So by putting together [6], Theorem 1, p. 201 and Lemmas 2.1, 2.2 and 3.1, 3.2, we obtain

THEOREM. *Let G be any compactly generated* LCAG. *Then $L^1(G)$ admits discontinuous translation invariant linear functionals.*

Remarks. (i) In order to extend this result to *all* locally compact abelian groups it would suffice to prove that $\Delta(L^1(G))$ is not closed for all *discrete* groups; alternatively one might try to prove an 'extension' lemma:

Let G_0 be a compact open subgroup of a locally compact abelian group G. Suppose that $\Delta(L^1(G_0))$ is not closed. Then also $\Delta(L^1(G))$ is not closed.

We would conjecture that the above is true, but so far have been unable to prove it.

(ii) It is interesting to note that in the case of compact groups K we have the following corollary:

There is a linear mapping $T: L^1(K)\to L^1(K)$ such that $T(\varepsilon_g * f) = \varepsilon_g * Tf$ $\forall g\in K$, $f\in L^1(K)$ and T is not continuous.

Proof. Let a be any discontinuous translation invariant linear functional, and let f_0 be the constant function $f_0(g) = 1$ $\forall g$. Then T: $L^1(K)\to L^1(K)$, given by $f\mapsto a(f)\cdot f_0$ plainly commutes with translations since $\varepsilon_g * f_0 = f_0$ $\forall g$ and is not continuous.

This is in contrast with the known fact that any linear map from $L^1(\boldsymbol{R})$ into $L^1(\boldsymbol{R})$ which commutes with translations is continuous, viz. [5].

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Power factorization in Banach algebras with a bounded approximate identity

by

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Abstract. Let A be a Banach algebra with a bounded left approximate identity, let X be a left Banach-A-module, let x be in the closure of $A\cdot X$, and let (a_n) be a sequence tending to infinity with $a_n > 1$ for all n. Then there is an a in A and a sequence (y_n) in X such that $x = a^n y_n$ and $\|y_n\| \leqslant a_n^n \|x\|$ for all n. This is used to show that a radical Banach algebra A with bounded approximate identity cannot have $\|x^n\|^{1/n}$ tending to zero uniformly in the unit ball of A.

1. Introduction. In this paper we show that an element x in a Banach algebra A with bounded left approximate identity may be factorized as $x = a^n y_n$ for some $a, y_1, y_2, \ldots$ in A with some control of the growth of the sequence of norms $\|y_1\|, \|y_2\|, \ldots$ Like all factorization results concerning bounded approximate identities (see [2], [6]) the method is an adaption of that of P. J. Cohen [3]. P. C. Curtis and H. Stetkaer [5] have shown that for each x in A and each positive integer n there are a, y in A such that $x = a^n y$. J. K. Miziotek, T. Müldner, and A. Rek [8] have proved that a radical Banach algebra with a bounded approximate identity cannot satisfy a condition that forces the growth of the products $\|x_1 x_2 \ldots \ldots x_n\|^{1/n}$ uniformly to zero for certain sequences (x_n). We strengthen this result by showing that if A is a radical Banach algebra for which there is a positive sequence (a_n) converging to zero such that, for each x in A, $\liminf \|x^n\|^{1/n}/a_n$ is finite, then A does not have a bounded left approximate identity. We obtain this from the factorization $x = a^n y_n$.

If A is a Banach algebra recall that A has a *bounded left approximate identity* [for a Banach-A-module X] *bounded by d* if for each finite subset $\{x_1, \ldots, x_n\}$ of A [of X] and $\varepsilon > 0$, there is an e in A such that $\|e\| \leqslant d$ and $\|x_j - ex_j\| < \varepsilon$ for $j = 1, 2, \ldots, n$. This form of the definition is equivalent to the usual form that there is a net $\{e(\lambda): \lambda\in\Gamma\}$ in A bounded by d such that $x = \lim e(\lambda)x$ for all x in A. The former definition is more convenient for our applications as it simplifies the notation slightly. For a discussion of bounded approximate identities and known results see E. Hewitt and K. A. Ross [6], § 32, or F. F. Bonsall and J. Duncan [2], § 11. A *left Banach-A-module* X is a Banach space X that is a left A-module and for which there is a constant K such that $\|ax\| \leqslant K\|a\|\cdot\|x\|$ for all a in A and x in X. We work with left structures throughout and the results for right Banach-A-modules and bounded approximate right identities may be obtained from these by considering the reversed product on the algebra and module ([2], p. 6).

