STUDIA MATHEMATICA, T. LVI. (1976)

W^*-algebras and invariant functionals

by

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Abstract. Let M be a W^*-algebra and let S be a semigroup of ultraweakly continuous *-homomorphisms from M into M. We say that M is *S-finite* in case M contains sufficiently many S-invariant ultraweakly continuous functionals to separate the positive elements in M from zero. Using an idea of K. Deleeuw and I. Glicksberg on almost periodic semigroup of operators, we have obtained various characterizations of S-finiteness of M, including some mild generalizations of results of Kovacs and Szucs.

1. Introduction. Let M be a W^*-algebra and let S be a semigroup of ultraweakly continuous *-homomorphisms from M into M. A linear functional φ on M is *S-invariant* if $\varphi\cdot s = \varphi$ for all $s \in S$. We say that M is *S-finite* if for each non-zero positive element x in M, there exists an S-invariant ultraweakly continuous linear functional φ on M such that $\varphi(x) \neq 0$.

In [8], Kovacs and Szucs, generalizing the notion of finite W^*-algebra, initiated the notion of S-finiteness when S is a group of *-automorphisms. They proved ([8], Theorems 1 and 2) among other things that if M is S-finite, then for each x in M, K_x, the weak*-closed convex hull of $\{s\cdot x,\ s \in S\}$ contains a unique fixed element $\bar{x}$ invariant under S. Furthermore, the map $x \dashrightarrow \bar{x}$ of M onto $\mathscr{F}(M, S)$, the space of all elements in M fixed under the action of S, has properties resembling that of the Dixmier tracial map $\natural$ on a finite W^*-algebra. Recently E. Størmer [9] proved that if S is a group, then M is S-finite if and only if S is relatively compact in the relative weak* operator topology on $\mathscr{B}_*(M)$, the space of ultraweakly continuous bounded linear operators on M.

Let $\bar{S}$ denote the closure of S in $\mathscr{B}(M)$, the space of bounded linear operators on M, with respect to the weak* operator topology. We show in this paper that for any semigroup S, if M is S-finite, then $\bar{S}$ with the weak* operator topology is a compact topological semigroup. Furthermore

* This work is supported by NRC Grant No. A-7679.

if $C(\bar{S})$, the space of bounded complex-valued functions on $\bar{S}$, has an invariant mean (which is the case when S is a group, or when S is amenable [2]), then Kovacs and Szucs' results stated above remain valid. Other characterizations of finiteness are also obtained. In particular, we show that a W^*-algebra M is finite if and only if the group of inner automorphisms is equicontinuous on the unit ball of M with respect to some relative topology of the dual pair (M, M_*).

The essential tool of our work is contained in Section 3, where we considered semigroups of weak*-continuous bounded linear operators on a dual Banach space. Also, the idea in considering the compact topological semigroup $\bar{S}$ is taken from the work of K. Deleeuw and I. Glicksberg on almost periodic semigroups of operators ([3], Section 4).

2. Preliminaries and some notations. Let E be a topological vector space and E^* be the continuous dual of E. If F is a linear subspace of E^*, then a topology $\mathcal{T}$ on E is called a *topology of the dual pair* (E, F) if $\mathcal{T}$ is compatible with respect to the structure of E and the continuous dual of E with respect to $\mathcal{T}$ is F.

If K is a subset of E, then $\mathrm{co}(K)$ will denote the convex hull of K and $\overline{\mathrm{co}}(K)$ will denote the closed convex hull of K.

If $(E, \mathcal{T})$ is a separated locally convex linear topological space and K is a subset of E, then a family of mappings S from K into K is *quasi-equicontinuous* (with respect to $\mathcal{T}$) on K if the closure of S in the product space $(K, \mathcal{T})^K$ is a set of continuous functions from $(K, \mathcal{T})$ into $(K, \mathcal{T})$. Furthermore, S is *equicontinuous* (with respect to $\mathcal{T}$) on K if there exists a family of continuous seminorms Q on E, determining the topology $\mathcal{T}$, such that for any $\varepsilon > 0$ and any $p \in Q$, there exist $\delta > 0$, $q \in Q$ such that $q(x-y) < \delta$, $x, y \in K$, implies $p(sx-sy) < \varepsilon$ for all $s \in S$. Clearly, if S is equicontinuous on K, then S is quasi-equicontinuous on K. But the converse is not true in general.

