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On projections and unconditional bases in direct sums of Banach spaces

by

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Abstract. We prove that a complemented subspace in a direct sum of totally incomparable Banach spaces is essentially a direct sum of complemented subspaces in summands. We describe unconditional bases in $\sum_{i=1}^{N} l_{p_i}$ as direct sums of bases in summands.

1. Introduction. This paper is a continuation of our research of [4] and [16]. It is divided into two parts. In Section 3 we prove Theorem 3.5 which says that any complemented subspace in a direct sum of totally incomparable Banach spaces is isomorphic to a direct sum of complemented subspaces in summands by an isomorphism which can be extended to an isomorphism of the whole direct sum. Particular cases of this result, concerning sums of l_p-spaces and not considering extensions, were proved in [4] and [16]. However, in the proofs presented in those papers we used an analogue of the decomposition method; this part of the proof contains a gap. In Section 4 we consider only sums of l_p-spaces, $1 \leqslant p \leqslant \infty$, where for notational reasons l_∞ denotes the space of null-sequences, usually denoted by c_0. Using the results about complemented subspaces we prove that an unconditional basis in a finite direct sum of l_p-spaces is equivalent to the direct sum of unconditional bases in summands. Some particular cases of two summands were considered in [4] and [16]. The method of the proof is a development of the methods of [4] and [16].

The above-mentioned result is not quite satisfactory since no good description of unconditional bases in l_p, $1 < p \neq 2 < \infty$, is known. However, it is known that in l_1, l_2 and l_∞ all normalized unconditional bases are equivalent (cf. [1] and [8]); thus we infer that in l_1+l_2, l_2+l_∞, l_1+l_∞ and $l_1+l_2+l_\infty$ all normalized unconditional bases are quasi-equivalent

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(for l_1+l_2 and l_2+l_∞ this was proved in [4]). The spaces listed above are the only known spaces possessing the above property.

2. Preliminaries. In this paper we employ the notation commonly used in the Banach space theory. The only exception is that *the symbol l_∞ denotes the space of sequences tending to zero*, equipped with the supremum norm, usually denoted by c_0.

A "space" always means "Banach space", a subspace is assumed to be closed and operators are assumed to be linear and bounded. If we have a sequence of spaces $(X_i)_{i=1}^N$, then $\sum_{i=1}^N X_i$ ($X_1+\ldots+X_N$ for a small number of summands) will denote the Cartesian product of X_i's with coordinatewise algebraic operations and some norm giving the product topology. If we have the space $X=\sum_{i=1}^N X_i$, then by P_{X_i} we always denote the projection from X onto X_i annihilating all X_j for $j\neq i$. On the other hand, if we have a projection P in the space X, then $X=\operatorname{Im}P+\ker P$. If we have $X=\sum_{i=1}^N X_i$, we will always identify X_i with a subspace of X in the natural way. Banach spaces X and Y are called *totally incomparable* iff they do not have isomorphic infinite-dimensional subspaces.

We consider real and complex scalars as well. If the scalar field is not specified, it means that the reasoning is valid for both cases.

The detailed proofs of all the required results about bases can be found in [15]. General information about Banach spaces can be found in [9] and [3]. For an elegant presentation of perturbation theory for linear operators in Banach spaces the reader is referred to [5].

3. Complemented subspaces. In the present section we consider complemented subspaces in direct sums of totally incomparable Banach spaces. We start with some definitions. Let $T\colon X\to Y$ be an operator with a closed range. We write

$$\alpha(T)=\dim\ker T \quad \text{and} \quad \beta(T)=\dim Y/\operatorname{Im}T.$$

We say that an operator T has an *index* if at most one of the above numbers is infinity and we define the index of T by

$$\mu(T)=\alpha(T)-\beta(T).$$

An operator with finite index is called a *Fredholm operator*.

An operator $S\colon X\to Y$ is called *strictly singular* if for any infinite-dimensional subspace $X_1\subset X$, $S|X_1$ is not an isomorphic embedding. It is well known (cf. [3]) that if Φ is a Fredholm operator with index k and S is a strictly singular operator, then $\Phi+S$ is a Fredholm operator

of index k. For a detailed exposition of the above facts the reader is referred to [5], Chap. V.2. For facts concerning the spectral theory of operators in Banach spaces the reader can consult [3], Chap. VII. If we have a Fredholm operator $\Phi\colon X\to X$ and X_1 is a complemented subspace of X, then $\Phi(X_1)$ is also a complemented subspace. This can easily be proved and is left to the reader.

First we consider the case of complex scalars and next we show how the real case can be deduced from the complex one.

LEMMA 3.1. *Let X be a complex Banach space and let $P\colon X\to X$ be a projection. Let us consider $Q=P+S$, where S is a strictly singular operator. Then $\sigma(Q)$ is a countable set and has at most two limit points,* 0 *and* 1. *Moreover, for numbers $\lambda\in\sigma(Q)$, $\lambda\neq 0,1$ the corresponding spectral projections are of finite rank.*

Proof. Let us define the set of complex numbers $U=\{\lambda\in C\colon P+S-\lambda I$ has an index$\}$. Since for $\lambda\neq 0,1$, $P-\lambda I$ is an isomorphism, we infer that $U\supset C\setminus\{0,1\}$. By [5], Theorem V.1.8, $\dim\ker(P+S-\lambda I)$ is constant in U except on a set of isolated points in U. Since for $|\lambda|$ big enough $P+S-\lambda I$ is an isomorphism and U is a connected set, this constant is equal to 0. To describe $\sigma(Q)$ let us recall that the point $\lambda\neq 0,1$ is in $\sigma(Q)$ iff $P+S-\lambda I$ is not an isomorphism. But for such values of λ the operator $P-\lambda I$ is an isomorphism, and so $P+S-\lambda I$ is a Fredholm operator with index 0. So it is not an isomorphism iff $\dim\ker(P+S-\lambda I)>0$ but by the previous remarks this proves that $\sigma(Q)\cap U$ is a set of isolated points. So $\sigma(Q)$ is countable and has at most two limit points, 0 and 1. Let $E(\lambda)$ be the spectral projection corresponding to the point λ, $\lambda\neq 0,1$. Then $P+S|E(\lambda)(X)$ and $(P-I)+S|E(\lambda)(X)$ are isomorphisms of $E(\lambda)(X)$ (cf. [3], VII. 3.20). So $[(P-I)+S](P+S)=S^2+SP+(P-I)S$ is an isomorphism of $E(\lambda)(X)$. But it is also a strictly singular operator, and so $E(\lambda)(X)$ is finite dimensional.