Our main result is the following theorem.

THEOREM 1. *Let A be a Banach algebra with a bounded left approximate identity bounded by d, and let X be a left Banach-A-module. Let (a_n) be a sequence of real numbers such that $a_n > 1$ for all n and $a_n \to \infty$ as $n \to \infty$, let $\delta > 0$, and let N be a positive integer. If x is in the closed linear span of the set $A\cdot X = \{bz\colon b \in A,\ z \in X\}$, then there is an a in A and $y_1, y_2, \ldots$ in X such that*

(i) $x = a^j y_j$ *for* $j = 1, 2, \ldots$,

(ii) $\|a\| \leqslant d$,

(iii) $y_j \in (A\cdot x)^-$ *for* $j = 1, 2, \ldots$,

(iv) $\|x - y_h\| \leqslant \delta$ *for* $h = 1, \ldots, N$, *and*

(v) $\|y_j\| \leqslant a_j^j \|x\|$ *for* $j = 1, 2, \ldots$

In Section 2 we prove this result (except for (iv)) for a commutative Banach algebra assuming that the factorization $ax = y$ holds for all Banach-A-modules. This proof does not extend to the non-commutative case. In Section 3 we prove Theorem 1 in full, and in Section 4 give an application and an example to show that the growth condition (v) is essentially best possible.

2. Proof of the commutative case. It is known that C^*-algebras [1] and group algebras $L^1(G)$ of a locally compact group G [7] have bounded left approximate identities that are commutative. The following proof will apply to such an algebra B, say, by regarding B as a module over the closed (commutative) subalgebra A generated by a commutative bounded left approximate identity of B. We assume the factorization theorem in the form due to M. A. Rieffel [11] (see [2], [6]) and state it as Lemma 1.

LEMMA 1. *Let A be a Banach algebra with bounded left approximate identity bounded by d, and let Y be a left Banach-A-module. If z is in the closed linear span of $A\cdot Y$ and if $\varepsilon > 0$, then there is an a in A and y in Y satisfying $z = ay$, $\|a\| \leqslant d$, and $\|z - y\| \leqslant \varepsilon$.*

The idea in the commutative proof of Theorem 1 is to write $x = u_1 w_1$, $w_1 = u_2 w_2$, $w_2 = u_3 w_3, \ldots$ using Lemma 1. Then $x = u_1 \ldots u_n w_n$ for all n. We now factorize the sequence (u_n) as $u_n = a v_n$ for some $a, v_1, v_2, \ldots$ in A. This implies that $x = a^n v_1 \ldots v_n w_n$ and choosing $y_n = v_1 \ldots v_n w_n$ gives Theorem 1(i). In the proof below we shall assume that $K = 1$ and that $d < a_n$ for all n to simplify the calculations. We shall not prove (iv) though it can be obtained by tight control of $\|z - y\| < \varepsilon$ in applying Lemma 1.

Proof of Theorem 1 (*when A is commutative*). By normalizing we may assume that $\|x\| = 1$. We choose $\eta > 0$ and a real sequence (γ_n) such that

$$\gamma_n > 1 \quad \text{for all } n, \tag{1}$$

$$(d+\eta)^n \gamma_1 \ldots \gamma_n < a_n^n \quad \text{for all } n, \tag{2}$$

$$(d+\eta)^{-n}(\gamma_1 \ldots \gamma_n)^{-1} a_n^n \to \infty \quad \text{as} \quad n \to \infty. \tag{3}$$

This choice may be made by choosing $\eta > 0$ so that $(d+\eta) < a_n$ for all n using the assumptions $d < a_n$ for all n and $a_n \to \infty$ as $n \to \infty$. Then $\mu = \inf\{a_n^n (d+\eta)^{-n}\colon n > 0\} > 1$. We choose the sequence (γ_n) so that $\gamma_n > 1$ and $\gamma_1 \ldots \gamma_n < \mu$ for all n. We then choose a sequence (β_n) so that

$$(d+\beta_1\eta) \ldots (d+\beta_n\eta)\gamma_1 \ldots \gamma_n < a_n^n \quad \text{for all } n, \tag{4}$$

$$1 \leqslant \beta_n \quad \text{for all } n, \tag{5}$$

$$\beta_n \to \infty \quad \text{as} \quad n \to \infty. \tag{6}$$

This choice is possible by (2) and (3).

