Proof. From Lusin's Theorem ([1], II, p. 95) there exists an absolutely continuous function $F(x)$ such that

(3.1) *the Fourier series of its derivative $f(x)$ and its conjugate converge almost everywhere,*

(3.2) *the function $\tilde{F}(x)$, conjugate to $F(x)$, is essentially unbounded in any interval $[a, b] \subset [0, 2\pi]$.*

So from (3.1) we have $f \in L$ and $\tilde{S}_n(f, x) = O(1)$ for a.e. x.

Suppose for this f that there exists an E satisfying (i) and (ii) in the theorem, i.e., for a.e. x, $\sum_{n=1}^{\infty} S_n(f, x) a_n(\varphi)$ converges for all $\varphi \in AC$.

Then similar to Corollary 1.1 we would get that for a.e. x,

$$\Big|\sum_{n=0}^{\infty} a_{2n+1}(\varphi) S_n(f, x)\Big| < \infty \quad \text{for all } \varphi \in AC. \tag{3.3}$$

Hence from Lemma 3.1, for a.e. x, $\exists M_x$ such that

$$\Big|\sum_{n=0}^{\infty} a_{2n+1}(\varphi) S_n(f, x)\Big| \leqslant M_x \|\varphi\|_{AC} \quad \text{for all } \varphi \in AC. \tag{3.4}$$

Hence from [4], p. 133, Th. 3, $\sum_{n=1}^{\infty}{}' \frac{1}{n} A_n(f, x) \sim -\tilde{F}(x)$ is equivalent to a function differentiable a.e. which contradicts (3.2). ■

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Best order conditions in linear spaces, with applications to limitation, inclusion, and high indices theorems for ordinary and absolute Riesz means

by

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Abstract. It is the first purpose of this paper to obtain simple order conditions which hold for certain sequences of continuous linear functionals on a Fréchet space with a Schauder basis, and to investigate best-possible order conditions. We then specialize the results to Banach spaces and to summability fields of matrices. By using results on summability fields and absolute summability fields of Riesz typical means and generalized Cesàro means, some of them new, we are able to obtain some best-possible order conditions in these fields; in particular, we can specify the best-possible limitation theorems for a sequence (or series) which is limitable, or absolutely limitable, by the Riesz method. We then apply our limitation theorems to obtain two equivalence theorems, of 'high-indices' type, for ordinary and absolute Riesz summability. Finally, we can obtain improved forms of two inclusion theorems which specify necessary and sufficient conditions for an arbitrary matrix method to include the ordinary or absolute Riesz methods.

1. Introduction. A locally convex linear topological (Hausdorff) space (over the complex number field) which is complete and metrizable has a topology generated by a countable set of seminorms $p = \{p_j\}$, and such a space (X, p) is a Fréchet space (F-*space*). We may assume without loss of generality that no seminorm p_j is identically zero. An F-space with a norm topology, $(X, \|\cdot\|_X)$, is a Banach space (B-*space*). A *sequence space* is a vector subspace of ω, the space of all complex-valued sequences. An FK-*space* (X, p) is a (locally convex) Fréchet sequence space for which the coordinate functionals (i.e., the maps $P_n(x) = x_n$, $n = 0, 1, \ldots$) are continuous; an FK-space has a unique FK-topology. A BK-*space* is a Banach sequence space with continuous coordinates. Examples of BK-spaces are the spaces m, c, c^0 of bounded, convergent, null sequences, respectively, all with

$$\|x\| = \sup_k |x_k|; \qquad l = \Big\{x\colon\ \|x\| = \sum_{k=0}^{\infty} |x_k| < \infty\Big\};$$

$$v = \Big\{x\colon\ \|x\| = \lim_k |x_k| + \sum_{k=0}^{\infty} |x_k - x_{k+1}| < \infty\Big\}; \qquad v^0 = c^0 \cap v.$$

A countable collection of points, $\{a^k\}$, of an F-space (X, p), is a (*Schauder*) *basis* for (X, p) if there are unique functionals f_k $(k = 0, 1, \ldots)$

such that $x = \sum_{k=0}^{\infty} f_k(x) a^k$ for each $x \in X$ (in the topology of X). If an FK-space has the basis $\{e^k\}$, where $e^k = (0, \ldots, 0, 1, 0, 0, \ldots)$ (with 1 in rank k), then necessarily $f_k(x) = x_k$ and (X, p) is then called an FK-AK *space*. Examples of BK-AK spaces are c^0, l, v^0. The spaces c, v have $e \cup \{e^k\}$ as basis, $e = (1, 1, 1, \ldots)$. Throughout this paper, "basis" always means "Schauder basis".