The next Lemma can be found in [5], Theorem V.2.1.

LEMMA 3.2. *Let $T\colon X\to Y$ be an operator with closed range and $\alpha(T)<\infty$ and let $S\colon X\to Y$ be a strictly singular operator. Then $\alpha(T+S)<\infty$ and $T+S$ has a closed range.*

Now we are ready to prove our key proposition.

PROPOSITION 3.3. *Let X and Y be totally incomparable complex Banach spaces. Let P be an infinite-dimensional projection in $X+Y$. Then there exists a Fredholm operator $\Phi\colon X+Y\to X+Y$ with index* 0 *and complemented subspaces $X_1\subset X$ and $Y_1\subset Y$ such that $\Phi(\operatorname{Im}P)$ is a subspace of finite codimension in X_1+Y_1.*

Proof. Let us define an operator $Q\colon X+Y\to X+Y$ by the formula $Q=P_XPP_X+P_YPP_Y$. We have $Q=P-P_XPP_Y-P_YPP_X=P+S$, where S is a strictly singular operator, and so we can apply

Lemma 3.1. Now let us consider a curve Γ which is the boundary of a rectangle symmetric with respect to the real axis such that $\Gamma\cap\sigma(Q)=\emptyset$ with 1 in the interior and 0 in the exterior of Γ. Define an operator

$$P_1 = -(2\pi i)^{-1}\int_{\Gamma} R(\lambda, Q)\,d\lambda,$$

where $R(\lambda, Q)$ is the resolvent of Q. By [8], Theorem VII.3.10, P_1 is a projection and $P_1Q=QP_1$. Since $R(\lambda,Q)$ commutes with P_X and P_Y, we have $P_1P_X=P_XP_1$ and $P_1P_Y=P_YP_1$. This shows that $\operatorname{Im}P_1 = X_1+Y_1$, where $X_1=\operatorname{Im}P_XP_1$ and $Y_1=\operatorname{Im}P_YP_1$. Moreover, $\sigma(P_1-Q)$ consists of a sequence of points tending to zero and for any point in $\sigma(P_1-Q)$ different from 0 the corresponding spectral projection is finite dimensional. To see this observe that $P_1-Q=(P_1-QP_1)-Q(I-P_1)$ has $\operatorname{Im}P_1$ and $\ker P_1$ as invariant subspaces and in both P_1-Q acts as an operator with the spectrum satisfying the desired properties. Thus $I+P_1-Q$ is a Fredholm operator of index 0 (cf. [3], Theorem VII.4.6). Since $P-Q$ is strictly singular, we infer that the operator

$$\Phi = I+(P_1-Q)-(P-Q) = I-P+P_1$$

is a Fredholm operator with index 0. So $\Phi(\operatorname{Im}P)$ is a closed complemented subspace. To finish the proof it is enough to show that $\Phi(\operatorname{Im}P)$ is a subspace of finite codimension in $\operatorname{Im}P_1$. Observe that $\Phi P = P_1P$, and so $\Phi(\operatorname{Im}P)\subset\operatorname{Im}P_1$ and $\Phi|\operatorname{Im}P = P_1|\operatorname{Im}P$. Suppose that $\operatorname{Im}P_1/\Phi(\operatorname{Im}P)$ is infinite dimensional. Then there exists an infinite-dimensional subspace $Z\subset\ker P$ such that $\Phi|Z$ is an isomorphism and $\Phi(Z)\subset\operatorname{Im}P_1$. But for $z\in Z$ we have $\Phi(z)=z+P_1(z)\in\operatorname{Im}P_1$, and so $z\in\operatorname{Im}P_1$. Hence $Z\subset\operatorname{Im}P_1$ but this implies that $Q|Z$ is an isomorphism (by [3], Theorem VII.3.20) and since $Q=P+S$ we infer that $S|Z$ is an isomorphism. This contradiction completes the proof.

Now we explain how the above proof can be used to obtain the result for real Banach spaces.

If we have a real Banach space X, then we can construct a complex space X_C such that X is a "real part" of X_C. It is the space of all pairs $x+iy$, where $x, y\in X$, the norm is defined by

$$\|x+iy\| = \sup\{\sqrt{\|ax-by\|^2+\|bx+ay\|^2}:\ a^2+b^2=1\}$$

and the scalar multiplication is defined by

$$(a+ib)(x+iy) = (ax-by)+i(bx+ay).$$

If we have a real operator $T\colon X\to X$, then it induces the complex operator $T_C\colon X_C\to X_C$ defined by $T_C(x+iy)=Tx+iTy$.

Proposition 3.3 for real Banach spaces follows from the fact that if we start from the real projection P in $X+Y$ and pass to P_C in $(X+Y)_C$ and apply the construction of the proof to obtain the complex projection P_1, it appears that $P_1=(\tilde P_1)_C$ for some real projection $\tilde P_1\colon X+Y\to X+Y$. Thus the constructed Fredholm operator is induced by some real Fredholm operator in $X+Y$ which has all the desired properties. Observe that if Z is a real Banach space, then operators in Z_C induced by operators in Z form a set closed under sums, taking inverses, composition and multiplication by a real scalar. This set is closed in the norm topology.