Using Lemma 1 applied to A and X repeatedly with suitable z and ε, at each stage we choose sequences (u_n) from A and (w_n) from X such that

$$\begin{aligned} x &= u_1 w_1, & \|u_1\| &\leqslant d, & \|w_1\| &\leqslant \gamma_1, & &\text{and} \\ w_n &= u_n w_{n+1}, & \|u_{n+1}\| &\leqslant d, & \|w_{n+1}\| &\leqslant \gamma_1 \ldots \gamma_{n+1} & &\text{for all } n. \end{aligned} \tag{7}$$

The condition $\|w_{n+1}\| \leqslant \gamma_1 \ldots \gamma_{n+1}$ follows from (1) and $\|w_{n+1}\| \leqslant \|w_n\| + \|w_{n+1} - w_n\|$ for small enough $\|w_{n+1} - w_n\|$.

Let $Y = \{(a_n)\colon a_n \in A,\ \|a_n\|/\beta_n \to 0 \text{ as } n \to \infty\}$, and define a norm $\|\cdot\|$ on Y by $\|(a_n)\| = \sup\{\|a_n\|/\beta_n\colon n \geqslant 1\}$. If we let $a\cdot(a_n) = (a\cdot a_n)$ for all a in A and (a_n) in Y, then Y is a Banach-A-module, and $A\cdot Y$ is dense in Y because $\|a_n\|/\beta_n \to 0$ for all (a_n) in Y and because A has a left approximate identity. Lemma 1 gives us an a in A and (v_n) in Y such that $a(v_n) = (u_n)$, $\|a\| \leqslant d$, and $\|(v_n) - (u_n)\| \leqslant \eta$. Thus $a\cdot v_n = u_n$ and $\|v_n\| \leqslant \|u_n\| + \eta\beta_n \leqslant d + \eta\beta_n$ for all n. Substituting back in our previous equations we obtain $x = u_n \ldots u_1 w_n = a^n v_1 \ldots v_n w_n$ for all n. Further,

$$\|v_1 \ldots v_n w_n\| \leqslant (d+\eta\beta_1) \ldots (d+\eta\beta_n)\gamma_1 \ldots \gamma_n \leqslant a_n^n$$

follows from (4) and (7). The proof is completed by taking $y_n = v_1 \ldots v_n w_n$ for all n.

3. Proof of Theorem 1. References could be given ([2], [6]) for the two lemmas but we include them in order to make this paper self contained, and to have the statements in exactly the right form. In the lemmas A is a Banach algebra with a bounded left approximate identity bounded by d, A_1 is the Banach algebra obtained from A by adjoining an identity 1, Y is a left Banach-A-module, and Z is the closed linear subspace (Banach-A-submodule) of Y spanned by $\{a\cdot y\colon a\in A,\ y\in Y\}$. We shall regard Y as a Banach-A_1-module by defining $(\lambda 1+a)y = \lambda y+ay$ for all scalars λ, all a in A, and y in Y.

LEMMA 2. *The Banach algebra A has a bounded left approximate identity for Z bounded by d.*

Proof. Let $z_1, \ldots, z_n$ be in Z, and let $\varepsilon > 0$. Then there are a_{ij} in A and y_{ij} in Y for $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$ such that

$$\Big\| z_i - \sum_{j=1}^{m} a_{ij} y_{ij}\Big\| < \varepsilon/2(1+Kd)$$

for all i. There is an e in A so that $\|e\|\leqslant d$ and

$$\|a_{ij}-ea_{ij}\| < \varepsilon/2mK(1+\|y_{ij}\|) \quad \text{for all } i, j.$$

From these inequalities we obtain

$$\begin{aligned}\|z_{i_m}-ez_j\| &\leqslant (1+Kd)\Big\| z_i-\sum_{j=1}^{m} a_{ij}y_{ij}\Big\| + K\sum_{j=1}^{m}\|a_{ij}-ea_{ij}\|\|y_{ij}\|\\ &\leqslant \varepsilon \quad \text{for} \quad i = 1,\ldots, m.\end{aligned}$$

This proves Lemma 1.