A *topological semigroup* is a set S with an associative operation and a Hausdorff topology $\mathcal{T}$ such that for each $a \in S$, the two mappings $s \to as$ and $s \to sa$ from S into S are continuous. The multiplication in S is said to be *jointly continuous* in case the map $(s, t) \to st$ from $S \times S$ into S is continuous when $S \times S$ has the product topology.

If S is a semigroup, and $l_\infty(S)$ is the space of bounded complex-valued functions on S with the supremum norm, we define the *left* and *right translation operators* by: $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$ for all $a, s \in S$. If A is a closed translation invariant subspace of $l_\infty(S)$ containing the constant one function, 1, then an element φ in A^* is a *mean* if $\varphi(1) = \|\varphi\| = 1$. Furthermore, φ is a *left* (respectively *right*) *invariant mean* if $\varphi(l_a f) = \varphi(f)$ (respectively $\varphi(r_a f) = \varphi(f)$) for all $a \in S$. Also, φ is a *two-sided invariant mean* if φ is both left and right invariant. The semigroup S is *left amenable* if $l_\infty(S)$ has a left invariant mean. Right amenable semigroup is defined similarly. We say that S is *amenable* if $l_\infty(S)$ has a two-sided invariant mean. (See [2] and [4].)

3. Ultraweakly almost periodic semigroup of operators. Let E be a conjugate Banach space with a *fixed* predual E_* (i.e. E_* is a Banach space and $(E_*)^* = E$). Let $\mathcal{B}(E)$ denote the space of bounded linear operators from E into E and let $\mathcal{B}_*(E)$ denote the subspace of $\mathcal{B}(E)$ consisting of all elements T in $\mathcal{B}(E)$ that are continuous from $(E, \sigma(E, E_*))$ into $(E, \sigma(E, E_*))$. By the weak*-operator topology (abbreviated as W^*OT) on $\mathcal{B}(E)$ we shall mean the separated locally convex topology determined by the seminorms $\{p_{x,\varphi};\ x \in E \text{ and } \varphi \in E_*\}$, where $p_{x,\varphi}(T) = |\varphi(Tx)|$ for all T in $\mathcal{B}(E)$. Then, as known ([6], p. 973) the unit ball of $\mathcal{B}(E)$ is compact with respect to the W^*OT.

Let S be a fixed semigroup of contractive linear operators on E (i.e. $\|sx\| \leqslant \|x\|$ for all $s \in S$ and $x \in E$) contained in $\mathcal{B}_*(E)$. We say that S is *ultraweakly* (respectively *ultrastrongly*) *almost periodic* if for each $\varphi \in E_*$, $O(f) = \{f \circ s;\ s \in S\}$ is relatively compact with respect to the $\sigma(E_*, E)$ (respectively norm) topology.

For each $\varphi \in E_*$, define

$$p_\varphi(x) = \sup\{|\varphi(s\cdot x)|,\ |\varphi(x)|;\ s \in S\}.$$

Then each p_φ is a seminorm on E and the topology on E determined by $\{p_\varphi;\ \varphi \in E_*\}$ is stronger then the $\sigma(E, E_*)$ topology. Furthermore, $p_\varphi(sx-sy) \leqslant p_\varphi(x-y)$ for all $x, y \in E$ and $s \in E$.

Let $\bar{S}$ denote the closure of S in $\mathcal{B}(E)$ with respect to the W^*OT Then $\bar{S}$ is a semigroup and $(\bar{S}, W^*OT)$ is a compact Hausdorff space such that

(1) $$\text{for each } h \in \bar{S}, \quad k \to kh$$

is a continuous map from $(\bar{S}, W^*OT)$ into $(\bar{S}, W^*OT)$,

(2) $$\text{for each } s \in S, \quad k \to sk$$

is a continuous map from $(\bar{S}, W^*OT)$ into $(\bar{S}, W^*OT)$.

