to a subspace of l_p and Y is isomorphic to a subspace of l_q. Moreover, we can assume that $X \cap l_q = \{0\}$ and $Y \cap l_p = \{0\}$. One can easily prove (cf. [6] and [13]) that $P_{l_p}|X$ and $P_{l_q}|Y$ are isomorphisms. Let P_X (resp. P_Y) denote the projection from $E^{(a)}$ onto X (resp. Y) annihilating Y (resp. X). Then $a_i u_i = P_{l_p}(z_i) = P_{l_p}P_X(z_i) + P_{l_p}P_Y(z_i)$. Since $P_{l_p}P_Y$ is a compact operator, $\|P_{l_p}P_Y(z_i)\| \to 0$ and so $\|P_{l_p}P_X(z_i)\| \to 0$, which implies $\|P_X(z_i)\| \to 0$. But this contradicts Remark 4.5.

The spaces $E^{(a)}$ for $p = 2$ and $2 < q$ were studied by H. P. Rosenthal in [14]. In this case the results of this remark can be derived from the theory of $\mathscr{L}_p$-spaces.

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On the inductive limit of $\bigcup l_p$, $0 < p < 1$

by

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Abstract. For each p, $0 < p < 1$, let l_p be the linear space of all scalar sequences $x = (x_n)$ such that $\|x\|_p = \sum |x_n|^p < \infty$. We define the q-topology on $\bigcup l_p$ to be the strongest linear topology on $\bigcup l_p$ such that each injection i_p: $l_p \to \bigcup l_p$ is continuous. This paper contains results about $\bigcup l_p$ with the q-topology.

0. Introduction. For each p, $0 < p < 1$, let l_p be the linear space of all scalar sequences $x = (x_n)$ such that $\|x\|_p = \sum |x_n|^p < \infty$. We define the q-topology on $\bigcup l_p$ to be the strongest linear topology on $\bigcup l_p$, such that each injection i_p: $l_p \to \bigcup l_p$ is continuous.

To investigate the properties of this topology, we will find it useful to use the following notation. The set R is the set of all sequences of positive numbers increasing to one and R_p, $0 < p < 1$, is the set of all sequences (a_n) in R such that $p \leqslant a_1$. The set Q is the set of all sequences of positive numbers less than one decreasing to zero, and Q_p is the set of all sequences of positive numbers (a_n) in Q such that $a_1 \leqslant p$. The space $l_{(r_n)}$ is the set of all scalar sequences $x = (x_n)$ such that $\|x\|_{(r_n)} = \sum |x_n|^{r_n} < \infty$. The vector e_n is the vector $(0, \ldots, 0, 1, 0, \ldots)$, where the non-zero entry is in nth position. The projection P_n is the mapping which maps $(x_1, x_2, \ldots)$ onto $(x_1, x_2, \ldots, x_n, 0, \ldots)$. The set S_p is the set $\{x \in l_p: \|x\|_p \leqslant 1\}$. The symbol $\mathrm{card}(A)$ represents the cardinality of the set A, and $\mathrm{supp}(x)$ is the support of the vector $x = (x_1, x_2, \ldots)$, i.e., the set of all integers n such that $x_n \neq 0$. The space Φ is the linear space of all scalar sequences with at most finitely many non-zero entries, and τ is the strongest linear topology on Φ. A block basic sequence $\{z_n\}$ is a sequence of non-zero vectors of the form $z_n = \sum_{i=n_{k-1}+1}^{n_k} a_i e_i$, where $\{n_k\}$ is a strictly increasing sequence of non-negative integers. A space has a block basis if it has a Schauder basis consisting of a block basic sequence. The symbol $\bigcup l_p$ denotes the space $\bigcup_{0<p<1} l_p$, and $[x_n]_q$ indicates the q-closed linear subspace of $\bigcup l_p$ generated by the set $\{x_n\}$. Finally, $\mathrm{co}^q\{A\}$ is the q-convex balanced hull of the set A, i.e., the set of all vectors of the form $\sum_{i=1} a_i a_i$, where a_i is in A and the scalars α_i satisfy $\sum_{i=1} |\alpha_i|^q \leqslant 1$, and a balanced set B is q-convex if and only if $\mathrm{co}^q\{B\} = B$.

We show that $\{e_n\}$ is a "symmetric" Schauder basis for $\bigcup l_p$, and that $\bigcup l_p$ is a complete separable non-locally convex linear topological space with the q-topology. We also show the following: A sequence converges in $\bigcup l_p$ if and only if the sequence is contained in and converges in some l_p; a set is compact in $\bigcup l_p$ if and only if the set is contained in and compact in some l_p; no closed infinite-dimensional subspace of $\bigcup l_p$ is contained in any l_p; each closed infinite-dimensional subspace of $\bigcup l_p$ contains an isomorphic copy of Φ; and no infinite-dimensional subspace of $\bigcup l_p$ is metrizable.