For an infinite matrix $A \equiv (a_{nk})_{n,k\geq 0}$ and an infinite sequence $x \equiv \{x_k\}_{k\geq 0}$, of complex numbers, denote $Ax \equiv \{(Ax)_n\}_{n\geq 0}$, $(Ax)_n \equiv \sum_{k=0}^{\infty} a_{nk} x_k$; the existence of Ax requires the convergence of this last series for each $n \geq 0$. If F is a sequence space, we write $F_A \equiv \{x\colon Ax \in F\}$; thus ω_A is the existence domain of the matrix A. The matrix A is called F-*reversible* when, for each $y \in F$, $y = Ax$ has a unique solution $x \in \omega$; a *normal* matrix A ($a_{nk} = 0$, $k > n$; $a_{nn} \neq 0$) is ω-reversible. A normal matrix A is said to have the *mean-value property* $M_K(A)$, $0 \leq K < \infty$, when

$$(1) \quad \forall x \in c_A^0, \qquad \Big|\sum_{k=0}^{m} a_{nk} x_k\Big| \leq K \cdot \max_{0 \leq \mu \leq m} \Big|\sum_{k=0}^{\mu} a_{\mu k} x_k\Big| \qquad (m, n = 0, 1, \ldots).$$

For a normal matrix with zero column-limits ($\{e^k\} \subset c_A^0$), (1) is equivalent to the property that c_A^0 has AK (see Wilansky and Zeller [29], §§ 3 and 4). We denote the unit matrix by I, and write $J = (j_{nk})$, $j_{nk} = 1$ $(0 \leq k \leq n)$, $j_{nk} = 0$ $(k > n)$, with inverse $J^{-1} \equiv (j_{nk}^{-1})$, $j_{nn}^{-1} = 1$, $j_{n,n-1}^{-1} = -1$, $j_{nk}^{-1} = 0$ otherwise; thus a series $\sum a_n$, $a = \{a_n\}$, is related to its sequence of partial sums $x = \{x_n\}$ by $x = Ja$, $a = J^{-1}x$.

2. General results.

THEOREM 1. *Let (X, p) be an F-space with Schauder basis $\{a^k\}_{k\geq 0}$, with the representation*

$$(2) \qquad x = \sum_{k=0}^{\infty} f_k(x) a^k \quad \text{for each } x \in X.$$

Let B be an infinite matrix of complex numbers whose only non-zero elements occur on a fixed finite number of diagonals, namely $b_{nk} = 0$ for $k < \max(0, n+r)$ and for $k > \max(0, n+s)$, $r \leq s$ (r, s fixed); for each $x \in X$ and $n = 0, 1, \ldots$, write

$$(3) \qquad u_n(x) = \sum_k b_{nk} f_k(x).$$

Then for each $j \geq 0$ and each $x \in X$:

$$(4) \qquad \lim_{k\to\infty} p_j(a^k) f_k(x) = 0,$$

$$(5) \qquad u_n(x) = o\Big\{\max_{n+r\leq k\leq n+s} \big(|b_{nk}|/p_j(a^k)\big)\Big\} \quad \text{as} \quad n \to \infty.$$

Proof. Since $\sum f_k(x) a^k$ is a convergent series (in the topology $p = \{p_j\}$), it follows that its terms tend to zero in this topology, which gives (4). Now, by hypothesis, since no seminorm is identically zero, we have, for each fixed j and for all sufficiently large n, $n + r \geq 0$ and $p_j(a^k) \neq 0$ for $k \geq n + r$. Thus, by (3) and (4),

$$\begin{aligned} |u_n(x)| &= \Big|\sum_{k=n+r}^{n+s} b_{nk} f_k(x)\Big| \\ &\leq \Big\{\max_{n+r\leq k\leq n+s} \big(|b_{nk}|/p_j(a^k)\big)\Big\} \sum_{k=n+r}^{n+s} p_j(a^k)|f_k(x)| \\ &= o\Big\{\max_{n+r\leq k\leq n+s} \big(|b_{nk}|/p_j(a^k)\big)\Big\} \quad \text{as} \quad n\to\infty. \end{aligned}$$

THEOREM 2. *Let (X, p) be an F-space, and $a^k \in X$, $a^k \neq 0$ $(k = 0, 1, \ldots)$. Let B be an infinite matrix with no zero row, satisfying the conditions in Theorem 1, and let $\{\theta_n\} \in \omega$, $\theta_n \neq O(1)$. Then $\exists v^* \in \omega$ and $\exists x^* \in X$ such that $x^* = \sum_{k=0}^{\infty} v_k^* a^k$ (with convergence in the topology of (X, p)), and*

$$(6) \qquad \theta_n (Bv^*)_n \neq o\Big\{\max_{n+r\leq k\leq n+s} \big(|b_{nk}|/P_k\big)\Big\} \quad \text{as} \quad n\to\infty,$$

where

$$P_k \equiv \max_{0\leq j\leq k^*} p_j(a^k), \qquad k^* = \max\big(k, \min\{j\colon p_j(a^k) \neq 0\}\big).$$

Proof. From $\{\theta_n\}$ choose a subsequence $\{\theta_{n_i}\}_{i\geq 0}$ of non-zero terms, such that

$$(7) \qquad n_{i+1} + r > n_i + s \geq 0 \qquad (i = 0, 1, \ldots),$$

$$(8) \qquad \sum_{i=0}^{\infty} |\theta_{n_i}|^{-1} < \infty;$$

then choose k_i in $n_i + r \leq k_i \leq n_i + s$ such that

$$(9) \qquad \max_{n_i+r\leq k\leq n_i+s} \big(|b_{n_i,k}|/P_k\big) = |b_{n_i,k_i}|/P_{k_i} \qquad (i = 0, 1, \ldots).$$