The only not obvious fact in the above considerations is contained in the following

Lemma 3.4. *Let Z be a real Banach space and $Q\colon Z\to Z$ be a linear operator. Let Γ be the boundary of a rectangle on a complex plane symmetric with respect to the real axis and disjoint with $\sigma(Q_C)$. Then*

$$-(2\pi i)^{-1}\int_{\Gamma} R(\lambda, Q_C)\,d\lambda$$

is induced by some real projection in Z.

Proof. To see the conclusion let us consider the approximating Riemann sums corresponding to the division of Γ symmetric with respect to the real axis and containing the vertices of Γ. Those sums have the form

$$-(2\pi i)^{-1}\sum a_k R(\lambda_k, Q_C)-\bar a_k R(\bar\lambda_k, Q_C),$$

where a_k are either real or purely imaginary.

In the first case

$$\begin{aligned} a_kR(\lambda_k, Q_C)-a_kR(\bar\lambda_k, Q_C) &= a_k\big((Q_C-\lambda_kI)^{-1}-(Q_C-\bar\lambda_kI)^{-1}\big)\\ &= a_k(\lambda_k-\bar\lambda_k)\,(Q_C-\lambda_kI)^{-1}(Q_C-\bar\lambda_kI)^{-1}\\ &= a_k(\lambda_k-\bar\lambda_k)\,\big(Q_C^2+|\lambda_k|^2I-(\lambda_k+\bar\lambda_k)Q_C\big)^{-1}.\end{aligned}$$

In the second case

$$\begin{aligned} a_kR(\lambda_k, Q_C)-\bar a_kR(\bar\lambda_k, Q_C) &= a_k\big(R(\lambda_k, Q_C)+R(\bar\lambda_k, Q_C)\big)\\ &= a_k(\lambda_k+\bar\lambda_k)\,\big(Q_C^2+|\lambda_k|^2I-(\lambda_k+\bar\lambda_k)Q_C\big)^{-1}.\end{aligned}$$

In both cases we obtain the purely imaginary scalar multiplied by an operator induced by a real one. So the whole sum is induced by a real operator and the same is true for the integral. Since this integral is a projection, it is induced by a real projection.

Thus we have obtained an analogue of Proposition 3.3 for real Banach spaces. From this fact we derive our main result.

Theorem 3.5. *Let X and Y be two totally incomparable Banach spaces (complex or real) and let Z be a complemented subspace in $X+Y$. Then there exists an isomorphism $\varphi\colon X+Y\xrightarrow{\text{onto}}X+Y$ such that*

$$\varphi(Z) = \varphi(Z)\cap X+\varphi(Z)\cap Y.$$

Proof. Let P be a projection from $X+Y$ onto Z. Apply Proposition 3.3 to P so as to obtain a Fredholm operator Φ and subspaces X_1 and Y_1. Consider the decomposition

$$X+Y = X_1+Y_1+W, \quad \text{where} \quad W = X_2+Y_2,\ X_2 \supset X \text{ and } Y_2 \subset Y.$$

Put $\Phi^{-1}(W) = V+\ker\Phi$. Observe that $\Phi|V\colon V \xrightarrow{\text{onto}} \Phi(V)$ is an isomorphism. Moreover, V and Z form a decomposition of $\operatorname{span}\{V, Z\}$ since $\Phi(V) \subset W$ and $\Phi(Z) \subset X_1+Y_1$. We can decompose $W = \Phi(V)+E_1$, where E_1 is finite-dimensional and $E_1 = E_1 \cap X + E_1 \cap Y$.

Thus we have two decompositions:

$$X+Y = \Phi(V)+E_1+X_1+Y_1,$$
$$X+Y = V+\operatorname{span}\{\ker\Phi, Z\}+E_2,$$

where E_2 is finite-dimensional and $\Phi(E_2) \subset X_1+Y_1$.

Since $\Phi|V$ is an isomorphism, we infer that

$$\Phi|\operatorname{span}\{\ker\Phi, Z\}+E_2\colon \operatorname{span}\{\ker\Phi, Z\}+E_2 \to X_1+Y_1+E_1$$

is a Fredholm operator with index 0. So $E_1+X_1+Y_1$ and $\operatorname{span}\{\ker\Phi, Z\}+ +E_2$ are isomorphic. Since Z is a subspace of finite codimension in $\operatorname{span}\{\ker\Phi, Z\}$, we can find an isomorphism

$$\tilde{\varphi}\colon \operatorname{span}\{\ker\Phi, Z\}+E_2 \xrightarrow{\text{onto}} E_1+X_1+Y_1$$

such that $\tilde{\varphi}(Z) = \tilde{\varphi}(Z)\cap X+\tilde{\varphi}(Z)\cap Y$. Now we define φ by the formulas

$$\varphi|\operatorname{span}\{\ker\Phi, Z\}+E_2 = \tilde{\varphi},$$
$$\varphi|V = \Phi.$$

This isomorphism has the desired properties.

Remark 3.6. It is an old problem posed by Banach whether a subspace of finite codimension in an infinite-dimensional Banach space is isomorphic to the whole space. If this problem has a positive solution, Theorem 3.5 will immediately follow from Proposition 3.3.

By an easy induction (cf. [13]) one can generalize the above Theorem 3.5 to the case of a finite number of pairwise totally incomparable summands. Let us state a special case of this result.

COROLLARY 3.7. *A complemented subspace of a direct sum of a finite number of totally incomparable Banach spaces is isomorphic to a direct sum of complemented subspaces in summands.*

Since l_p and l_q, $1 \leqslant p < q \leqslant \infty$ are totally incomparable Banach spaces (cf. eg. [9]) and by the theorem of Pełczyński [11] an infinite-dimensional complemented subspace in l_p is isomorphic to l_p, we have

COROLLARY 3.8. *An infinite-dimensional complemented subspace in $\sum_{i=1}^{N} l_{p_i}$ is isomorphic to $\sum_k l_{p_{i_k}}$ for some subsequence (i_k) of the sequence* $1, \ldots, N$.