1. Main results.

THEOREM 1. *$\{e_j\}$ forms a Schauder basis for $\bigcup l_p$ with the q-topology.*

Proof. Suppose $x = (x_1, x_2, \ldots)$ is in $\bigcup l_p$; then x is in l_p for some $p \in (0, 1)$. Since $\sum_{k=1}^{n} x_k e_k$ converges to x in l_p, and the l_p-topology is weaker than the q-topology on l_p, $\sum_{k=1}^{n} x_k e_k$ converges to x in the q-topology. Since the q-topology is stronger than the l_1-topology, the representation is unique and the coefficient functionals are continuous.

LEMMA 2. *Let A and B be balanced q-convex sets; then*

$$\mathrm{co}^q(A \cup B) = \{\lambda x + \mu y \colon |\lambda|^q + |\mu|^q \leqslant 1,\ x \in A,\ y \in B\}.$$

Proof. Let $z = \lambda_1 x_1 + \ldots + \lambda_m x_m + \mu_1 y_1 + \ldots + \mu_n y_n$ be an element of $\mathrm{co}^q(A \cup B)$, where x_i is in A, y_i is in B and $\sum |\lambda_i|^q + \sum |\mu_i|^q \leqslant 1$. Let $\lambda = \left(\sum |\lambda_i|^q\right)^{1/q}$ and $\mu = \left(\sum |\mu_i|^q\right)^{1/q}$. Then $z = \lambda x + \mu y$, where $\frac{\lambda_1}{\lambda} x_1 + \ldots \ldots + \frac{\lambda_m}{\lambda} x_m$ is in A, $y = \frac{\mu_1}{\mu} y_1 + \ldots + \frac{\mu_n}{\mu} y_n$ is in B, and $|\lambda|^q + |\mu|^q \leqslant 1$.

Remark. Lemma 2 generalizes easily to the case when there are finitely many sets. This fact will be useful.

THEOREM 3. *In $\bigcup l_p$, the q-topology restricted to each l_p is strictly weaker than the l_p-topology.*

Proof. Let N be a q-neighborhood of zero. Choose p' such that $p < p' < 1$. Then there exists $\alpha > 0$ such that $\alpha S_{p'} \subset N$. Since $(\alpha S_{p'}) \cap l_p$ is not contained in any multiple of S_p, N restricted to l_p is not contained in any multiple of the S_p.

The following theorem will show that, in order to define the q-topology, it suffices to choose any sequence (p_k) in R and define the topology on $\bigcup l_p$ to be the strongest linear topology for which each injection $i_k \colon l_{p_k} \to \bigcup l_p$ $= \bigcup_{k=1}^{\infty} l_{p_k}$ is continuous. This fact will prove useful in our later development, and we will use it frequently without any discussion.

THEOREM 4. *Let (p_k) be any sequence of positive numbers increasing to one. Then the q-topology is the strongest vector topology on $\bigcup l_p$ which is weaker than the l_{p_k}-topology on each l_{p_k}.*

Proof. Let $\mathcal{T}$ denote the strongest linear topology on $\bigcup l_p$ such that each injection from l_{p_k} is continuous. Since each injection from l_{p_k} is continuous in the q-topology, the q-topology is weaker than $\mathcal{T}$. Also, if $0 < p < 1$, there exists p_k such that $p < p_k < 1$. Since the l_{p_k}-topology on l_p is weaker than the l_p-topology on l_p, the injection from l_p must be $\mathcal{T}$-continuous. Hence $\mathcal{T}$ is weaker than the q-topology.

THEOREM 5. *The linear topology on Φ generated by the collection of paranorms, $\{\|\ \|_{(q_n)} \colon (q_n) \in Q\}$, is the strongest linear topology, τ.*

Proof. Let $\mathcal{T}$ be the linear topology generated by the collection $\{\|\ \|_{(q_n)} \colon (q_n) \in Q\}$. Clearly, $\mathcal{T} \subset \tau$. The topology τ is known to be locally convex (cf. [2]). Let N be a convex balanced τ-neighborhood of 0. Choose t_n, $0 < t_n < 1$ such that $t_n e_n \in N$. For some ε, $0 < \varepsilon < 1$, choose (q_n) in Q such that $\sum_{n=1}^{\infty} \frac{1/q}{t_n} < 1$. Then $\|\lambda\|_{(q_n)} < \varepsilon$ implies λ is in N. Hence $\tau \subset \mathcal{T}$.

THEOREM 6. *Let N be a q-neighborhood of 0 in $\bigcup l_p$, let (p_k) be a sequence of positive numbers increasing to one, and let*

$$A_N = \{\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots) \in \Phi \colon \lambda_1 x_1 + \ldots + \lambda_n x_n \in N \text{ for every } x_k \in S_{p_k}\}.$$

Then A_N is a neighborhood of 0 in Φ.