Define $v_k^* = (|\theta_{n_i}| P_{k_i})^{-1}$ if $k = k_i$ $(i = 0, 1, \ldots)$ and $v_k^* = 0$ otherwise. For a fixed j it then follows from the definitions of v^* and P_k, and using (8), that for all sufficiently large integers l, m,

$$\begin{aligned} p_j\Big(\sum_{k=l}^{m} v_k^* a^k\Big) &\leq \sum_{k=l}^{m} v_k^* p_j(a^k) = \sum_{l\leq k_i\leq m} v_{k_i}^* p_j(a^{k_i}) \\ &= \sum_{l\leq k_i\leq m} |\theta_{n_i}|^{-1} P_{k_i}^{-1} p_j(a^{k_i}) \leq \sum_{l\leq k_i\leq m} |\theta_{n_i}|^{-1} \to 0 \end{aligned}$$

as $m > l \to \infty$. Thus the series $\sum_k v_k^* a^k$ is Cauchy, and hence convergent to a value $x^* \in X$. Now, by (7) and the definition of v^*,

$$(Bv^*)_{n_i} = \sum_{k=n_i+r}^{n_i+s} b_{n_i,k} v_k^* = b_{n_i,k_i} v_{k_i}^* = b_{n_i,k_i} |\theta_{n_i}|^{-1} P_{k_i}^{-1}.$$

Thus, by (9),

$$|\theta_{n_i}(Bv^*)_{n_i}| = \max_{n_i+r \leq k \leq n_i+s} (|b_{n_i,k}|/P_k) \quad (i = 0, 1, \ldots),$$

and (6) follows.

Combining Theorems 1 and 2, and specializing to a Banach space with a basis, it follows in the notation of these theorems that $v_k^* = f_k(x^*)$, and we get:

THEOREM 3. *Let X be a B-space with Schauder basis $\{a^k\}_{k \geq 0}$ and representation* (2). *Let B be an infinite matrix satisfying the conditions of Theorem* 2, *with $u_n(x)$ defined as in* (3). *Then for each $x \in X$ we have*

$$(10) \qquad u_n(x) = o\{\max_{n+r \leq k \leq n+s} (|b_{nk}| \cdot \|a^k\|^{-1})\} \quad \text{as} \quad n \to \infty,$$

and (10) *is best-possible, in the sense that given any unbounded complex sequence $\{\theta_n\}$, there is an element $x^* \in X$ such that*

$$(11) \qquad \theta_n u_n(x^*) \neq o\{\max_{n+r \leq k \leq n+s} (|b_{nk}| \cdot \|a^k\|^{-1})\} \quad \text{as} \quad n \to \infty.$$

Remark 1. We can apply the above theorems to sequence spaces containing $\{e^k\}$, with continuous coordinates. It then follows, in Theorems 1 and 3, that $f_k(x) = x_k$ and so $u_n(x) = (Bx)_n$. While in Theorem 2 we get $v_k^* = x_k^*$ and so $Bv^* = Bx^*$. The simplest choice of B is of course $B = I$, but if, for example, we choose $B = J^{-1}$ we obtain limitation theorems for $x_n - x_{n-1}$; that is, for the terms of a series $\sum a_n$, with partial sums $\{x_n\}$.

COROLLARY 1. *Let $(X, \|\cdot\|)$ be a BK-space containing $\{e^k\}$.*

(a) *Given any $\{\theta_n\} \in \omega$, $\theta_n \neq O(1)$, $\exists x' = \sum_{k=0}^{\infty} x_k' e^k \in X$, $\exists x'' = \sum_{k=0}^{\infty} x_k'' e^k \in X$ such that*

$$\theta_n x_n' \neq o(\|e^n\|^{-1}), \qquad \theta_n(x_n'' - x_{n-1}'') \neq o\{\max(\|e^n\|^{-1}, \|e^{n-1}\|^{-1})\}.$$

(b) *If $\{e^k\}$ is a basis for X, then*

$$\forall x \in X, \ x_n = o(\|e^n\|^{-1}) \quad \text{and} \quad x_n - x_{n-1} = o\{\max(\|e^n\|^{-1}, \|e^{n-1}\|^{-1})\}.$$

Proof. Use Remark 1 and substitute the two choices of B mentioned there into Theorems 1 and 2.

The application to matrix fields depends on the following lemma.

LEMMA 1. (a) *Let (F, p) be an FK-space and A an infinite matrix. Then F_A is an FK-space with the seminorms*

$$(12) \qquad \begin{gathered} p_j(Ax) \ (j = 0, 1, \ldots); \quad |x_i| \ (i = 0, 1, \ldots); \\ \sup_{m \geq 0} \Big| \sum_{k=0}^{m} a_{nk} x_k \Big| \ (n = 0, 1, \ldots). \end{gathered}$$

If A is row-finite, the last set of seminorms can be discarded. If A is F-reversible, the last two sets can be discarded (leaving only $\{p_j(Ax)\}$).

(b) *If F is a BK-space and A is F-reversible, then F_A is a BK-space and $\|x\|_{F_A} = \|Ax\|_F$.*

(c) *If (F, p) is an FK-space with basis $\{a^k\}$ and representation* (2), *and if the matrix T is F-reversible, then the sequence $\{d^k\}$, given by $Td^k = a^k$ $(k \geq 0)$, is a basis for F_T, and $\forall x \in F_T$, $x = \sum_{k=0}^{\infty} f_k(Tx) d^k$.*

Proof. Part (a) is due to Zeller ([30], Satz 4.10 — see also Wilansky [27], § 12.4); and (b) is an obvious deduction from (a). Part (c) is given in Jakimovski and Livne [7], Theorem 2.7 (it is stated there for $a^k = e^k$, but the proof is the same).