This corollary will be used many times in Section 4. It was stated without proof in [16].

4. Unconditional bases. In this section we use Corollary 3.8 to investigate the form of unconditional bases in $\sum_{i=1}^{N} l_{p_i}$, $1 \leqslant p_i \leqslant \infty$. (Recall once more that l_∞ means the space of null sequences.) We start with some definitions and well-known facts.

Let (x_n) be a basis in the space X. We call the sequence (z_k) *almost disjoint with respect to the basis* (x_n) if there is a sequence of indices n_k such that

$$z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i x_i + v_k,$$

where $\|v_k\| \leqslant 2^{-k-1}\|z_k\|$ for $k = 1, 2, \ldots$

It is well known (cf. [2]) that a sequence almost disjoint with respect to some basis is equivalent to a block basic sequence with respect to that basis.

If we have two isomorphic Banach spaces X and Y, $d(X, Y)$ will denote the Banach–Mazur distance, i.e.,

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|, \text{ where } T \text{ is an isomorphism from } X \text{ onto } Y\}.$$

If we have an unconditional basis (x_n) in a space X we can always introduce an equivalent norm such that this basis will be unconditionally monotone, i.e., for any subset σ of natural numbers the projection P_σ defined by $P_\sigma\big(\sum_{i=1}^{\infty} a_i x_i\big) = \sum_{i\in\sigma} a_i x_i$ will have norm one.

Recall that the basis (x_n) in a space X is shrinking if the biorthogonal functionals (x_n^*) form a basis in X^* and it is boundedly complete if X is the dual space to $\operatorname{span}\{x_n^*\}$ in the natural duality. By a theorem of R. C. James (cf. [15]) if (x_n) is an unconditional basis in a space X, it is shrinking iff X does not contain a subspace isomorphic to l_1 and (x_n) is boundedly complete iff X does not contain a subspace isomorphic to l_∞.

Detailed information about l_p spaces can be found in [9]. In particular, [9] contains a good exposition of the "gliding hump" technique. The following lemma can be easily proved using the "gliding hump" technique.

LEMMA 4.0. *Let X be a subspace of $\sum_{i=1}^{N} l_{p_i}$ and let Y be a subspace of $\sum_{i=1}^{N} l_{q_i}$, where $p_i < q_j$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. Then any operator from Y into X is compact.*

Our aim now is to prove Theorem 4.11. The main point in the proof is Proposition 4.1. This proposition and a duality argument are sufficient to prove the theorem for $1 < p_i < \infty$. To deal with the non-reflexive case we use Lemma 4.7.

PROPOSITION 4.1. *Let* $X = \sum_{i=1}^{M} l_{p_i}$ *and* $Y = \sum_{j=1}^{N} l_{q_j}$, *where* $1 \leqslant p_i < q_j \leqslant \infty$ *for* $i = 1, 2, \ldots, M$ *and* $j = 1, 2, \ldots, N$. *Let* $z_n = (x_n, y_n)$ *be an unconditionally monotone normalized basis for* $X+Y$. *Suppose that* (n_s) *is a sequence of indices such that* $\operatorname{span}\{z_{n_s}\}_{s=1}^{\infty}$ *contains a subspace isomorphic to* l_{p_i} *for some* i. *Then* $\overline{\lim}\, \|x_{n_s}\| > 0$.

Proof. Suppose that the assertion of the proposition is false, i.e., that $\|x_{n_s}\| \to 0$. Then we can construct a sequence of pairs of natural numbers (p_s, k_s) and elements $g_s \in X+Y$ such that the following are satisfied:

1. $p_s < k_s < p_{s+1}$ for $s = 1, 2, \ldots$;
2. $0 < \inf \|g_s\| \leqslant \sup \|g_s\| < \infty$ and $g_s = \sum_{i=p_s}^{k_s} a_i z_{n_i}$;
3. (g_s) is equivalent to the unit vector basis in l_{p_i} for some i;
4. there is a positive μ such that $\|P_X(g_s)\| \geqslant \mu$ for $s = 1, 2, \ldots$;
5. $\|P_X(g)\| \leqslant 2^{-1}\|g\|$ for all g of the form $g = \sum_{i=p_s}^{k_s} \beta_i z_{n_i}$;
6. $P_Y(g_s)$ is almost disjoint with respect to (z_n).

In the construction we will use the following lemmas.

LEMMA 4.2. *If* $g_s = \sum_{p_s}^{k_s} a_i z_{n_i}$, $s = 1, 2, \ldots$ *and* 1–5 *are satisfied, then there exists a* $\mu_1 > 0$ *such that for any sequence of numbers* (ε_i), $|\varepsilon_i| = 1$, $g_s(\varepsilon) = \sum_{p_s}^{k_s} \varepsilon_i a_i z_{n_i}$ *also satisfies* 1–5, *where* μ *is replaced by* μ_1.

Proof. Conditions 1–3 are satisfied since for any $\varepsilon = (\varepsilon_i)$ the operator T_ε defined by $T_\varepsilon(z_n) = \varepsilon_n z_n$ is an isomorphism and there is a constant C such that $\|T_\varepsilon\|\,\|T_\varepsilon^{-1}\| \leqslant C$ for all ε. If condition 4 is not satisfied for any $\mu_1 > 0$, then there is $\varepsilon = (\varepsilon_i)$ and a subsequence (s_k) such that $\|P_X(g_{s_k}(\varepsilon))\| \leqslant 2^{-k}$ and then by stability theorems (cf. [15]) $g_{s_k}(\varepsilon)$ is equivalent to a basic sequence $\big(P_Y(g_{s_k}(\varepsilon))\big)$ in Y, which in view of condition 3 contradicts the fact that Y does not contain l_{p_i}. Condition 5 is obviously satisfied.