Proof. Consider the collection $\mathcal{A} = \{A_N \colon N$ a balanced q-neighborhood of 0$\}$. If $A_N \in \mathcal{A}$, then A_N is balanced because N is balanced. Also $A_M + A_M \subset A_N$ if $M + M \subset N$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots)$ be any element in Φ. If t is a scalar, then $t \sum_{j=1}^{n} \lambda_j S_{p_j} \subset t \sum_{j=1}^{n} \lambda_j S_{p_j}$. Thus $t \sum_{j=1}^{n} \lambda_j S_{p_j}$ is contained in N for all sufficiently small t. This implies that A_N is absorbing.

Hence $\mathcal{A}$ is a local base at zero for a linear topology (cf. e.g. [1]). Thus A_N is a τ-neighborhood of 0.

THEOREM 7. *Let N and A_N be as above and let $\hat{N} = \{\lambda_1 x_1 + \ldots + \lambda_n x_n \colon \lambda \in A_N$ and $x_k \in S_{p_k}\}$. Then $\{\hat{N} \colon N \in \mathcal{B}\}$, where $\mathcal{B}$ is a local base for the q-topology, is a local base for the q-topology in $\bigcup l_p$.*

Proof. Clearly, $\hat{N} \subset N$. Also if $\hat{M}$ is in $\{\hat{N} \colon N \in \mathcal{B}\}$, then $\hat{M}$ is absorbing and also $\hat{M}$ is balanced because S_{p_k} is balanced. Furthermore, if $\hat{M}$ is in $\{\hat{N} \colon N \in \mathcal{B}\}$ we will construct a $\hat{G}$ in $\{\hat{N} \colon N \in \mathcal{B}\}$ such that $\hat{G} + \hat{G} \subset M$. There exists $k > 0$ such that $(|\lambda|^q + |\mu|^q)^{1/q} \leqslant K(|\lambda| + |\mu|)$ for all $q \geqslant p$, and for all λ and μ. Choose G in $\mathscr{B}$ such that $KG+KG\subset M$. Let Z be in $\hat{G}+\hat{G}$. Then there exists $\lambda=(\lambda_1,\ldots,\lambda_n,0,\ldots)$ and $\mu=(\mu_1,\ldots,\mu_n,0,\ldots)$ in A_G such that $Z=(\lambda_1x_1+\ldots+\lambda_nx_n)+(\mu_1y_1+\ldots+\mu_ny_n)$, where x_i and y_i are in S_{p_i}. Then

$$Z=(|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1}\left(\frac{\lambda_1}{(|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1}}x_1+\frac{\mu_1}{(|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1}}y_1\right)+\ldots$$

$$\ldots+(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n}\left(\frac{\lambda_n}{(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n}}x_n+\frac{\mu_n}{(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n}}y_n\right).$$

Since

$$\left\|\frac{\lambda_jx_j+\mu_jy_j}{(|\lambda_j|^{p_j}+|\mu_j|^{p_j})^{1/p_j}}\right\|_{p_j}\leqslant 1,$$

we need only show that

$$((|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1},\ldots,(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n},0,\ldots)$$

is in A_M. To see this, let z_k be in S_{p_k}. Then

$$(|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1}z_1+\ldots+(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n}z_n$$
$$=(|\lambda_1|+|\mu_1|)\frac{(|\lambda_1|^{p_1}+|\mu_1|^{p_1})^{1/p_1}}{|\lambda_1|+|\mu_1|}z_1+\ldots+(|\lambda_n|+|\mu_n|)\frac{(|\lambda_n|^{p_n}+|\mu_n|^{p_n})^{1/p_n}}{|\lambda_n|+|\mu_n|}z_n$$
$$\in KG+KG\subset M.$$

From the above considerations we see that the collection $\{\hat{N}\colon N\in\mathscr{B}\}$ is a local base at zero for a linear topology which is stronger than the q-topology. Since the injections, i_n, are continuous in this topology, the topology must equal the q-topology.

PROPOSITION 8. *If (p_n) is a sequence of positive numbers increasing to one, then the collection of all sets $\{\sum\lambda_ix_i\colon\sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi$, and $x_i\in a_iS_{p_i}\}$, where (a_i) and (q_i) are in Q_{p_1} form a local base at zero for a linear topology in $\bigcup l_p$. This topology also has a local base at zero given by the collection of all sets $\{\sum_{i=1}^{n}x_i\colon x_i\in a_iS_{p_i}$, n is a natural number$\}$, where (a_i) is an Q_{p_1}.*

Proof. The set $N=\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi,$ and $x_i\in S_{p_i}\}$ is clearly balanced and absorbing. Also if $G=\{\sum\lambda_ix_i;\ \sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi$, and $x_i\in\beta_iS_{p_i}\}$, where $q_i\leqslant p_1$, and $\beta_i\leqslant a_i/2^{1/q_i}$, then $G+G\subset N$. Hence the collection of sets $\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi,$ and $x_i\in a_iS_{p_i}\}$ forms a local base for a linear topology.

