E}(z)\big)$, is in Q. We can now draw a commutative diagram of the following form:

$$\begin{tikzcd}
P(\boldsymbol{D}^A) \arrow[r, "\varphi"] \arrow[d, "p"] & \boldsymbol{I}^A \arrow[r, "\varrho"] \arrow[d, "\pi_{B\cup E}"] & P(\boldsymbol{D}^A) \arrow[d, "p"] \\
Q \arrow[r, "\psi"] & \boldsymbol{I}^{B\cup E} \arrow[r, "\sigma"] & Q
\end{tikzcd}$$

Here σ and ψ are as defined above and $p(\lambda) = \big(\tilde{\pi}_B(\lambda), \tilde{\pi}_E(\lambda)\big)$. We have $\varrho \circ \varphi = \iota_{P(\boldsymbol{D}^A)}$ and $\sigma \circ \psi = \iota_Q$, so that, by Lemma 1 of [4], p is an open mapping. To finish the proof with a contradiction we need just one more lemma, which tells us that p is not open.

LEMMA 3. *Let R, S, T be compact spaces, with* $\operatorname{card} R$, $\operatorname{card} T \geqslant 2$ *and* $\operatorname{card} S \geqslant \omega$.

Let the mapping

$$p\colon P(R\times S\times T)\to Q$$
$$= \{(\mu, \nu)\colon \mu \in P(R\times S),\ \nu \in P(S\times T) \text{ and } \tilde{\Pi}_S(\mu) = \tilde{\Pi}_S(\nu)\}$$

be defined by $p(\lambda) = \big(\tilde{\Pi}_{R\times S}(\lambda), \tilde{\Pi}_{S\times T}(\lambda)\big)$, *where* $\tilde{\Pi}_S$, $\tilde{\Pi}_{R\times S}$ *and* $\tilde{\Pi}_{S\times T}$ *have the obvious interpretations. Then p is not an open mapping.*

Proof. Choose an accumulation point s_0 of S and distinct points $r_0, r_1 \in R$ and $t_0, t_1 \in T$.

Define $\lambda \in P(R\times S\times T)$ to be

$$\tfrac{1}{2}\big(\delta(r_0, s_0, t_0) + \delta(r_1, s_0, t_1)\big).$$

Choose disjoint open neighbourhoods U_0, U_1 of r_0, r_1 in R and W_0, W_1 of t_0, t_1 in T, and let V be the neighbourhood of λ in $P(R\times S\times T)$ defined by

$$V = \{\lambda'\colon \lambda'\big((U_0\times S\times W_0)\cup(U_1\times S\times W_1)\big) > 0\}.$$

Now $p(\lambda) = (\mu, \nu)$, where

$$\mu = \tfrac{1}{2}\big(\delta(r_0, s_0) + \delta(r_1, s_0)\big)$$

and

$$\nu = \tfrac{1}{2}\big(\delta(s_0, t_0) + \delta(s_0, t_1)\big).$$

Since s_0 is an accumulation point of S, any neighbourhood of (μ, ν) in Q contains points (μ', ν'), where

$$\mu' = \tfrac{1}{2}\big(\delta(r_0, s_0) + \delta(r_1, s_1)\big),$$
$$\nu' = \tfrac{1}{2}\big(\delta(s_1, t_0) + \delta(s_0, t_1)\big),$$

and s_1 is distinct from s_0 (but lies in a sufficiently small neighbourhood of it). The unique $\lambda' \in P(R\times S\times T)$ for which $p(\lambda') = (\mu', \nu')$ is

$$\lambda' = \tfrac{1}{2}\big(\delta(r_0, s_0, t_1) + \delta(r_1, s_1, t_0)\big),$$

which is not in V. Thus $p[V]$ is not a neighbourhood of $p(\lambda)$ and p is not an open mapping.

THEOREM 4. *Let S be a compact space. The space $P(S)$ is an absolute retract if and only if S is a Dugundji space and $w(S) \leqslant \omega_1$.*

Proof. If S is a Dugundji space and $w(S) \leqslant \omega_1$, then $P(S)$ is AR by Corollary 2. On the other hand, suppose that $P(S)$ is an absolute retract. Then S is a Dugundji space by Proposition 1. If $w(S)$ were to exceed ω_1, then by Theorem 5.6 and Proposition 6.3 of [5], there would be a continuous injection $\varphi\colon \boldsymbol{D}^{\omega_2}\to S$. Now $\boldsymbol{D}^{\omega_2}$ is a Dugundji space ([8], Theorem 6.6), so there would be a regular extension operator u for φ. We should have

$$\tilde{\varphi}\colon P(\boldsymbol{D}^{\omega_2})\to P(S)$$

and

$$u'\colon P(S)\to P(\boldsymbol{D}^{\omega_2})$$

with $u'\circ\tilde{\varphi} = \iota_{P(\boldsymbol{D}^{\omega_2})}$. Thus $P(\boldsymbol{D}^{\omega_2})$ would be a retract of $P(S)$, and hence an absolute retract, contradicting Theorem 3.

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