COROLLARY 2. *Let F be a BK-space, A an F-reversible matrix, and $\{e^k\} \subset F_A$.*

(a) *Let $\theta_n \neq O(1)$. Then $\exists x^* \in F_A$ such that $\theta_n x_n^* \neq o(\|Ae^n\|_F^{-1})$.*

(b) *If F_A is AK, then $\forall x \in F_A$, $x_n = o(\|Ae^n\|_F^{-1})$.*

Proof. This follows from Corollary 1 (with $X = F_A$) and Lemma 1(b).

Remark 2. To shorten the exposition, the phrase "$\forall x \in X$, $u_n(x) = o(\varrho_n)$ is a best-possible limitation theorem" will mean that $\forall x \in X$, $u_n(x) = o(\varrho_n)$ as $n \to \infty$, and that, given any unbounded complex sequence $\{\theta_n\}$, $\exists x^* \in X$ such that $\theta_n u_n(x^*) \neq o(\varrho_n)$.

COROLLARY 3. (a) *Let A be a c^0-reversible matrix and c_A^0 be AK. Then we have the best-possible limitation theorem: $\forall x \in c_A^0$, $x_n = o(1/\sup_m |a_{mn}|)$.*

(b) *If A is normal, with the mean-value property $M_K(A)$, and $\lim_m a_{mn} = 0$ $(n = 0, 1, \ldots)$, then $|a_{nn}| \leq \sup_m |a_{mn}| \leq K|a_{nn}|$, and so the best limitation theorem is: $\forall x \in c_A^0$, $x_n = o(1/|a_{nn}|)$.*

Proof. (a) Put $F = c^0$ in Corollary 2. (b) The hypotheses ensure that c_A^0 is AK (see the remarks following (1) above).

Corollary 3 (b) is given in Wilansky and Zeller [29], p. 263; see also Jurkat and Peyerimhoff [12], Sätze 4, 5, and Wilansky [28], Theorem 11.1.

3. Application to Riesz means: definitions and lemmas. We need first the definitions of the Riesz summability method $(R, \lambda, \varkappa)$ and the (discrete) generalized Cesàro method $(C^*, \lambda, \varkappa)$ (see Bosanquet and Russell [4]). Here, and throughout, $\lambda = \{\lambda_n\}$ is any fixed sequence satisfying $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$ (we take $\lambda_0 = 0$ without loss of generality) and $\varkappa \geqslant 0$; when $\varkappa > 0$ we write $\varkappa = p + \eta$ $(p = 0, 1, \ldots;\ 0 < \eta \leqslant 1)$. For real sequences, we shall also use the notation $a_n \asymp b_n$ to mean that there are finite positive constants k_1, k_2 such that ultimately $k_1 b_n \leqslant a_n \leqslant k_2 b_n$.

For any $x = \{x_n\} \in \omega$ write, for $n = 0, 1, \ldots,$

$$C_n^0 \equiv x_n, \quad C_n^p \equiv \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \ldots (\lambda_{n+p} - \lambda_\nu)(x_\nu - x_{\nu-1}) \qquad (x_{-1} = 0,\ p = 1, 2, \ldots),$$

$$E_n^0 \equiv 1, \quad E_n^p \equiv \lambda_{n+1} \ldots \lambda_{n+p} \quad (p = 1, 2, \ldots);$$

$$h_{n\nu} \equiv h_{n\nu}^{p\eta} \equiv (-1)^{p+1}(\lambda_{\nu+p+1} - \lambda_\nu) f_n[\lambda_\nu \ldots \lambda_{\nu+p+1}] \quad (0 \leqslant \nu \leqslant n),$$

where $f_n[\lambda_\nu \ldots \lambda_{\nu+p+1}]$ is a divided difference of the function

$$f_n(u) \equiv (\lambda_{n+p+1} - u)^{p+\eta} \quad (u \leqslant \lambda_{n+p+1})$$

(for definitions and basic properties of divided differences, see [18], Chapter 1, [22], § 4, or [4], § 2). Also write

$$C_n^{p+\eta} \equiv \sum_{\nu=0}^{n} h_{n\nu} C_\nu^p, \quad E_n^{p+\eta} \equiv \sum_{\nu=0}^{n} h_{n\nu} E_\nu^p \quad (p = 0, 1, \ldots;\ 0 < \eta \leqslant 1);$$

thus $E_n^\varkappa$ is the value of $C_n^\varkappa$ for the sequence $x = e = (1, 1, 1, \ldots)$. We then define the $(C^*, \lambda, \varkappa)$-means of x as

$$t_n^\varkappa \equiv t_n^\varkappa(x) \equiv C_n^\varkappa / E_n^\varkappa. \tag{13}$$