The collection of sets $\{\sum_{i=1}^{n}x_i\colon x_i\in a_iS_i\}$ obviously forms a local base for a linear topology which is weaker than the topology given above.

Given N as above, choose $(\gamma_i)\in Q_1$ such that $\sum_{i=1}^{\infty}(\gamma_i/a_i)^{1/q_i}<1$, and let $M=\{\sum_{i=1}^{n}x_i\colon x_i\in\gamma_iS_{p_i}\}$. Then $M\subset N$. Hence the two topologies agree.

THEOREM 9. *The q-topology has each of the following as a local base at zero*

(i) $\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi$, *and* $x_i\in a_iS_{p_i}\}$, *where* $(a_i)\in Q$ *and* $(q_n)\in Q$.

(ii) $\{\sum x_i\colon x_i\in a_iS_{p_i}\}$, *where* (a_i) *is in* Q.

(iii) $\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q}\leqslant 1,\ (\lambda_i)\in\Phi$, *and* $x_i\in a_iS_{p_i}\}$, *where* (a_i) *is in* Q *and q is any (fixed) number between 0 and p_1.*

Proof. The collection of sets given in (iii) forms a local base at zero for a linear topology on $\bigcup l_p$ by a proof similar to the proof of the previous theorem. It is clear that the topologies generated by the sets in (i), (ii), and (iii) are the same. Since the injections i_n are continuous in these topologies, we need only show that they are stronger than the q-topology. By Theorem 7, $\{\hat{N}\colon N\in\mathscr{B}\}$ is a local base at zero for the q-topology. Let $\hat{N}$ be in $\{N\colon N\in\mathscr{B}\}$. By Theorem 6, A_N is a τ-neighborhood of 0 in Φ. By Theorem 5, there exist $(q_n)\in Q_{p_1}$ and ε, $0<\varepsilon<1$, such that $\{\lambda\in\Phi\colon \|\lambda\|_{(q_n)}\leqslant\varepsilon\}\subset A_N$. Hence

$$\hat{N}\supset\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q_i}\leqslant\varepsilon,\ (\lambda_i)\in\Phi,\ \text{and}\ x_i\in S_{p_i}\}$$
$$\supset\{\sum\lambda_ix_i\colon\ \sum|\lambda_i|^{q_i}\leqslant 1,\ (\lambda_i)\in\Phi,\ \text{and}\ x_i\in\varepsilon^{1/q_i}S_{p_i}\}.$$

Remark. Sets of the form $\bigcup a_nS_{p_n}$, where $(a_n)\in Q$ cannot be q-neighborhoods of 0 since for any (a_n) and (β_n) in Q there exist an x and y in $\bigcup\beta_nS_{p_n}$ such that $x+y\notin\bigcup(a_nS_{p_n})$.

COROLLARY 10. *There exists a local base, B, at zero for the q-topology on $\bigcup l_p$ such that for every N in B if $\sum_1^m a_ke_k$ is in N, $\pi(n)$ is a permutation of the natural numbers, and $|\eta_k|\leqslant 1$, then $\sum_1^m\eta_ka_ke_{\pi(k)}$ is in N.*

Proof. Let B be the local base at zero given in (ii) of Theorem 9. Let $x=\sum_1^m a_ke_k$ be in N and N in B. Then $x=\sum_1^n x_j$, where x_j is in $a_jS_{p_j}$. We can assume that $x_j=\sum_{k=1}^{n}X_{j_k}e_k$. Since x_j is in $a_jS_{p_j}$, $y_j=\sum_{k=1}^{m}\eta_kX_{j_k}e_{\pi(k)}$ is in $a_jS_{p_j}$. Hence $\sum_{k=1}^{m}\eta_ka_ke_{\pi(k)}=\sum_{j=1}^{n}y_j$ is in N.

Remark. Let (c_n) be any bounded sequence of numbers which is bounded away from zero, and let $\pi(n)$ be any permutation of the natural numbers. If T is the linear mapping of $\bigcup l_p$ onto itself such that $T(e_n)=c_ne_{\pi(n)}$, then T is a q-isomorphism. Moreover, if T_π is the linear mapping of $\bigcup l_p$ onto itself such that $T_\pi(e_n)=e_{\pi(n)}$, then the family $\{T_\pi\colon\pi$ a permutation of the natural numbers} is equi-continuous. These facts follow from Corollary 10, and they show the "symmetry" of the basis $\{e_n\}$ with respect to the q-topology.