If $\bar{S} \subseteq \mathcal{B}_*(E)$, then $(\bar{S}, W^*OT)$ is even a compact topological semigroup.

LEMMA 3.1. *The following are equivalent:*

(a) *S is ultraweakly almost periodic.*

(b) *The topology on E determined by the seminorms $\{p_\varphi;\ \varphi \in E_*\}$ is a topology of the dual pair (E, E_*).*

(c) *S is equicontinuous on the unit ball B with respect to some topology $\mathcal{T}$ of the dual pair (E, E_*).*

(d) *S is quasi-equicontinuous on B with respect to the $\sigma(E, E_*)$ topology.*

(e) $\bar{S} \subseteq \mathscr{B}_*(E)$.

Proof. That (a) implies (b) follows from the Mackey–Arens theorem and (b) implies (c) is clear.

Assume (c) holds. Let $\{s_a\}$ be a net in S such that $s_a(x)$ converges to $s(x)$ for all x in B in the weak*-topology. Let $\tilde{s}$ be the extension of s to E. Then $\tilde{s}$ is in the weak closure of $\operatorname{co}(S)$ in the product topological vector space $(E, \mathscr{T})^E$. By Mazur's theorem, there exists a net $\{h_\beta\}$ in $\operatorname{co}(S)$ such that $h_\beta(x)$ converges to $\tilde{s}(x)$ in the $\mathscr{T}$-topology for each x in E. Since $\operatorname{co}(S)$ is an equicontinuous family of maps from $(B, \mathscr{T})$ to $(B, \mathscr{T})$, it follows that $\tilde{s}$ is also a continuous map from $(B, \mathscr{T})$ into $(B, \mathscr{T})$. Since B is closed and convex, it follows from [7], Corollary 17.3, that s is also a continuous map from $(B$, weak*) into $(B$, weak*). Hence (d) holds.

That (d) implies (e) is trivial. Finally, if (e) holds, then for each $\varphi \in E_*$, the set $\{\varphi \cdot t;\ t \in \bar{S}\}$ equals to the $\sigma(E_*, E)$-closure of $\{\varphi \cdot s;\ s \in S\}$. Since the map $t \to \varphi \cdot t$ from $\mathscr{B}_*(E)$ into E_* is continuous when $\mathscr{B}_*(E)$ has the induced W^*OT and when E_* has the $\sigma(E_*, E)$ topology, it follows that $\{\varphi \cdot s;\ s \in S\}$ is relatively compact in E_* with respect to the $\sigma(E_*, E)$ topology.

The next result is an analogue of Lemma 3.1.

LEMMA 3.2. *The following are equivalent:*

(a) *S is ultrastrongly almost periodic.*

(b) *The topology on E determined by $\{p_\varphi;\ \varphi \in E_*\}$ agrees with the $\sigma(E, E_*)$ on the unit ball B.*

(c) *S is equicontinuous on B with respect to the $\sigma(E, E_*)$ topology on B.*

Proof. Assume that (a) holds; then B can be considered as an equicontinuous family of linear maps on E_*. Hence by [7], Theorem 8.17, the weak*-topology on B agrees with the topology of uniform convergence on the totally bounded subsets of E_*. Consequently (b) holds.

That (b) implies (c) is clear.

Finally, if (c) holds, then for each $\varphi \in E_*$, $\{\varphi \circ s;\ s \in S\}$ is an equicontinuous family of continuous functions on the compact Hausdorff space $(B, \sigma(E, E_*))$. An easy application of the Ascoli's theorem shows that $\{\varphi \circ s;\ s \in S\}$ is relatively compact in the norm topology of E_*.

From now on, the semigroups S and $\bar{S}$ are equipped with the W^*OT. Then S is a topological semigroup.

COROLLARY 3.3. *If S is ultraweakly almost periodic, then $\bar{S}$ is a compact topological semigroup. Furthermore, if S is strongly almost periodic, then the multiplication in $\bar{S}$ is even jointly continuous.*

Proof. The first statement follows immediately from Lemma 3.1 (a) $\Leftrightarrow$ (e). The second statement also follows from Lemma 3.2 (a) $\Leftrightarrow$ (c) and [5], 14.1.