Thus if $T^\varkappa$ is the (sequence to sequence) matrix of the $(C^*, \lambda, \varkappa)$-transform, and if the matrix $A^{p\eta} \equiv (a_{n\nu}^{p\eta})$ is defined by

$$a_{n\nu}^{p\eta} \equiv h_{n\nu}^{p\eta} E_\nu^p / E_n^{p+\eta} \quad (0 \leqslant \nu \leqslant n), \quad p = 0, 1, \ldots, \ 0 < \eta \leqslant 1, \qquad A^{p0} \equiv I \ \text{(unit matrix)}, \quad p = 0, 1, \ldots; \tag{14}$$

then $t_n^{p+\eta} = \sum_{\nu=0}^{n} a_{n\nu}^{p\eta} t_\nu^p$, and so we have the matrix product $T^{p+\eta} = A^{p\eta} T^p$, valid for $p = 0, 1, \ldots$ and $0 \leqslant \eta \leqslant 1$ (see Bosanquet and Russell [4], § 3). It is proved in [4] (see § 3 and (46)) that $A^{p\eta}$ is a normal, regular $c \to c$ matrix possessing $M_1(A)$, namely the mean-value property (1) with constant $K = 1$. Also it is an easy deduction from [4], inequalities (24), (25), (37) that

$$\frac{\eta}{p+1} \Lambda_{np}^{-\eta} \leqslant a_{nn}^{p\eta} \leqslant \frac{p+1}{\eta} \Lambda_{np}^{-\eta}, \quad \Lambda_{np} \equiv \lambda_{n+p+1} / (\lambda_{n+p+1} - \lambda_n). \tag{15}$$

If $\overline{T}^\varkappa \equiv T^\varkappa J$ and $x \equiv Ja$, then $\overline{T}^\varkappa a = T^\varkappa x$, so that $\overline{T}^\varkappa$ is the series-to-sequence $(C^*, \lambda, \varkappa)$ matrix. Writing $(\overline{T}^p)^{-1} \equiv (\bar{\tau}_{nk}^p)$ we have (see [26], pp. 297–298)

$$\bar{\tau}_{nk}^p = (-1)^{p+1}(\lambda_{k+p+1} - \lambda_k)\lambda_{k+1} \ldots \lambda_{k+p} / \beta_{nk}^p \quad (0 \leqslant k \leqslant n \leqslant k+p+1), \qquad \bar{\tau}_{nk}^p = 0 \ \text{otherwise}, \tag{16}$$

where

$$\beta_{nk}^p \equiv \prod_{k \leqslant \nu \leqslant k+p+1, \nu \neq n} (\lambda_n - \lambda_\nu), \quad p = 1, 2, \ldots;$$

another way of expressing β_{nk}^p is $\beta_{nk}^p = w_k'(\lambda_n)$, $w_k(x) = (x - \lambda_k) \ldots (x - \lambda_{k+p+1})$. The inverse of T^p is given by $(T^p)^{-1} \equiv (\tau_{nk}^p) = J(\overline{T}^p)^{-1}$; thus

$$\tau_{nk}^p = \sum_{i=k}^{n} \bar{\tau}_{ik}^p \quad (0 \leqslant k \leqslant n \leqslant k+p), \qquad \tau_{nk}^p = 0 \ \text{otherwise}, \quad p = 1, 2, \ldots \tag{17}$$

For $p = 0$ we have $(\tau_{nk}^0) = I$, $(\bar{\tau}_{nk}^0) = J^{-1}$; and, for $p = 1$,

$$\tau_{nn}^1 = \Lambda_n, \quad \tau_{n,n-1}^1 = 1 - \Lambda_n, \quad \tau_{nk}^1 = 0 \ \text{otherwise}, \qquad \Lambda_n = \Lambda_{n0} = \lambda_{n+1} / (\lambda_{n+1} - \lambda_n).$$

The Riesz $(R, \lambda, \varkappa)$-means of x are defined, as usual, by

$$R^\varkappa(\omega) \equiv R^\varkappa(x, \omega) \equiv \sum_{\lambda_\nu < \omega} (1 - \lambda_\nu/\omega)^\varkappa (x_\nu - x_{\nu-1}) \quad (\omega > 0), \quad R^\varkappa(0) \equiv x_0, \tag{18}$$

where $x_{-1} = 0$. The $(R^*, \lambda, \varkappa)$-means of x are $R_n^\varkappa = R^\varkappa(x, \lambda_{n+1})$.

We now define the sequence spaces:

$$R_{\lambda\varkappa} = \{x\colon \exists \lim_{\omega \to \infty} R^\varkappa(x, \omega)\}, \qquad R_{\lambda\varkappa}^0 = \{x\colon \lim_{\omega \to \infty} R^\varkappa(x, \omega) = 0\},$$

$$R_{|\lambda\varkappa|} = \Big\{x\colon \int_0^\infty |dR^\varkappa(x, \omega)| < \infty\Big\}, \qquad R_{|\lambda\varkappa|}^0 = R_{|\lambda\varkappa|} \cap R_{\lambda\varkappa}^0,$$

$$R_{\lambda\varkappa}^* = \{x\colon \exists \lim_{n \to \infty} R^\varkappa(x, \lambda_{n+1})\}, \qquad R_{|\lambda\varkappa|}^* = \Big\{x\colon \sum_{n=0}^{\infty} |R^\varkappa(x, \lambda_{n+1}) - R^\varkappa(x, \lambda_{n+2})| < \infty\Big\};$$

$$C_{\lambda\varkappa}^* = c_{T^\varkappa} = \{x\colon \exists \lim_{n \to \infty} t_n^\varkappa(x)\}, \qquad C_{\lambda\varkappa}^{*0} = c_{T^\varkappa}^{0} = \{x\colon \lim_{n \to \infty} t_n^\varkappa(x) = 0\},$$

$$C_{|\lambda\varkappa|}^* = v_{T^\varkappa} = \Big\{x\colon \sum_{n=0}^{\infty} |t_n^\varkappa(x) - t_{n+1}^\varkappa(x)| < \infty\Big\}, \qquad C_{|\lambda\varkappa|}^{*0} = v_{T^\varkappa}^0 = C_{|\lambda\varkappa|}^* \cap C_{\lambda\varkappa}^{*0}.$$