THEOREM 11. *Let B be a subset of $\bigcup l_p$ such that $P_n B \subset B$ for $n = 1, 2, 3, \ldots$; then if B is q-closed, B is l_1-closed (in $\bigcup l_p$).*

Proof. Let x be any point in $\bigcup l_p$ which is in the l_1-closure of B, and let N be any open q-neighborhood of x. Choose a sequence $\{y_n\}$ such that y_n is in B and converges to x in the l_1-topology. By Theorem 1, $P_m x$ converges to x in the q-topology. Hence there exists an n_0 such that $P_{n_0} x$ is in N. Clearly, $P_{n_0} y_n$ converges to $P_{n_0} x$ in the q-topology. Hence $P_{n_0} y_n$ is eventually in N. Since $P_{n_0} y_n$ is also in B, this implies that x is in the q-closure of B and hence in B.

COROLLARY 12. *The q-topology has a local base at zero whose members are l_1-closed.*

Proof. The q-closure of any of the sets given in (i), (ii), or (iii) of Theorem 9 satisfy the conditions of the previous theorem.

Remark. It is not necessarily true that the sets given in (i), (ii), or (iii) of Theorem 9 are closed. For example, if $\sum_i^\infty |a_i|^{p_1} < \infty$, then $x = (a_1, a_2, \ldots)$ is in the closure of $B = \{\sum_1^n x_i \colon x_i \in a_i S_{p_i}\}$ but x is not in B.

DEFINITION. The *r-topology* is defined to be the linear topology on $\bigcup l_p$ generated by the sets $\{x \colon \|x\|_{(r_n)} < \varepsilon\}$, where $\varepsilon > 0$, $\|x\|_{(r_n)} = \sum |x_n|^{r_n}$, and (r_n) is any sequence of numbers increasing to one.

THEOREM 13. *The set $\bigcup l_p$ equals the set $\bigcap \{l_{(r_n)} \colon (r_n) \in R\}$.*

Proof. It is clear that the set $\bigcup l_p$ is contained in the set $\bigcap \{l_{(r_n)} \colon (r_n) \in R\}$. Conversely, if x is not in $\bigcup l_p$, choose a sequence (n_k) of positive integers such that $n_1 < n_2 < \ldots$ and $\|P_{n_k}(x) - P_{n_{k-1}}(x)\|_{p_k} \geqslant 1$, where $P_{n_0}(x) = 0$. If $(r_n) = (p_1, \ldots, p_1, p_2, \ldots, p_2, p_3, \ldots)$, where p_k appears $n_k - n_{k-1}$ times, then x is not in $l_{(r_n)}$.

THEOREM 14. *The space $\bigcup l_p$ is complete for the q-topology.*

Proof. To see the space is complete for the q-topology, let $\{y_d \colon d \in D\}$ be a q-Cauchy net in $\bigcup l_p$. Since the net is q-Cauchy, it is r-Cauchy, and hence r-converges to some point y in $\bigcup l_p$ (cf. [1]). Since $\lim y_d = y$ in the r-topology, $\lim y_d = y$ in the l_1-topology.

By Corollary 12, the q-topology has a local base consisting of l_1-closed sets, and this implies that $\lim y_d = y$ in the q-topology.

THEOREM 15. *The q-topology is not locally convex.*

Proof. The topology is strictly stronger than the l_1-topology. Furthermore, it is weaker than the l_p-topology on each l_p, $0 < p < 1$. Hence any locally convex topology weaker than the q-topology must be weaker than the l_1-topology. This implies that the topology is not locally convex.

PROPOSITION 16. *Suppose $\{x_n\}$ is a sequence which is l_p-bounded in l_p for some p, $0 < p < 1$. If $1 \geqslant q > p$ and if $\lim_{n\to\infty} \|x_n\|_\infty = 0$, then $\lim \|x_n\|_q = 0$.*

Proof. Suppose that the result is not true. Then there are positive numbers A and B and a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A > B$, $\|y_n\|_p < A$ and $B < \|y_n\|_q$. By applying the mean value theorem to $f(s) = t^s$ for $0 < t \leqslant 1$, we find that $\sum_{k=1}^\infty |z_k|^p - \sum_{k=1}^\infty |z_k|^q \geqslant \sum_{k=1}^\infty (p-q)\ln(\|z\|_\infty)|z_k|^q$ for every sequence of scalars $z = (z_k)$ such that $\|z\|_\infty \leqslant 1$. If we replace (z_k) by $\{y_n\}$ in this last inequality, we obtain $(p-q)\ln(\|y_n\|_\infty)\|y_n\|_q \leqslant \|y_n\|_p - \|y_n\|_q$. Hence $A - B \geqslant (p-q)\ln(\|y_n\|_\infty) B$ holds for $n = 1, 2, \ldots$ Since $\lim \|y_n\|_\infty = 0$, this is impossible.