Let $C(S, E_*)$ denote the smallest uniformly closed subalgebra of $C(S)$ closed under complex conjugation and containing 1 and all functions f of the form

$$f(s) = \varphi(sx), \quad x \in E, \quad \varphi \in E_* \quad \text{and} \quad s \in S.$$

Then $C(S, E_*)$ is translation invariant.

The first part of the next lemma is an analogue of Lemma 4.8 in [3] and the proof is almost identical. The second part also follows from Lemma 2.10 in [3]. We omit the details.

LEMMA 3.4. *If S is ultraweakly almost periodic and i: $S \to \bar{S}$ is the injection map, then the adjoint map i^*: $C(\bar{S}) \to C(S)$ defined by $i^*(F) = f \circ i$ a linear isometry mapping $C(\bar{S})$ onto $C(S, E_*)$. Furthermore, $C(S)$ has a left (respectively right) invariant mean if and only if $C(\bar{S})$ has a left (respectively right) invariant mean.*

Let $WAP(S)$ denote the closed subspace of *weakly almost periodic functions* in $C(S)$, i.e., all $f \in C(S)$ such that $\{r_a f;\ a \in S\}$ is relatively compact in the weak topology of $C(S)$. Also let $AP(S)$ denote the closed subspace of *strongly almost periodic functions* in $C(S)$, i.e., all f in $C(S)$ such that $\{r_a f;\ a \in S\}$ is relatively compact in the norm topology of $C(S)$ (see [3]).

COROLLARY 3.5. *If S is ultraweakly almost periodic, then $C(S, E_*) \subseteq WAP(S)$. Also if S is ultrastrongly almost periodic, then $C(S, E_*) \subseteq AP(S)$.*

Proof. If S is ultraweakly almost periodic, then $\bar{S}$ is a compact topological semigroup and $C(\bar{S}) = WAP(\bar{S})$ ([3], Theorem 2.7) Hence $i^*(C(\bar{S})) = C(S, E_*) \subseteq WAP(S)$ by [3], Lemma 5.2. The proof of the second statement is similar.

Let (E, S) denote all elements x in E such that $sx = x$ for all $s \in S$, and let $\mathscr{F}(E_*, S)$ denote all $\varphi \in E_*$ such that $\varphi \circ s = \varphi$ for all $s \in S$. Then $\mathscr{F}(E, S)$ is a $\sigma(E, E_*)$-closed subspace of E. Furthermore, if we let $\mathscr{F}(E, S)_*$ denote the restriction of all $\varphi \in E_*$ to $\mathscr{F}(E, S)$, then $\mathscr{F}(E, S)$ is the continuous dual of $\mathscr{F}(E, S)_*$. We say that $\mathscr{F}(E_*, S)$ *separates* $\mathscr{F}(E, S)$ if for each non-zero element x in $\mathscr{F}(E, S)$, there exists $f \in \mathscr{F}(E_*, S)$ such that $f(x) \neq 0$.

PROPOSITION 3.6. *If S is ultraweakly almost periodic on E and $C(S, E_*)$ has a right invariant mean, then $\mathscr{F}(E_*, S)$ separates $\mathscr{F}(E, S)$.*

Proof. If x is a non-zero element in $\mathscr{F}(E, S)$, pick an element φ in E_* such that $\varphi(x) \neq 0$. Let $K = \overline{\operatorname{co}}\{\varphi \circ s;\ s \in S\}$. By Lemma 3.1, K is a $\sigma(E_*, E)$-compact convex subset of E_*. Then for each $t \in S$, $\psi \to \psi \circ t$ defines a continuous affine map from $(K, \sigma(E_*, E))$ into $(K, \sigma(E_*, E))$. Since $\bar{S}$ is a compact topological semigroup, by Lemma 3.4, and [3], Theorem 2.3 and Lemma 2.8, $\bar{S}$ contains a compact topological group G which is also a right ideal. Let λ be the unique normalized Haar measure on G. For each $\psi \in E_*$, define $\hat{\psi}$: $G \in E_*$ by $\hat{\psi}(t) = \psi \circ t$. Then the vector-valued integral $\int_G \hat{\psi} d\lambda$ defines an element ψ_0 in K. Then $\psi_0 \in \mathscr{F}(E_*, S)$ and $\psi_0(x) = \varphi(x) \neq 0$.