LEMMA 3. *Let $0<\lambda<(d+1)^{-1}$. If e is in A with $\|e\|\leqslant d$, then $(1-\lambda+\lambda e)^{-1}$ exists in A_1 and $\|(1-\lambda+\lambda e)^{-1}\|\leqslant(1-\lambda-d\lambda)^{-1}$. If $\varepsilon>0$, then there is an $\eta>0$ such that $e\in A$, $\|e\|\leqslant d$, $y\in Y$, $\|y-ey\|<\eta$ implies that $\|y-(1-\lambda+\lambda e)^{-1}y\|<\varepsilon$.*

Proof. Since $1-\lambda$ is positive and

$$\|\lambda(1-\lambda)^{-1}e\|\leqslant\lambda(1-\lambda)^{-1}d<1,$$

the element $\big(1+\lambda(1-\lambda)^{-1}e\big)$ is invertible in A_1. Standard estimates using geometric series imply that

$$\|(1-\lambda+\lambda e)^{-1}\| = (1-\lambda)^{-1}\big\|\big(1+\lambda(1-\lambda)^{-1}e\big)^{-1}\big\|\leqslant(1-\lambda-\lambda d)^{-1}.$$

Thus

$$\begin{aligned}\|y-(1-\lambda+\lambda e)^{-1}y\| &\leqslant K(1-\lambda-\lambda d)^{-1}\|(1-\lambda+\lambda e)y-y\|\\ &\leqslant K\lambda(1-\lambda-\lambda d)^{-1}\|y-ey\|.\end{aligned}$$

Taking $\eta = \varepsilon\cdot K^{-1}\lambda^{-1}(1-\lambda-\lambda d)$ completes the proof.

The idea behind the proof of Theorem 1 is to apply Lemma 3 inductively to construct a sequence (b_n) in A_1 that converges to an element a in A, and such that $(b_n^{-j}x)$ is Cauchy in X for each fixed j even though the sequence $(\|b_n^{-j}\|)$ is unbounded for each fixed j. The control on the growth of the sequence (y_n) is obtained by considering a subsequence $(a_{H(n)})$ of (a_n) that diverges to infinity fast, and doing the construction for the jth powers on the intervals $[H(n), H(n+1)]$.

Proof of Theorem 1. We shall assume that $\|x\| = 1$, and that

$$\delta\leqslant\min\{1,\ a_n^n-1\colon\ n = 1, 2, \ldots\}.$$

We choose and fix a λ satisfying $0<\lambda<(d+1)^{-1}$. We choose $H(0)$ so that $H(0)\geqslant N$ and for all $j\geqslant H(0)$ the inequality

$$a_j > 2K(1-\lambda-\lambda d)^{-1}+1$$

holds. The sequence $\big(H(n)\big)$ of positive integers is now chosen so that $H(n)$ is the maximum of $H(n-1)+1$ and $\inf\{j\colon a_h\geqslant K\cdot 2^n(1-\lambda-\lambda d)^{-n}+1$ for all $h\geqslant j\}$. The choice of the sequence $\big(H(n)\big)$ satisfying these conditions is possible because $a_n>1$ for all n and $a_n\to\infty$ as $n\to\infty$.

We shall inductively define a sequence (e_n) in A and a sequence (b_n) of invertible elements in A_1 such that $b_0 = 1$, $\|e_n\|\leqslant d$,

$$b_n = \sum_{k=1}^{n}\lambda(1-\lambda)^{k-1}e_k+(1-\lambda)^n, \tag{1}$$

and

$$\|b_n^{-j}\cdot x-b_{n-1}^{-j}\cdot x\| < \delta/2^n \tag{2}$$

for all $j\leqslant H(n)$ and all positive integers n. We may choose e_1 and b_1 to satisfy (1) and (2) by applying Lemmas 2 and 3 to X.