COROLLARY 17. *If $0 < p < q$ and $0 < \|x\|_\infty < 1$, then $(p-q)\|x\|_q \ln(\|x\|_\infty) \leqslant \|x\|_p - \|x\|_q$.*

Proof. The proof follows immediately from the proof of the preceding proposition.

THEOREM 18. *A sequence $\{x_n\}$ in $\bigcup l_p$ converges in the q-topology if and only if there exists a p, $0 < p < 1$, such that $\{x_n\} \subset l_p$ and $\{x_n\}$ converges in the l_p-topology.*

Proof. We first prove that an r-convergent sequence l_p-converges in some l_p. In so doing, we can assume that the sequence $\{x_n\}$ r-converges to zero. First of all $\{x_n\}$ must be l_p-bounded in some l_p, $0 < p < 1$, for, otherwise one can find a subsequence $\{x_{n_k}\}$ and a strictly increasing sequence $\{m_k\}$ of non-negative integers such that $\|(P_{m_k} - P_{m_{k-1}+1}) x_{n_k}\|_{p_k} \geqslant k$, and this contradicts the fact that $\{x_n\}$ r-converges to zero. Since $\{x_n\}$ r-converges to zero, $\{x_n\}$ converges to zero in the l_∞-topology. Thus, by Proposition 16, $\{x_n\}$ converges to zero in any l_q when $q > p$. This completes the proof for the r-topology.

If a sequence converges in the q-topology, then the sequence must converge in the r-topology. Hence it must converge in some l_p, $0 < p < 1$.

COROLLARY 19. *A set B is q-bounded if and only if there exists a p, $0 < p < 1$, such that B is contained in l_p and B is bounded in the l_p-topology.*

Proof. This corollary follows from the preceding theorem and the fact that a set B is bounded in a linear topological space if and only if for every sequence $\{b_n\}$ contained in B and every sequence (a_n) of scalars, $\lim a_n b_n = 0$.

THEOREM 20. *Let B be a subset of $\bigcup l_p$. Then B is q-totally bounded if and only if B is l_p-totally bounded in some l_p, $0 < p < 1$.*

Proof. Since B is totally bounded, B is bounded. Hence by Theorem 18 B is contained in l_{p_0} for some p_0, $0 < p_0 < 1$. We will show that B is l_p-totally bounded for every p, $p_0 < p < 1$. If B is not totally bounded in l_p, then by [4] there exist a sequence $\{x_n\}$ in l_p, an $\varepsilon > 0$, and a strictly increasing sequence (N_k) of positive integers such that $\sum_{k=N_n}^{\infty} |x_{n,k}|^p \geqslant \varepsilon$, where $x_n = (x_{n,1}, x_{n,2}, \ldots)$. Since B is totally bounded, B is totally bounded in l_1. Hence $\lim_{n\to\infty} \sum_{k=N_n}^{\infty} |x_{n,k}| = 0$. Proposition 16 then implies that $\lim_{n\to\infty} \sum_{k=N_n}^{\infty} |x_{n,k}|^p = 0$, and this is a contradiction.

THEOREM 21. *Let B be a subset of $\bigcup l_p$. Then B is q-compact if and only if B is l_p-compact in some l_p, $0 < p < 1$.*

Proof. Since B is compact, B is totally bounded. Thus Theorem 20 implies that B is totally bounded in some l_p, $0 < p < 1$. Since B is closed, B is closed in l_p. Hence B is l_p-compact.

THEOREM 22. *Any subset of $\bigcup l_p$ is separable in the q-topology.*

Proof. Let X be any subset of $\bigcup l_p$, and let $X_n = X \cap l_{p_n}$. Select an l_{p_n}-dense countable subset, Y_n, of X_n. Then $Y = \bigcup Y_n$ is a countable dense subset of X.

Remark. It is not necessarily true that a subset of a separable linear topological space is separable — even when the subset is a closed subspace. An example of such a space is given in [3].

THEOREM 23. *Let X be an infinite-dimensional subspace of $\bigcup l_p$ which is closed in the q-topology. If (p_k) is in R, there exists a sequence $\{x_k\}$ such that x_k is in $(l_{p_k} \cap X) \setminus l_{p_{k-1}}$.*

Proof. Suppose the theorem is not true. Then there exists a p, $0 < p < 1$, such that X lies entirely in l_p. Since the l_p-topology is stronger than the q-topology, X is l_p-closed. Also X is l_q-closed in l_q for any q, $p < q \leqslant 1$ for the same reason. The identity mapping of X onto itself is l_p-to-l_q continuous, and therefore the identity mapping of X onto itself is an l_p-to-l_q isomorphism by the open mapping theorem. But l_p and l_q contain no infinite-dimensional isomorphic subspaces (cf. [5]).