Clearly,

$$R^0_{|\lambda\varkappa|} \subseteq R_{|\lambda\varkappa|} \subseteq R^*_{|\lambda\varkappa|}, \qquad R_{|\lambda\varkappa|} \subseteq R_{\lambda\varkappa} \subseteq R^*_{\lambda\varkappa}, \qquad C^{*0}_{|\lambda\varkappa|} \subseteq C^*_{|\lambda\varkappa|} \subseteq C^*_{\lambda\varkappa}.$$

Also it is well known that

(19) $$c \subseteq R_{\lambda\varkappa} \subseteq R_{\lambda\mu} \quad \text{and} \quad v \subseteq R_{|\lambda\varkappa|} \subseteq R_{|\lambda\mu|}, \quad \text{for} \quad 0 \leqslant \varkappa \leqslant \mu.$$

LEMMA 2. (a) $R_{\lambda\varkappa}$, $R^0_{\lambda\varkappa}$ *are* BK-*spaces with norm* $\|x\|_{\lambda\varkappa} = \sup_{\omega\geqslant 0}|R^\varkappa(x,\omega)|$; $R_{|\lambda\varkappa|}$, $R^0_{|\lambda\varkappa|}$ *are* BK-*spaces with norm*

$$\|x\|_{|\lambda\varkappa|} = \lim_{\omega\to\infty}|R^\varkappa(x,\omega)| + \int_0^\infty |dR^\varkappa(x,\omega)|.$$

(b) $C^*_{\lambda\varkappa}$, $C^{*0}_{\lambda\varkappa}$ *are* BK-*spaces with norm* $\|x\|'_{\lambda\varkappa} = \sup_{n\geqslant 0}|t^\varkappa_n(x)|$; $C^*_{|\lambda\varkappa|}$, $C^{*0}_{|\lambda\varkappa|}$ *are* BK-*spaces with norm*

$$\|x\|'_{|\lambda\varkappa|} = \lim_{n\to\infty}|t^\varkappa_n(x)| + \sum_{n=0}^\infty |t^\varkappa_n(x) - t^\varkappa_{n+1}(x)|.$$

Proof. (a) The result for $R_{\lambda\varkappa}$ is given in Peyerimhoff [19], §8; $R^0_{\lambda\varkappa}$ follows immediately, and we can follow through a similar argument to establish the results for $R_{|\lambda\varkappa|}$ and $R^0_{|\lambda\varkappa|}$. (b) Since $T^\varkappa$ is a normal matrix, the results for the C^* spaces follow at once from Lemma 1(b). There are obvious analogues for the R^* spaces, which also follow from Lemma 1(b).

LEMMA 3. (a) $R_{\lambda\varkappa} = C^*_{\lambda\varkappa}$ $(\varkappa \geqslant 0)$;
(b) $R_{|\lambda p|} = C^*_{|\lambda p|}$ $(p = 0, 1, 2, \ldots)$.

Proof. (a) is due to Russell [22] and Meir [17] when $\varkappa$ is an integer and to Bosanquet and Russell [4], Theorem 2, in the general case. (b) is due to Körle [15].

Remark 3. It is not at present known whether a definition of $|C^*, \lambda, \varkappa|$ summability can be formulated, which will be equivalent to $|R, \lambda, \varkappa|$ summability for every $\varkappa \geqslant 0$ and every $\lambda_n \nearrow \infty$. When $\varkappa = \eta$, $0 < \eta \leqslant 1$, then $C^*_{|\lambda\eta|} \equiv R^*_{|\lambda\eta|}$, but even the equivalence $R_{|\lambda\eta|} = R^*_{|\lambda\eta|}$ $(0 < \eta < 1)$ is apparently known only for restricted λ (see Körle [13], Satz 3).

LEMMA 4. *For* $p = 0, 1, \ldots$, $0 \leqslant \eta_1 < \eta \leqslant 1$, $D \equiv A^{p\eta}(A^{p\eta_1})^{-1}$ *is a normal, regular* $c\to c$ *matrix with the mean-value property* $M_1(D)$.

Proof. Note that, from (14), for $0 < \eta_1 < \eta \leqslant 1$,

(20) $$a^{p\eta}_{n\nu}/a^{p\eta_1}_{n\nu} = h^{p\eta}_{n\nu}/h^{p\eta_1}_{n\nu} \qquad (0 \leqslant \nu \leqslant n);$$

we can then employ a method of proof of Bosanquet and Russell [4] to show that the ratios in (20) decrease when $0 \leqslant \nu \nearrow \leqslant n$ and $0 < \eta_1 < \eta \leqslant 1$. (We use the two functions $f(x) = (\omega - x)^{p+\eta}$, $f_1(y) = (\omega - y)^{p+\eta_1}$, verify that

$$\left(\frac{\partial}{\partial x}\right)^{p+2}\left(\frac{\partial}{\partial y}\right)^{p+2}\{f(x)(x-y)f_1(y)\} \geqslant c(\omega - x)^{\eta-2}(\omega - y)^{\eta_1-2}(y-x),$$

where $c > 0$, and follow through the proof as in [4], Lemma 2.)