Suppose that $e_0, \ldots, e_n$ have been chosen. Let $E\colon \{e\in A\colon \|e\|\leqslant d\}\to A_1$ be defined by

$$E(e) = \sum_{k=1}^{n}\lambda(1-\lambda)^{k-1}(1-\lambda+\lambda e)^{-1}e_k+(1-\lambda)^n.$$

Then

$$E(e)-b_n = \sum_{k=1}^{n}\lambda(1-\lambda)^{k-1}\big((1-\lambda+\lambda e)^{-1}e_k-e_k\big).$$

By Lemma 3 applied with Y equal to the direct sum of n copies of A regarded as a left Banach-A-module in the natural manner, $\|E(e)-b_n\|$ may be made arbitrarily small provided that $\|ee_k-e_k\|$ are sufficiently small for $k = 1, \ldots, n$. Since $\mathrm{Inv}(A_1)$ is open and the mapping $g\mapsto g^{-1}$ is continuous on $\mathrm{Inv}(A_1)$, it follows that $E(e)$ is invertible and $\|E(e)^{-1}-$ $-b_n^{-1}\|$ is arbitrarily small provided that $\|ee_k-e_k\|$ are sufficiently small for $k = 1, \ldots, n$. We now apply Lemma 2, and Lemma 3 with Y equal to the direct sum of n copies of A and $H(n+1)$ copies of Z. We choose e_{n+1} with $\|e_{n+1}\| \leqslant d$ so that $\|e_k - e_{n+1}e_k\|$ $(k = 1, \ldots, n)$ and $\|b_n^{-j}x - e_{n+1}b_n^{-j}x\|$ $(j = 1, \ldots, H(n))$ are so small that $E(e_{n+1})$ is invertible in A_1, $\|E(e_{n+1})^{-1}\| \leqslant \|b_n^{-1}\| + 1$,

$$(3) \qquad \|E(e_{n+1})^{-1} - b_n^{-1}\| \leqslant \eta K^{-1}\big(2^{-n-1}(1-\lambda-\lambda d)^{n+1}\big)^{H(n)},$$

and

$$(4) \qquad \|\{(1-\lambda+\lambda e_{n+1})^{-1} - 1\}b_n^{-j}x\| \leqslant \eta K^{-1}2^{-1}(\|b_n^{-1}\|+1)^{-1}$$

for $j = 1, \ldots, H(n+1)$ where η to be chosen later does not depend on e_{n+1} or b_{n+1}. Then

$$\begin{aligned} b_{n+1} &= (1-\lambda)^{n+1} + \lambda(1-\lambda)^{n+1}e_{n+1} + \sum_{k=1}^{n}\lambda(1-\lambda)^{k-1}e_k \\ &= (1-\lambda+\lambda e_{n+1})E(e_{n+1}) \end{aligned}$$

and so

$$\|b_{n+1}^{-1}\| \leqslant \|(1-\lambda+\lambda e_{n+1})^{-1}\| \cdot \|E(e_{n+1})^{-1}\| \leqslant (1-\lambda-\lambda d)^{-1} \cdot (\|b_n^{-1}\|+1)^{-1}.$$

Because of the term $(1-\lambda)^{-n}$ in b_n^{-1} we have $\|b_n^{-1}\| \geqslant (1-\lambda)^{-n} \geqslant 1$ which gives $\|b_{n+1}^{-1}\| \leqslant 2(1-\lambda-\lambda d)^{-1}\|b_n^{-1}\|$. Repeated use of this formula and $b_0 = 1$ leads to

$$\|b_{n+1}^{-1}\| \leqslant 2^{n+1}(1-\lambda-\lambda d)^{-n-1}.$$

Let $1 \leqslant r \leqslant H(n)$. Then

$$\begin{aligned} \|(b_{n+1}^{-1} - b_n^{-1})b_n^{-r}x\| &= \|\{E(e_{n+1})^{-1}(1-\lambda+\lambda e_{n+1})^{-1} - b_n^{-1}\}b_n^{-r}x\| \\ &\leqslant K\|E(e_{n+1})^{-1}\| \cdot \|\{(1-\lambda+\lambda e_{n+1})^{-1} - 1\}b_n^{-r}x\| + \\ &\quad + K\|E(e_{n+1})^{-1} - b_n^{-1}\| \cdot \|b_n^{-r}\| \leqslant \eta \end{aligned}$$