For each $x \in E$, let K_x denote the $\sigma(E, E_*)$-closure of the convex hull of $\{s \cdot x;\ s \in S\}$. The next result, which we shall need, is due to Yeadon [10]. For the sake of completeness, we give a different and simple proof.

LEMMA 3.7 (Yeadon). *If $K_x \cap \mathscr{F}(E, S) \neq \emptyset$ for each $x \in E$, then there exists $P \in \overline{\operatorname{co}}\, S$, where $\overline{\operatorname{co}}\, S$ is the closure of $\operatorname{co} S$ in $\mathscr{B}(E)$ with respect to the W^*OT, such that $P(x) \in K_x \cap \mathscr{F}(E, S)$ for each $x \in E$.*

Proof. For each $x \in E$, let $F(x) = \{t \in \overline{\operatorname{co}}\, S;\ t(x) \in \mathscr{F}(E, S)\}$. Then $F(x)$ is a non-empty closed subset of compact space $(\overline{\operatorname{co}}\, S, W^*OT)$. Furthermore, the family $\{F(x);\ x \in E\}$ has finite intersection property. Indeed, if $x_1, \ldots, x_n \in E$ and $t \in \bigcap \{F(x_i);\ i = 1, \ldots, n-1\}$, choose $t' \in \overline{\operatorname{co}}\, S$ such that $t'\big(t(x_n)\big) \in \mathscr{F}(E, S)$, then $t' \cdot t \in \bigcap \{F(x_i);\ i = 1, \ldots, n\}$. It follows from the compactness of $\overline{\operatorname{co}}\, S$ that $\bigcap \{F(x);\ x \in E\}$ is non-empty, and any P in the intersection satisfies the required property of the lemma.

PROPOSITION 3.8. *If S is ultraweakly almost periodic and $C(S, E_*)$ has a left invariant mean, then there exists a projection $P \in \mathscr{B}_*(E)$ mapping E onto $\mathscr{F}(E, S)$ such that*

(1) $\qquad P(x) \in K_x \cap \mathscr{F}(E, S) \quad$ *for* $\quad x \in E$,

(2) $\qquad P \in \overline{\operatorname{co}}\, S$.

Proof. By Lemma 3.1 and Lemma 3.7, it suffices to show that $K_x \cap \mathscr{F}(E, S) \neq \emptyset$ for each x in E. Indeed, using Lemma 3.1 (e), $\bar{S}$ is a semigroup of continuous affine mappings from the compact convex set $\big(K_x, \sigma(E, E_*)\big)$ into itself. Then exactly as the proof of Proposition 3.6, let G be compact topological group in $\bar{S}$ which is also a left ideal and let $\hat{x}$ define a vector-valued mapping $G \to E$ by $\hat{x}(t) = t(x)$. Then if λ is the normalized Haar measure on G, then the element defined by the vector-valued integral $\int_G \hat{x} d\lambda$ is an element in $K_x \cap \mathscr{F}(E, S)$.

In case $\mathscr{F}(E_*, S)$ separates $\mathscr{F}(E, S)$, then $\varphi(z) = \varphi(x)$ for all elements z in the intersection of $\mathscr{F}(E, S)$ and K_x, and $\varphi \in \mathscr{F}(E_*, S)$. Consequently, $P(x)$ is unique. Hence $P(s \cdot x) = P(x)$ for all $s \in S$. Summarizing:

PROPOSITION 3.9. *If S is ultraweakly almost periodic, $\mathscr{F}(E_*, S)$ separates $\mathscr{F}(E, S)$ and $C(S, E_*)$ has a left invariant mean, then there exists a projection $P \in \mathscr{B}_*(E)$ such that*

(1) *$P(x)$ is the unique element in $K_x \cap \mathscr{F}(E, S)$ for each $x \in E$,*

(2) *$P(s \cdot x) = P(x)$ for all $s \in S$, $x \in E$,*

(3) *$P \in \overline{\operatorname{co}}\, S$,*

(4) *$P(x) = \int_G \hat{x} d\lambda$, where G is a compact topological left ideal group in $\bar{S}$, λ is the normalized Haar measure of G and $\hat{x}(t) = t(x)$ for all $t \in G$.*

Remark 3.10. (a) If S is amenable (which is the case when S is commutative, or when S is a solvable group), then $C(S, E_*)$ always has an invariant mean.