LEMMA 4.3. *For any natural number* N *there exists a constant* $\gamma(N)$ *such that for any* N*-dimensional Banach space* E *and any finite set of vectors* $(x_i) \subset E$

$$\min\Big\{\Big\|\sum \varepsilon_i x_i\Big\| = |\varepsilon_i| = 1\Big\} \leqslant \gamma(N) \max \|x_i\|.$$

Proof. Since the Banach–Mazur distance between l_2^N and any N-dimensional space is bounded by some constant depending only on N, it is

enough to consider the Hilbert space only. Moreover, it is enough to prove the lemma in the case where

$$\|x_i + x_j\| \geqslant \max(\|x_i\|, \|x_j\|)$$

and

$$\|x_i - x_j\| \geqslant \max(\|x_i\|, \|x_j\|) \quad \text{for all } i, j;\ i \neq j,$$

since in the opposite case we can replace x_i and x_j by $x_i \pm x_j$ for a suitable choice of the sign. If the above conditions are satisfied, we easily conclude that $|\langle x_i \|x_i\|^{-1}, x_j \|x_j\|^{-1}\rangle| \geqslant 2^{-1}$ and this implies that the cardinality of our set is bounded by some constant depending only on N. This proves the lemma.

LEMMA 4.4. *Let* $X = \sum_{i=1}^{M} l_{p_i}$ *and* $Y = \sum_{j=1}^{N} l_{q_j}$, *where* $1 \leqslant p_i < q_j \leqslant \infty$ *for all* i *and* j. *Let* $\|z_n\| \geqslant \mu > 0$, *let* (z_n) *weakly converge to zero and let* $\|P_X(z_n)\| \geqslant \mu_1 > 0$. *Then there is a subsequence* (z_{n_s}) *equivalent to the unit vector basis in* l_{p_i} *for some* i.

The proof uses a standard "gliding hump" argument (cf. [2] and [9]).

Now we can proceed to the construction.

Using the condition $\|x_{n_s}\| \to 0$ we can easily construct by induction a sequence (p_s, k_s) and $\bar{g}_s$ satisfying 1, 2, 4 and 5 with $\bar{g}_s$ instead of g_s. Observe that μ can be chosen as close to 1/2 as we wish. Passing to a subsequence (g_s), we can insure also 3. (If $\bar{g}_s \xrightarrow{w} 0$, this can be done by Lemma 4.4. The opposite case is possible only if some p_i are equal to 1 and then $\bar{g}_s$ is an unconditional basic sequence which does not converge weakly to zero, so there is a functional $f \in (X+Y)^*$ such that $|f(g_{s_k})| \geqslant \delta > 0$ for some subsequence (s_k) and the unconditionality implies that (g_{s_k}) is equivalent to the unit vector basis in l_1.)

Now we inductively choose a subsequence (s_r) and $\varepsilon = (\varepsilon_i)$, $|\varepsilon_i| = 1$, such that $g_{s_r}(\varepsilon)$ satisfies condition 6. Suppose we have chosen $s_1, \ldots, s_N$ and $(\varepsilon_i)_{i=1}^{k_{s_N}}$, $|\varepsilon_i| = 1$. Let K be such that $P_Y(\operatorname{span}\{g_{s_r}(\varepsilon)\}_{r=1}^{N})$ is almost contained in $\operatorname{span}\{z_i\}_{i=1}^{K}$ and let Q_K be the partial sum projection onto $\operatorname{span}\{z_i\}_{i=1}^{K}$. Since $z_n \xrightarrow{w} 0$, we have $\|Q_K(z_n)\| \to 0$. So we can find s_{N+1} such that for $n > p_{s_{N+1}}$ we have

$$\|Q_K P_Y(z_n)\| \leqslant 2^{-(N+1)} \gamma(K)^{-1} \beta,$$

where $\beta > 0$ satisfies the relation $\|P_Y(g_s(\varepsilon))\| \geqslant \beta$ for all s and all ε. It follows from condition 5 that such a β exists. By Lemma 3.4 we can find ε_i, $i = k_{s_N}+1, \ldots, k_{s_{N+1}}$, $|\varepsilon_i| = 1$ such that

$$\Big\|Q_K P_Y \sum_{p_{s_{N+1}}}^{k_{s_{N+1}}} \varepsilon_i a_i z_{n_i}\Big\| \leqslant 2^{-(N+1)}\beta \leqslant 2^{-(N+1)} \Big\|P_Y\Big(\sum_{p_{s_{N+1}}}^{k_{s_{N+1}}} \varepsilon_i a_i z_{n_i}\Big)\Big\|.$$

In view of Lemma 4.2 this finishes the construction.

Using (p_s, k_s) and (g_s) satisfying 1–6, we will obtain the contradiction. We have

$$P_X(g_s) = a_s + b_s, \quad \text{where} \quad a_s = \sum_{i=p_s}^{k_s} z_{n_i}^*\big(P_Y(g_s)\big) z_{n_i}.$$

Moreover,

$$1 = \frac{\|P_X(a_s)+P_X(b_s)\|}{\|a_s+b_s\|} \leqslant \frac{\|P_X(a_s)\|+\|P_X(b_s)\|}{\|a_s+b_s\|} \leqslant \frac{\|P_X(a_s)\|}{\|a_s\|} + \frac{\|P_X(b_s)\|}{\|P_X(g_s)\|}$$

$$\leqslant 2^{-1} + \frac{\|P_X(b_s)\|}{\|P_X(g_s)\|},$$

which implies

$$\|P_X(b_s)\| \geqslant 2^{-1}\|P_X(g_s)\| \geqslant 2^{-1}\mu.$$

But $P_Y(g_s)$ is almost disjoint with respect to the basis (z_n) so it is an unconditional basic sequence in Y. Therefore the operator $P_XQ\,|\,\mathrm{span}\{P_Y(g_s)\}_{s=1}^{\infty}$, where Q is the natural projection in the basis (z_n) onto the space $\mathrm{span}\,\{z_n\colon n \neq n_i \text{ for } p_s \leqslant i \leqslant k_s,\ s = 1, 2, \ldots\}$, is a non-compact operator from a subspace of Y into X, which contradicts Lemma 4.0.