by (3), (4), and the bound for $\|b_n^{-1}\|$. Hence

$$\begin{aligned} \|b_{n+1}^{-j}x - b_n^{-j}x\| &\leqslant \sum_{r=0}^{j-1} K\|b_{n+1}^{-j+1+r}\| \cdot \|(b_{n+1}^{-1} - b_n^{-1})b_n^{-r}x\| \\ &\leqslant \sum_{r=0}^{j-1} K \cdot \{2^{n+1}(1-\lambda-\lambda d)^{-n-1}\}^{H(n)}\eta \\ &= jK \cdot \big(2^{n+1}(1-\lambda-\lambda d)^{-n-1}\big)^{H(n)}\eta \leqslant \delta 2^{-n-1}, \end{aligned}$$

provided η is small enough, for $j = 1, \ldots, H(n)$. This completes the inductive construction of the sequences (e_n) and (b_n) satisfying (1) and (2).

The sequence (b_n) is Cauchy in A_1, and $\lim b_n = a$ is in A because $\lim b_n = \sum_{1}^{\infty}\lambda(1-\lambda)^{k-1}e_k$. Further $\|a\| \leqslant d$. The sequence $(b_n^{-j}x)$ is Cauchy in n for each j, since $n > m$ and $H(m+1) \geqslant j$ imply that

$$(5) \qquad \|b_n^{-j}x - b_m^{-j}x\| \leqslant \sum_{k=m}^{n-1}\|b_{k+1}^{-j}x - b_k^{-j}x\| \leqslant \delta/2^m$$

by (2). We let $y_j = \lim b_n^{-j}x$ for each j. Then $x = a^j y_j$ for each j. Since x is in $(A \cdot x)^-$, by Lemma 1 we have $(A_1 \cdot x)^- = (A \cdot x)^-$ so that $y_j \in (A \cdot x)^-$ for all j. By (5) we have $\|y_j - b_m^{-j}x\| \leqslant \delta/2^m$ if $H(m+1) \geqslant j$. Hence $\|y_j - x\| \leqslant \delta$ for $j = 1, \ldots, N$ because $H(0) \geqslant N$ and $b_0 = 1$.

If $1 \leqslant j \leqslant H(0)$, then $\|y_j\| \leqslant 1 + \delta$ so that $\|y_j\| \leqslant \alpha_j^j$ by the restriction on δ. Now suppose that $H(m) \leqslant j < H(m+1)$. Then the choice of $H(m)$ implies that $\alpha_j > K \cdot 2^m(1-\lambda-\lambda d)^{-m} + 1$. Also

$$\|y_j\| \leqslant \|b_m^{-j}x\| + \delta \leqslant K\|b_m^{-1}\|^j + 1 \leqslant K \cdot 2^{mj}(1-\lambda-\lambda d)^{-jm} + 1$$

by the bound on $\|b_m^{-1}\|$. Hence

$$\|y_j\| \leqslant \{K2^m(1-\lambda-\lambda d) + 1\}^j \leqslant \alpha_j^j$$

by the choice of $H(m+1) > j$. This completes the proof of Theorem 1.

4. Applications.

COROLLARY 1. *Let A be a Banach algebra with a bounded left approximate identity, and let B be a radical Banach algebra such that there is a sequence (γ_n) of positive real numbers tending to zero so that $\liminf \|b^n\|^{1/n}\gamma_n^{-1}$ is finite for all b in B. Then the zero homomorphism is the only continuous homomorphism from A into B.*

Proof. Let (α_n) be a sequence tending to infinity with $\alpha_n > 1$ for all n, and $\lim \alpha_n\gamma_n = 0$. Suppose that θ is a continuous non-zero homomorphism from A into B, and let x be in A with $\|\theta(x)\| = 1$. Regarding A as a left Banach-A-module we apply Theorem 1. Then there are $a, y_1, y_2, \ldots$ in A such that $x = a^j y_j$ and $\|y_j\| \leqslant \alpha_j^j\|x\|$ for all j. Hence $1 \leqslant \|\theta(a)^j\|\,\|\theta\|\|x\|\alpha_j^j$ for all j so that $1 \leqslant \liminf \|\theta(a)^j\|^{1/j}\gamma_j^{-1} \lim \alpha_n\gamma_n = 0$. This contradiction completes the proof.