(b) If S is a group, then $WAP(S)$ has a unique invariant mean (see [4], p. 37). Consequently, $C(S, E_*)$ also has an invariant mean by Corollary 3.5 when S is ultraweakly almost periodic in E.

(c). If S has finite intersection property for right ideals, then $AP(S)$ has an invariant mean. Hence $C(S, E_*)$ also has an invariant mean by Corollary 3.5 in case S is ultrastrongly almost periodic on E.

4. Applications to W^*-algebra. In this section M will denote a W^*-algebra and M_* will denote its unique predual. If $N \subseteq M$, let N^+ denote the collection of all positive elements in N. Also if $\mathscr{F} \subseteq M_*$, then $\mathscr{F}^+$ will denote the collection of all positive linear functionals in $\mathscr{F}$.

Let S be a fixed semigroup of ultraweakly continuous positive contractions from M into M. M is called *S-finite* if for each non-zero element in M^+, there exists a functional φ in $\mathscr{F}(M_*, S)^+$ such that $\varphi(x) \neq 0$.

LEMMA 4.1. *If M is S-finite, then S is ultraweakly almost periodic on M.*

Proof. Let $\mathscr{T}$ be the topology on M defined by the seminorms $\{q_\varphi,\ \varphi \in \mathscr{F}(M_*, S)^+\}$, where $q_\varphi(x) = |\varphi(x)|$ and let $B^+ = \{x \in M;\ \|x\| \leqslant 1,\ x \geqslant 0\}$. Then B^+ is a $\sigma(M, M_*)$-compact subset of M. We first show that for each φ in E_*, the restriction of φ to $(B^*, \mathscr{T})$ is continuous at 0. Otherwise, we can find $\varepsilon > 0$ such that for each $U \in \mathscr{N}$, where $\mathscr{N}$ is a $\mathscr{T}$-neighbourhood base of 0, there exists $x_u \in U \cap B^+$ and $\varphi(x_u) \geqslant \varepsilon$ (we can assume that $\varphi \geqslant 0$). By $\sigma(M, M_*)$-compactness of B^+, there exists a subset $\{x_{u_\alpha}\}$ of the net $\{x_u;\ U \in \mathscr{N}\}$ which converges to some x_0 in B^+ in the $\sigma(M, M_*)$ topology. It follows that $\varphi(x_{u_\alpha})$ converges to $\varphi(x_0)$ and $\varphi(x_0) \geqslant \varepsilon$. Hence $x_0 \neq 0$. By assumption, there exists $\psi \in \mathscr{F}(M_*, S)$ such that $\psi(x_0) > 0$. But the net $\{x_{u_\alpha}\}$ converges to 0 in the $\mathscr{T}$-topology, and ψ is $\mathscr{T}$-continuous. Hence $\psi(x_{u_\alpha})$ converges to 0, i.e., $\psi(x_0) = 0$ which is impossible.

To finish the proof, let $\varphi \in M_*$ and let $\{p_n\}$ be an orthogonal sequence of projections in M. Given $\varepsilon > 0$, choose $\psi_1, \ldots, \psi_m$ in $\mathscr{F}(M_*, S)^+$ such that whenever $x \in U$, where $U = \bigcap_{i=1}^{m} \{x;\ x \in B^+$ and $\psi_i(x) < \delta\}$ we have $\varphi(x) < \varepsilon$. Since $\{p_n\}$ is orthogonal, $\psi_i(p_n)$ converges to zero for each $i = 1, 2, \ldots, m$. Hence there exists N such that $\psi_i(p_n) < \delta$ for all $i = 1, \ldots, m$ and all $n \geqslant N$. It follows that $\psi_i(sp_n) < \delta$ for all $i = 1, \ldots, m$, $n \geqslant N$ and $s \in S$ (as each ψ_i is S-invariant). Hence $\varphi(s \cdot p_n) < \varepsilon$ for all $s \in S$, $n \geqslant N$ (note that $s \cdot p_n \in B^+$). Consequently, $\varphi(sp_n)$ converges to 0 uniformly in s. By a theorem of Akemann [1], p. 288, the set $\{\varphi \circ s;\ s \in S\}$ is relatively $\sigma(M_*, M)$-compact.