LEMMA 24. *Let $\{x_1, \ldots, x_m)$ be linearly independent elements of l_p and let $\{y_1, \ldots, y_n\}$ be linearly independent elements of $l_q \setminus l_p$, where $0 < p < q \leqslant 1$. Assume that $(\operatorname{span}\{y_1, \ldots, y_n\}) \cap l_p = \{0\}$. Then given any $\varepsilon > 0$ there exists a positive integer k such that $\|P_k(\lambda_1 x_1 + \ldots + \lambda_m x_m + \mu_1 y_1 + \ldots + \mu_n y_n)\|_p < 1$ implies that $|\mu_i| < \varepsilon$ for $i = 1, \ldots, n$.*

Proof. Suppose the statement is not true. Then there exist an $\varepsilon > 0$ and sequences $\{\lambda_{i_k}\}_{k=1}^{\infty}$, $i = 1, \ldots, m$, and $\{\mu_{j_k}\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, such that $\max(|\mu_{1_k}|, \ldots, |\mu_{n_k}|) \geqslant \varepsilon$ and $\|P_k(\lambda_{1_k} x + \ldots + \lambda_{m_k} x_m + \mu_{1_k} y_1 + \ldots + \mu_{n_k} y_n)\|_p < 1$. By dividing by suitable constants if necessary, we may assume that $\varepsilon = \max(|\mu_{1_k}|, \ldots, |\mu_{n_k}|)$, and by selecting subsequences if necessary, we may assume that $\lim_{k\to\infty} \mu_{i_k} = \mu_i$ for $i = 1, \ldots, n$ and $\varepsilon = \max(|\mu_1|, \ldots, |\mu_n|)$.

Case 1. The sequence $\{(\lambda_{1_k}, \ldots, \lambda_{n_k})\}_{k=1}^{\infty}$ is bounded (in l_n^{∞}).

Since the y_j's are linearly independent, $\varepsilon = \max(|\mu_1|, \ldots, |\mu_n|)$, and $(\operatorname{span}\{y_1, \ldots, y_n\}) \cap l_p = \{0\}$, it follows that $\lim_{k\to\infty} \|P_k(\mu_1 y_1 + \ldots + \mu_n y_n)\|_p = \infty$. Hence there exists a positive integer k such that

$$\|P_k(\lambda_{1_k} x_1 + \ldots + \lambda_{m_k} x_m + \mu_{1_k} y_1 + \ldots + \mu_{n_k} y_n\|_p > 1.$$

Case 2. The sequence $\{(\lambda_{1_k}, \ldots, \lambda_{n_k})\}_{k=1}^{\infty}$ is unbounded (in l_n^{∞}). Fix k_0 such that the vectors $P_{k_0} x_1, \ldots, P_{k_0} x_m$ are linearly independent. Then $\{\|P_{k_0}(\lambda_{1_k} x_1 + \ldots + \lambda_{m_k} x_m)\|_p\}_{k=1}^{\infty}$ is unbounded. Since $\max(|\mu_{1_k}|, \ldots, |\mu_{1_n}|) = \varepsilon$ it follows that there exists an integer $k \geqslant k_0$ such that

$$\begin{aligned} &\|P_k(\lambda_{1_k} x_1 + \ldots + \lambda_{m_k} x_m + \mu_{1_k} y_1 + \ldots + \mu_{n_k} y_n)\|_p \\ &\qquad > \|P_{k_0}(\lambda_{1_k} + \ldots + \lambda_{m_k} x_m + \mu_{1_k} y_1 + \ldots + \mu_{n_k} y_n)\|_p \\ &\qquad > 1. \end{aligned}$$

THEOREM 25. *If X is an infinite-dimensional subspace of $\bigcup l_p$ and if $X \cap l_{p_k}$ is finite dimensional for every k (or equivalently, $X \cap l_p$ is finite dimensional for each p, $0 < p < 1$), then X is r-isomorphic to Φ.*

Proof. Choose p_0 in the interval $(0, 1)$ and let $\{x_1, x_2, \ldots, x_{k_1}\}$ be a basis for $X \cap l_{p_0}$. By induction choose (p_n) in R such that $\dim(X \cap l_{p_n}) > \dim(X \cap l_{p_{n-1}})$, choose a strictly increasing sequence (k_n) of positive integers and choose a sequence of vectors $\{x_j\}_{j=k_1+1}^{\infty}$ such that the set $\{x_1, \ldots, x_{k_n}\}$ forms a basis for $X \cap l_{p_n}$.

Define a linear map T from the span of $\{x_n\}$ onto Φ by letting $T(x_n) = e_n$ and then extending by linearity. Since T is one-to-one and onto Φ, and since T^{-1} is continuous, we need only show that T is continuous. To show this, let N be any balanced convex neighborhood of 0 in Φ. Choose a sequence (α_n) such that $\alpha_n > 0$ and $\alpha_n e_n \in N$, and let $\beta_n = \frac{1}{2n}\alpha_n$. We will construct a sequence (r_n) in R and find a $\delta > 0$ such that if $x = \sum_{j=1}^{\infty} \lambda_j x_j$ and $\|x\|_{(r_n)} < \delta$, then $|\lambda_j| < \beta_j$ for $j = 1, \ldots, n$. This will show that T is continuous.