Remark 4.5. The construction of this proposition can be done in much greater generality. We present the result in the form we need in the proof of our main theorem. Let us only remark that the proposition remains true if we consider an unconditional basis in $X+Y$, where X is a subspace of l_p and Y is a subspace of l_q, $p<q$.

LEMMA 4.6. *Let $z_n = (x_n, y_n)$ be an unconditional basis in $l_1 + Y$, where Y has an unconditional shrinking basis. If $\|x_{n_s}\| \geqslant \mu > 0$ for some subsequence (n_s), then $\underline{\lim}\,|x_{n_s}^*(x_{n_s})| > 0$ (where $(z_n^*) = (x_n^*, y_n^*)$ is the dual basis to (z_n)).*

Proof. If the lemma is not true, then for some subsequence $x_{n_s}^*(x_{n_s}) \to 0$ and z_{n_s} is equivalent to the unit vector basis in l_1. So the biorthogonal system $(x_{n_s}^*, y_{n_s}^*)$ is equivalent to the unit vector basis in l_∞, which implies that $\sum a_s y_{n_s}^*$ is a convergent series for any null-sequence (a_s). Since $1 = x_{n_s}^*(x_{n_s}) + y_{n_s}^*(y_{n_s})$, we conclude that $\|y_{n_s}^*\| \geqslant a > 0$ for $s = 1, 2, \ldots$ but this implies that Y^* contains a subspace isomorphic to l_∞ which contradicts the fact that Y^* is separable (cf. [2]).

LEMMA 4.7. *Let $z_n = (x_n, y_n)$ be an unconditional normalized basis in $l_1 + Y$. Suppose that for some subsequence (n_s) we have $|x_{n_s}^*(x_{n_s})| \geqslant \mu > 0$ for $s = 1, 2, \ldots$ Then (z_{n_s}) is equivalent to the unit vector basis in l_1.*

Proof. Let us consider a convergent series $\sum_{s=1}^{\infty} a_s z_{n_s}$. Then $\sum_{s=1}^{\infty} a_s x_{n_s}$ is an unconditionally convergent series in l_1 and by the Orlicz theorem [10] $\sum_{s=1}^{\infty} |a_s|^2 \|x_{n_s}\|^2 < \infty$. But $\|x_{n_s}\| \geqslant \mu_1 > 0$ for $s = 1, 2, \ldots$ so $\sum_{s=1}^{\infty} |a_s|^2 < \infty$. Let us now define the operator $S\colon l_1 \to l_2$ by the formula

$$S(x) = \sum_s z_{n_s}^*(x, 0)e_s = \sum_s x_{n_s}^*(x)e_s,$$

where (e_s) denotes the unit vector basis in l_2. S is a continuous operator by the above considerations and by the Grothendieck inequality (cf. [8]) it is absolutely summing. But this implies that $\sum_s a_s S(x_{n_s})$ is an absolutely convergent series. Since $\|S(x_{n_s})\| = \sqrt{\sum_k |x_{n_k}^*(x_{n_s})|^2} \geqslant \mu$, we conclude that $\sum_{s=1}^{\infty} |a_s| < \infty$. But this means that (z_{n_s}) is equivalent to the unit vector basis in l_1.

Remark 4.8. The last part of this proof shows that any one-dimensional unconditional expansion of identity in l_1 is absolute. It is interesting to compare this remark with the remark after Lemma 1.3. in [12].

LEMMA 4.9. *Let $X = \sum_{i=1}^{M} l_{p_i}$ and $Y = \sum_{j=1}^{N} l_{q_j}$, where $1 < p_i < q_j \leqslant \infty$ for all i, j. Let $z_n = (x_n, y_n)$ be an unconditionally monotone basis for $X+Y$. Let (n_s) be such that $\mathrm{span}\{z_{n_s}\}$ contains a subspace isomorphic to some l_q. Then $\underline{\lim}\,\|x_{n_s}\| = 0$.*

Proof. The biorthogonal functionals $z_n^* = (x_n^*, y_n^*)$ form a basis in $(X+Y)^*$. By Proposition 4.1 $\underline{\lim}\,\|y_{n_s}^*\| > 0$. So by Lemma 4.4 we can find a subsequence of (n_s) (denote it also by n_s) such that $(z_{n_s}^*)$ is equivalent to the unit vector basis in some $l_{q_j}^*$, so (z_{n_s}) is equivalent to the unit vector basis in l_{q_j}. If $\|x_{n_s}\| \geqslant \mu > 0$, then by Lemma 4.4 we can find a subsequence equivalent to the unit vector basis in some l_{p_i}. This contradiction proves the lemma.

COROLLARY 4.10. *Let X, Y and (z_n) be as in Proposition 4.1. If (n_s) is a subsequence of indices such that $\|x_{n_s}\| \to 0$, then $\mathrm{span}(z_{n_s})$ is isomorphic to a complemented subspace of Y.*

Proof. Proposition 4.1 implies that $\mathrm{span}(z_{n_s})$ does not contain a subspace isomorphic to any l_{p_i}. So by Corollary 3.8 $\mathrm{span}(z_{n_s})$ is isomorphic to a complemented subspace of Y.