Now substitute $A = A^{p\eta_1}$, $B = A^{p\eta}$ in Peyerimhoff [20], Theorem II.21, from which it follows that $D \equiv BA^{-1}$ is regular (hence $c_A \subseteq c_B$), and that D has a normal inverse D^{-1} whose only positive elements are on the leading diagonal. This is enough to ensure that $M_1(D)$ holds ([29], p. 261; or see [20], Theorem II.16). For the case $\eta_1 = 0$, the definition $A^{p0} = I$ gives $D = A^{p\eta}$, which we already know to have the required properties.

LEMMA 5. *Let* $\varkappa = p + \eta$, *where* p *is a non-negative integer and* $0 < \eta \leqslant 1$, *and let* δ^{pj} *be defined by* $T^p\delta^{pj} = e^j$ $(j = 0, 1, \ldots)$. *If* X *denotes either one of the spaces* $R^0_{\lambda\varkappa}$ *or* $R^0_{|\lambda\varkappa|}$, *then* $\{\delta^{pj}\}_{j\geqslant 0}$ *is a basis for* X, *and we have the representation* $x = \sum_{j=0}^\infty t^p_j(x)\delta^{pj}$ $(\forall x \in X)$, *with convergence in the norm of* X, *where* $t^p_j(x)$ *are the* (C^*, λ, p)-*means of* x.

Proof. These results are due to Jakimovski and Tzimbalario; for $R^0_{\lambda\varkappa}$ see [9], Theorem 5, and for $R^0_{|\lambda\varkappa|}$ see [10], Theorem 5.

Remark 4. In the case $\varkappa = \eta$, $0 < \eta \leqslant 1$ (that is, for $p = 0$), we have $\delta^{0j} = e^j$, and Lemma 5 shows that $R^0_{\lambda\eta}$ and $R^0_{|\lambda\eta|}$ are then AK-spaces. This result for $R^0_{\lambda\eta}$ is due to Peyerimhoff [19], Satz 8.2.

LEMMA 5A. *Let* $\varkappa = p + \eta$, $\varkappa_1 = p + \eta_1$, *where* p *is a non-negative integer, and* $0 \leqslant \eta_1 < \eta \leqslant 1$, *and let* $\delta^{\varkappa_1 j}$ *be defined by* $T^{\varkappa_1}\delta^{\varkappa_1 j} = e^j$ $(j = 0, 1, \ldots)$. *Then* $\{\delta^{\varkappa_1 j}\}_{j\geqslant 0}$ *is a basis for* $R^0_{\lambda\varkappa}$, *and*

(21) $$\forall x \in R^0_{\lambda\varkappa}, \quad x = \sum_{j=0}^\infty t^{\varkappa_1}_j(x)\delta^{\varkappa_1 j}.$$

Proof. For $0 \leqslant \eta_1 < \eta \leqslant 1$ the matrix $D = A^{p\eta}(A^{p\eta_1})^{-1}$ is, by Lemma 4, normal and regular and possesses $M_1(D)$; hence c^0_D is AK and so $x = \sum_{j=0}^\infty x_j e^j$ $(x \in c^0_D)$. In the notation of Lemma 1(c), take $F = c^0_D$, $a^j = e^j$, $T = T^{\varkappa_1}$; then $F_T = (c^0_D)_{T^{\varkappa_1}} = c^0_{T^\varkappa}$, $d^j = \delta^{\varkappa_1 j}$, and $f_j(Tx) = (T^{\varkappa_1}x)_j = t^{\varkappa_1}_j(x)$. Lemma 1(c) then shows that $\{\delta^{\varkappa_1 j}\}$ is a basis for $c^0_{T^\varkappa}$, and $x = \sum_{j=0}^\infty t^{\varkappa_1}_j(x)\delta^{\varkappa_1 j}$ $(x \in c^0_{T^\varkappa})$. Since, by Lemma 3(a), $c^0_{T^\varkappa} \equiv C^{*0}_{\lambda k} = R^0_{\lambda\varkappa}$, the result follows.

Note that by taking $\eta_1 = 0$ in Lemma 5A, we obtain an alternative proof of the case $X = R^0_{\lambda\varkappa}$ of Lemma 5.

LEMMA 6. *Write* $g(t) = g_{\varkappa j}(t) = R^{\varkappa}(\delta^{pj}, t^{-1})$ (*where* $\varkappa = p+\eta$, $T^p\delta^{pj} = e^j$). *Then*

(22) $\quad g\in C^p[0,\infty), \quad g^{(i)}(0+) = g^{(i)}(\lambda_j^{-1}) = 0 \quad (i = 0,1,\dots,p);$

(23) $g^{(i)}(t)$ $(i = 0,1,\dots,p)$ *has exactly* i *distinct zeros in* $0<t<\lambda_j^{-1}$;

(24) g *has exactly one relative maximum point* $\zeta\in(0,\lambda_j^{-1})$; g *is strictly increasing in* $(0,\zeta]$, *strictly decreasing in* $[\zeta,\lambda_j^{-1}]$, *zero in* $[\lambda_j^{-1},\infty)$.