LEMMA 4.2. *If $\mathscr{F}(M_*, S)$ separates $\mathscr{F}(M, S)$ and $1 \in \mathscr{F}(M, S)$, then for each non-zero $x \in \mathscr{F}(M, S)^+$, there exists $\varphi \in \mathscr{F}(M_*, S)^+$ such that $\varphi(x) \neq 0$.*

Proof. Let $\psi \in \mathscr{F}(M_*, S)$ such that $\psi(x) \neq 0$, $\|\psi\| = 1$.

Let $\psi = \psi^+ - \psi^-$ be the unique orthogonal decomposition of ψ such that ψ^+, ψ^- are positive and normal, and $\|\psi^+\| + \|\psi^-\| = \|\psi\|$. Now $\psi = \psi^+ \circ s - \psi^- \circ s$, $\|\psi^+ \circ s\| = (\psi^+ \circ s)(1) = \psi^+(1)$ and $\|\psi^- \circ s\| = (\psi^- \circ s)(1) = \psi^+(1)$. Hence $\psi^+ \circ s = \psi^+$ and $\psi^- \circ s = \psi^-$ for all $s \in S$. Consequently, $\psi^+, \psi^- \in \mathscr{F}(M_*, S)^+$, and either $\psi^+(x) \neq 0$ or $\psi^-(x) \neq 0$.

THEOREM 4.3. *If S is a group of *-automorphisms, then the following are equivalent:*

(a) *M is S-finite,*

(b) *S is ultraweakly almost periodic,*

(c) *S is equicontinuous on the unit ball B of M with respect to some topology of the dual pair,*

(d) *S is quasi-equicontinuous on B with respect to the $\sigma(M, M_*)$ topology,*

(e) $\bar{S} \subseteq \mathscr{B}_*(M)$.

Proof. Use Proposition 1 in [8], Lemma 3.1, 4.1 and 4.2 and Proposition 3.1.

Using Proposition 3.9, we also have:

THEOREM 4.4. *Assume that $C(S, M_*)$ has an invariant mean. Then M is S-finite if and only if there exists a map $P\colon x \to \bar{x}$ from M onto $\mathscr{F}(M, S)$ satisfying the following properties:*

(a) *P is linear and strictly positive,*

(b) *P is ultraweakly and ultrastrongly continuous,*

(c) *$P(s \cdot x) = x$ for all $s \in S$, $x \in M$,*

(d) *$P(x) = x$ for all $x \in \mathscr{F}(M, S)$.*

In this case, there exists a compact left ideal topological group G in $\bar{S}$ such that $P(x) = \int_G \hat{x}\,d\lambda$, *where λ is the normalised Haar measure on G and* $\hat{x}(g) = g(x)$ for all $g \in G$. Furthermore, $P(x)$ is the unique element in $K_x \cap \mathscr{F}(M, S)$.

Remark. Theorem 4.3 (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (e) is proved implicitly in [9] by Størmer. In case S is the group of inner automorphisms, Theorem 4.3 (a) $\Leftrightarrow$ (b) is proved in Yeadon [11]. Also, when S is a group of *-automorphisms, Theorem 4.4 is proved by Kovacs and Szucs in [8] (except for the existence of G).

Finally, the following example shows that the converse of Lemma 4.1 is false. Let T be a finite semigroup containing more than one element and multiplication defined by $ab = a$ for any $a, b \in T$. Let $\mathscr{S} = \{l_t;\ t \in T\}$ and $M = l_\infty(T)$. Then M is not $\mathscr{S}$-finite, but $\mathscr{S}$ is ultraweakly almost periodic on M.

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Received September 11, 1974 (880)

Revised version January 22, 1975