THEOREM 4.11. *Let $X = \sum_{i=1}^{N} l_{p_i}$, $1 \leqslant p_1 < p_2 < \ldots < p_N \leqslant \infty$ and let (z_n) be an unconditional basis for X. Then one can divide the set of natural numbers into N parts N_i in such a way that $\overline{\mathrm{span}}\{z_n\}_{n\in N_i} \sim l_{p_i}$.*

Proof. We will use induction on the number of summands. If $N=1$, there is nothing to prove. Suppose we have proved the theorem for all sums of $(N-1)$ summands and consider $l_{p_1} + \sum_{i=2}^{N} l_{p_i}$. Introducing an equivalent norm, we can assume that (z_n) is an unconditionally monotone basis. We begin with the following lemma.

LEMMA 4.12. *Let $p_1 \neq 2$ and let Theorem 4.11 hold for $(N-1)$ summands. Then, for any unconditionally monotone basis $z_n = (x_n, y_n)$ for $l_{p_1} + \sum_{i=2}^{N} l_{p_i}$, 0 is an isolated limit point of the set $\{\|x_n\|\}$.*

Proof. If 0 is not an isolated limit point of the set $\{\|x_n\|\}$, then there is a sequence of numbers a_n, $a_n \searrow 0$ and a sequence of infinite sets N_n of natural numbers such that $a_{n+1} < \|x_k\| < a_n$ for $k \in N_n$. We can assume that $(z_k)_{k \in N_n}$ is equivalent to the unit vector basis in l_{p_1} with the constant d_n. (If $p_1 = 1$, this follows from Lemmas 4.6 and 4.7, and if $p_1 > 1$, it follows from Lemma 4.4.) Let $\overline{N}_n$ be a subset of N_n of cardinality $r(n) \cdot N$, where $r(n)$ is such that $d_n^{-1} \lambda(l_{p_1}^{r(n)}, l_{p_i}) \to \infty$ for $i = 2, 3, \ldots, N$, where

$$\lambda(l_p^s, l_q) = \inf\{\|T\| \, \|T^{-1}\| \, \|P\|, \text{ where } Y \text{ is an isomorphic embedding of } l_p^s \text{ into } l_q \text{ and } P \text{ is a projection from } l_q \text{ onto } T(l_p^s)\}.$$

It is possible to choose such an $r(n)$ since $p \neq 2$ (cf. [7] and [8]). Let us now consider the space $V = \overline{\operatorname{span}}\{z_k \colon k \in \bigcup_{n=1}^{\infty} \overline{N}_n\}$. By Corollary 4.10, V is isomorphic to $\sum_j l_{p_{i_j}}$ for some subsequence (i_j) of the sequence $2, 3, \ldots, N$ and by the inductive hypothesis $\{z_k \colon k \in \bigcup_{n=1}^{\infty} \overline{N}_n\}$ can be divided into less than N parts, each of them spanning some l_{p_i}. But then one of those parts contains $r(n_s)$ elements of the set $\overline{N}_{n_s}$ for some subsequence (n_s) of indices. But this contradicts the choice of $r(n)$. This contradiction proves the lemma.

Using this lemma, we can make the inductive step in the proof of the theorem. We will consider the following cases:

1° $p_1 = 1$.

By Proposition 4.1 $\overline{\lim} \|P_1(z_n)\| > 0$ (once more P_i means $P_{l_{p_i}}$) and by Lemma 4.12 there is no limit point of the set $\{\|P_1(z_n)\|\}$ in the interval $(0, \varepsilon)$ for some ε. Since $l_1 + \sum_{i=2}^{N} l_{p_i}$ is not isomorphic to l_1, Lemmas 4.6 and 4.7 imply that $\underline{\lim} \|P_1(z_n)\| = 0$. So we can divide the set of natural numbers into two parts, N_1 and N_2, in such a way that $\|P_1(z_n)\| \geqslant \varepsilon$ for $n \in N_1$ and $\|P_1(z_n)\| \to 0$ for $n \in N_2$. By Lemma 4.6 $\overline{\operatorname{span}}\{z_n \colon n \in N_1\} \sim l_1$ and by Corollary 4.10 and Corollary 3.8 $\operatorname{span}\{z_n \colon n \in N_2\} \sim \sum_{i=2}^{N} l_{p_i}$. To obtain the conclusion we have to apply the inductive hypothesis to the basis $\{z_n \colon n \in N_2\}$.

2° $1 < p_1$ and $p_N = \infty$.

In this case the basis is shrinking, the biorthogonal functionals span $l_{p_N}^* + \sum_{i=1}^{N-1} l_{p_i}^*$ and $l_{p_N}^* = l_1$. If we apply case 1° to biorthogonal functionals, we obtain the desired decomposition of the set of natural numbers.

3° $1 < p_1 < \ldots < p_N < \infty$ and $p_1 \neq 2$.

We proceed like in case 1° using Proposition 4.1, Lemma 4.9 and Lemma 4.12 to obtain the decomposition of the set of natural numbers. To reach the conclusion we apply Lemma 4.9 and Corollary 4.10.

4° $1 < p_1 < \ldots < p_N \leqslant \infty$ and $p_1 = 2$.

We apply case 3° or 1° for the dual space and biorthogonal functional, using $l_{p_N}^*$ instead of l_{p_1}. Thus the theorem is proved.

DEFINITION 4.13. Two unconditional bases (x_n) and (y_n) are called *quasi-equivalent* if there exists an isomorphism $T \colon \operatorname{span}(x_n) \xrightarrow{\text{onto}} \operatorname{span}(y_n)$ such that $T(x_n) = y_{\sigma(n)}$ for some permutation σ. If the permutation σ can be chosen to be the identity, we call such bases *equivalent*.