Remarks. (i) The growth condition $\|y_j\| \leqslant \alpha_j^j\|x_j\|$ is essentially the best possible. If A is a radical algebra with bounded approximate identity (e.g. $L_1[0, 1]$; see [10], p. 316, [2]), and $x = a^n y_n$ with $\|y_n\| \leqslant \beta_n\|x\|$ for all n, then $\|x\|^{1/n} \leqslant \|a^n\|^{1/n} \cdot \|x\|^{1/n}\beta_n^{1/n}$ implies that $\beta_n^{1/n} \to \infty$ as $n \to \infty$. Taking $\alpha_n = \beta_n^{1/n}$ gives the form of Theorem 1.

(ii) By applying Theorem 1 to a radical Banach algebra A with bounded approximate identity and using working similar to (i) for each sequence $\alpha_n \to \infty$ there is an a in A such that $\|a^n\|^{1/n}\alpha_n \to \infty$. So there are elements with $\|a^n\|^{1/n}$ tending to zero arbitrarily slowly.

(iii) *Let A be a semisimple regular commutative Banach algebra, and let $x, a, y_1, y_2, \ldots$ in A satisfy $x = a^n y_n$ for $n = 1, 2, \ldots$ If $\|y_n\|^{1/n}$ does not tend to infinity as n tends to infinity, then there is an f in A such that $fx = x$.*

For let F be the support of x in the carrier space Φ of A. The remark will follow if we show that the closure F^- of F is compact ([10], Corollary 3.7.3; Theorem 3.6.13).

If ψ is in F, then

$$1 = \liminf |\psi(x)|^{1/n} \leqslant |\psi(a)| \cdot \liminf \|y_n\|^{1/n}.$$

Thus there is a $\delta > 0$ such that $|\psi(a)| \geqslant \delta$ for all ψ in F, and so for all ψ in F^-. Therefore F^- is compact, completing the proof of (iii).

(iv) Corollary 1 and Remark (iii) lead to the following question. If x in a (commutative) Banach algebra A may be written as $x = a^n y_n$ for all n and if $\|y_n\|^{1/n}$ does not tend to infinity is there an f in A such that $fx = x$?

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On the minimum time control problem and continuous families of convex sets

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Abstract. Let a linear system with time variable be given

$$(X \xrightarrow{A} \square \xrightarrow{B_t} Y),$$

where X, $\square$, Y are Banach spaces, A and B_t, $0 \leqslant t \leqslant T$, are continuous linear operators. Let U be a convex closed set in X containing 0 in its interior. Let $\|x\| = \inf\{t > 0: x/t \in U\}$ be the Minkowski norm generated by U. By $\varphi(t)$ we denote

$$\varphi(t) = \inf\{\|x\|: B_t A x \in Y(t)\},$$

where $Y(t)$ is a given continuous family of closed sets.

We prove that if $B_t A(U)$ is a continuous family of sets at t_0 and the set $B_{t_0} A(U)$ has an interior, the $\varphi(t)$ is a continuous function at t_0.

By a *time control linear system* we shall understand a system of three Banach spaces over reals, X being called the *space of input*, $\square$ the *space of trajectories*, and Y the *space of output*, of a continuous linear operator A: $X \to \square$, called the *operator of input*, and of a family of continuous linear operators B_t: $\square \to Y$, t being real, $0 \leqslant t \leqslant T$.

Let U be a convex closed bounded set in X. Let $Y(t)$, $0 \leqslant t \leqslant T$, be a family of sets in Y. In the minimum time control problem we are looking for

$$T_0 = \inf\{t > 0: B_t A(U) \cap Y(t) \neq 0\} \tag{1}$$

and we ask when

$$B_{T_0} A(U) \cap Y(T_0) \neq 0. \tag{2}$$

The problem has been investigated in papers [2], [3], [4], where the respective sufficient conditions for (2) were given. Those conditions were only of the existence type.

In many problems which appear in the theory of control another, more effective, approach to the problem is used.

Namely, we assume that the set U has an interior. Of course, without loss of generality, we may assume that $0 \in \operatorname{Int} U$. Let $\| \;\|$ be the Minkowski