There exist a positive integer n_1 and a positive number ε such that $\|P_{n_1}(\lambda_1 x_1 + \ldots + \lambda_{k_1} x_{k_1})\|_{p_0} < \varepsilon$ implies $|\lambda_i| \leqslant \beta_i$, $i = 1, \ldots, k$. There exists a sequence $(\gamma_{1i})_{i=k_1+1}^{\infty}$ of positive numbers such that $|\lambda_i| < \gamma_{1j}$ implies

$$\|P_{n_1}(\lambda_{k_1+1} x_{k_1+1} + \lambda_{k_1+2} x_{k_1+2} + \ldots)\|_{p_0} < \frac{\varepsilon}{2}.$$

By Lemma 14 we can select a positive integer $n_2 > n_1$ such that $\|(P_{n_2} - P_{n_1})(\lambda_1 x_1 + \ldots + \lambda_{k_2} x_{k_2})\|_{p_1} < 1$ implies $|\lambda_j| < \beta_j$, γ_{1j} for $j = k_1 + 1, \ldots, k_2$. There exists a sequence $(\gamma_{2i})_{i=k_2+1}^{\infty}$ of positive numbers such that $|\lambda_i| < \gamma_{2i}$ implies $\|(P_{n_2} - P_{n_1})(\lambda_{k_2+1} x_{k_2+1} + \lambda_{k_2+2} x_{k_2+2} + \ldots)\|_{p_1} < \frac{1}{2}$. By Lemma 24 we can select a positive integer $n_3 > n_2$ such that

$$\|(P_{n_3}-P_{n_2})(\lambda_1 x_1+\ldots+\lambda_{k_3}x_{k_3})\|_{p_2} < 1 \text{ implies } |\lambda_j| < \beta_j,\ \gamma_{1j},\ \gamma_{2j}$$

for $j = k_2+1, \ldots, k_3$. There exists a sequence $(\gamma_{3i})_{i=k_3+1}^{\infty}$ of positive numbers such that $|\lambda_i| < \gamma_{3i}$ implies $\|(P_{n_3}-P_{n_2})(\lambda_{k_3+1}x_{k_3+1}+\lambda_{k_3+2}\lambda_{k_3+2}+\ldots)\|_{p_2} < \frac{1}{2}$. Continue this process inductively and obtain sequences $(n_k)_{k=1}^{\infty}$ and $(\gamma_{ij})_{j=k_i+1}^{\infty}$, $i = 1, 2, \ldots$ Choose a sequence (r_n) in R such that $r_n = p_{k-1}$ for $n_{k-1} < n < n_k$ $(n_0 = 0)$, $k = 1, 2, \ldots$, and let $\delta = \min(\frac{1}{2}, \frac{1}{2}\varepsilon)$.

To complete the proof, suppose that $x = \sum_{j=1}^{k_m} \lambda_j x_j$ and $\|x\|_{(r_n)} < \delta$. By the choice of (r_n), we have

$$\|P_{n_1}x\|_{p_0}+\|(P_{n_2}-P_{n_1})x\|_{p_1}+\ldots+\|(P_{n_m}-P_{n_{m-1}})x\|_{p_{m-1}} < \delta;$$

hence $\|(P_{n_j}-P_{n_{j-1}})x\|_{p_{j-1}} < \frac{1}{2}$ for $j = 2, \ldots, m$, and $\|P_{n_1}x\|_{p_0} < \frac{1}{2}\varepsilon$. It follows from the above that $|\lambda_j| \leqslant \beta_j$ for $j = 1, 2, \ldots, k_m$.

THEOREM 26. *Let X be an infinite-dimensional subspace of $\bigcup l_p$ which is closed in the q-topology. Then X contains a subspace which is isomorphic to Φ.*

Proof. By Theorem 23 there exists a sequence $\{x_k\}$ in X such that $x_k \in l_{p_k} \setminus l_{p_{k-1}}$. Let Y be the subspace spanned by $\{x_k\}$. Since $\dim(Y \cap l_{p_k}) = k$, Theorem 25 implies that Y is r-isomorphic to Φ. Since the q-topology, stronger than the r-topology, Y, is q-isomorphic to Φ also.

THEOREM 27. *$\bigcup l_p$ contains no infinite-dimensional metrizable subspaces in the q-topology.*

Proof. Suppose X is a metrizable subspace of $\bigcup l_p$. Then, Y, the closure of X, is also metrizable. By Theorem 26, Y contains a copy of Φ. Since Φ is not metrizable, this is contradiction.

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