It was proved by Bari [1] that all normalized unconditional bases in l_2 are equivalent and by Lindenstrauss and Pełczyński [8] that the same holds in l_1 and l_∞. From those facts and our Theorem 4.11 we have

COROLLARY 4.14. *All normalized unconditional bases in $l_1 + l_2$, $l_1 + l_\infty$, $l_2 + l_\infty$ and $l_1 + l_2 + l_\infty$ are quasi-equivalent.*

Remark 4.15. It seems to be an interesting problem to describe all spaces with unconditional bases in which all normalized unconditional bases are quasi-equivalent. In particular, we do not know an example of two non-quasi-equivalent normalized unconditional bases in the space $(\Sigma l_2)_1$, the space of all absolutely convergent series in l_2.

Remark 4.16. If we consider conditional bases, a theorem like Theorem 4.11 is not true. It follows from the results of [17] that $l_1 + l_2$ has a normalized basis weakly convergent to zero. Since weak convergence and norm convergence for sequences coincide in l_1, this basis is not a direct sum of bases in summands.

Remark 4.17. In this remark we show that Theorem 3.5 is not true for arbitrary subspaces. Let us take a sequence $(a) = (a_i)_{i=1}^{\infty}$ such that $\lim a_i = 0$ and $|a_i| \leqslant 1$. Let us consider the space $l_p + l_q$, $1 \leqslant p < q < \infty$, and denote by (u_i) the unit vector basis in l_p and by (v_i) the unit vector basis in l_q. Consider the space $E^{(a)} \subset l_p + l_q$ defined by

$$E^{(a)} = \operatorname{span}\{a_i u_i + v_i \colon i = 1, 2, \ldots\}.$$

If $\sum |a_i|^{pq/(q-p)} < \infty$, then using the Hölder inequality we obtain $E^{(a)} \sim l_p$. If $\sum |a_i|^{pq/(q-p)} = \infty$, one checks that $E^{(a)}$ contains subspaces isomorphic to l_p and l_q. Moreover, the sequence $z_i = a_i u_i + v_i$ is an unconditional basis in $E^{(a)}$. Suppose that $E^{(a)} = X + Y$, where X is isomorphic to a subspace of l_p and Y is isomorphic to a subspace of l_q. Moreover, we can assume that $X \cap l_q = \{0\}$ and $Y \cap l_p = \{0\}$. One can easily prove (cf. [6] and [13]) that $P_{l_p}|X$ and $P_{l_q}|Y$ are isomorphisms. Let P_X (resp. P_Y) denote the projection from $E^{(a)}$ onto X (resp. Y) annihilating Y (resp. X). Then $a_i u_i = P_{l_p}(z_i) = P_{l_p}P_X(z_i) + P_{l_p}P_Y(z_i)$. Since $P_{l_p}P_Y$ is a compact operator, $\|P_{l_p}P_Y(z_i)\| \to 0$ and so $\|P_{l_p}P_X(z_i)\| \to 0$, which implies $\|P_X(z_i)\| \to 0$. But this contradicts Remark 4.5.

The spaces $E^{(a)}$ for $p = 2$ and $2 < q$ were studied by H. P. Rosenthal in [14]. In this case the results of this remark can be derived from the theory of $\mathscr{L}_p$-spaces.

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On the inductive limit of $\bigcup l_p$, $0 < p < 1$

by

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Abstract. For each p, $0 < p < 1$, let l_p be the linear space of all scalar sequences $x = (x_n)$ such that $\|x\|_p = \sum |x_n|^p < \infty$. We define the q-topology on $\bigcup l_p$ to be the strongest linear topology on $\bigcup l_p$ such that each injection i_p: $l_p \to \bigcup l_p$ is continuous. This paper contains results about $\bigcup l_p$ with the q-topology.

0. Introduction. For each p, $0 < p < 1$, let l_p be the linear space of all scalar sequences $x = (x_n)$ such that $\|x\|_p = \sum |x_n|^p < \infty$. We define the q-topology on $\bigcup l_p$ to be the strongest linear topology on $\bigcup l_p$, such that each injection i_p: $l_p \to \bigcup l_p$ is continuous.

To investigate the properties of this topology, we will find it useful to use the following notation. The set R is the set of all sequences of positive numbers increasing to one and R_p, $0 < p < 1$, is the set of all sequences (a_n) in R such that $p \leqslant a_1$. The set Q is the set of all sequences of positive numbers less than one decreasing to zero, and Q_p is the set of all sequences of positive numbers (a_n) in Q such that $a_1 \leqslant p$. The space $l_{(r_n)}$ is the set of all scalar sequences $x = (x_n)$ such that $\|x\|_{(r_n)} = \sum |x_n|^{r_n} < \infty$. The vector e_n is the vector $(0, \ldots, 0, 1, 0, \ldots)$, where the non-zero entry is in nth position. The projection P_n is the mapping which maps $(x_1, x_2, \ldots)$ onto $(x_1, x_2, \ldots, x_n, 0, \ldots)$. The set S_p is the set $\{x \in l_p\colon \|x\|_p \leqslant 1\}$. The symbol $\operatorname{card}(A)$ represents the cardinality of the set A, and $\operatorname{supp}(x)$ is the support of the vector $x = (x_1, x_2, \ldots)$, i.e., the set of all integers n such that $x_n \neq 0$. The space Φ is the linear space of all scalar sequences with at most finitely many non-zero entries, and τ is the strongest linear topology on Φ. A block basic sequence $\{z_n\}$ is a sequence of non-zero vectors of the form $z_n = \sum_{i=n_{k-1}+1}^{n_k} a_i e_i$, where $\{n_k\}$ is a strictly increasing sequence of non-negative integers. A space has a block basis if it has a Schauder basis consisting of a block basic sequence. The symbol $\bigcup l_p$ denotes the space $\bigcup_{0<p<1} l_p$, and $[x_n]_q$ indicates the q-closed linear subspace of $\bigcup l_p$ generated by the set $\{x_n\}$. Finally, $\operatorname{co}^q\{A\}$ is the q-convex balanced hull of the set A, i.e., the set of all vectors of the form $\sum_{i=1} a_i a_i$,