Proof. Let λ, $\varkappa$ $(=p+\eta)$, j be fixed, and write

$$\mu_r = \lambda_{j+r} \quad (r = 0,1,\dots,p+1), \qquad \mu_{p+2} = +\infty;$$

$$k = (-1)^{p+1}(\mu_{p+1}-\mu_0)\mu_1\dots\mu_p;$$

$$d_t(x) = (1-xt)^{p+\eta} \quad (xt<1), \qquad d_t(x) = 0 \quad (xt\geqslant 1);$$

$$h_t^r(x) = \frac{(1-xt)^{p+\eta}}{(x-\mu_{r+1})\dots(x-\mu_{p+1})} \qquad (\mu_0\leqslant x\leqslant\mu_r<t^{-1}\leqslant\mu_{r+1}),$$

the denominator of $h_t^r(x)$ being interpreted as 1 if $r = p+1$ (as with every empty product). The (C^*,λ,p)-transform of δ^{pj} is, by definition, $\{t_\nu^p\} = T^p\delta^{pj} = e^j$. By a slight modification of the argument in Russell [22], pp. 424–425 (use $c_\omega(x) = (\omega-x)^\varkappa$ for $x<\omega$ instead of $(\omega-x)^p$, write $t = 1/\omega$, and omit the last five lines of p. 424, which are not then valid for $\varkappa>p$), it follows that

(25) $$g(t) \equiv R^k(\delta^{pj}, t^{-1}) \equiv k\cdot d_t[\mu_0\dots\mu_{p+1}] \quad \text{for} \quad t\geqslant 0,$$

and that this can also be written as

(26) $$g(t) = \begin{cases} k\cdot h_t^r[\mu_0\dots\mu_r] & \text{for} \quad \mu_{r+1}^{-1}\leqslant t<\mu_r^{-1} \quad (r = 0,\dots,p+1),\\ 0 & \text{for} \quad t\geqslant\mu_0^{-1}.\end{cases}$$

The deduction of (26) from (25) is a consequence of the method of deleting points from a divided difference (e.g. see [22], § 4 or [4], § 2) by means of the relation

$$f[x_0\dots x_n] = f_1[x_0\dots x_{n-1}], \qquad f_1(x) = \{f(x)-f(x_n)\}/(x-x_n).$$

The expansion formula for divided differences now shows that

(27) $$d_t[\mu_0\dots\mu_{p+1}] = \sum_{\varrho=0}^{p+1}\pi_\varrho d_t(\mu_\varrho), \qquad \pi_\varrho = \prod_{0\leqslant\nu\leqslant p+1,\nu\neq\varrho}(\mu_\varrho-\mu_\nu)^{-1},$$

and since, for each fixed $x\geqslant\mu_0$, $d_t(x)$ has a continuous pth derivative with respect to t, in $t\geqslant 0$, the same is therefore true of $g(t)$.

Now for $0\leqslant t<\mu_{p+1}^{-1}$, $d_t(\mu_\varrho) = (1-\mu_\varrho t)^{p+\eta}$ $(\varrho = 0,1,\dots,p+1)$, and so, differentiating (25) and (27) i times, we see that

$$g^{(i)}(0+) = (-1)^i(p+\eta)\dots(p+\eta-i+1)k\sum_{\varrho=0}^{p+1}\pi_\varrho\mu_\varrho^i = 0 \quad (0\leqslant i\leqslant p),$$

because this last sum is a $(p+1)$th order difference of the polynomial x^i.

Also, by (26), $g(t)\equiv 0$ for $t\geqslant\mu_0^{-1}$, while

$$g(t) = kh_t^0(\mu_0) = k'(1-\mu_0 t)^{p+\eta}>0 \quad \text{for} \quad \mu_1^{-1}\leqslant t<\mu_0^{-1},$$

where $k' = \mu_1\dots\mu_p/\{(\mu_1-\mu_0)\dots(\mu_p-\mu_0)\}$, from which we see that

$$g^{(i)}(\mu_0^{-1}) = 0 \quad \text{for} \quad 0\leqslant i\leqslant p.$$

To examine $g^{(p+1)}(t)$ we have, on differentiating (26) $p+1$ times,

(28) $$g^{(p+1)}(t) = \hat{k}\cdot\hat{h}_t^r[\mu_0\dots\mu_r], \qquad \mu_{r+1}^{-1}<t<\mu_r^{-1},$$

where

$$\hat{k} = k(p+\eta)\dots(1+\eta)\eta$$

and

$$\hat{h}_t^r(x) = (-1)^r x^{p+1}H_t(x), \qquad H_t(x) = \{(\mu_{r+1}-x)\dots(\mu_{p+1}-x)(1-xt)^{1-\eta}\}^{-1}.$$

But any derivative with respect to x (of any order) of any of the functions

$$(\mu_{r+1}-x)^{-1},\ \dots,\ (\mu_{p+1}-x)^{-1},\ (1-xt)^{\eta-1}$$

is certainly positive for $x<t^{-1}<\mu_{r+1}$, and it therefore follows by Leibniz' theorem that the product of these functions, $H_t(x)$, has positive derivatives (with respect to x) of all orders — hence the same is true of $x^{p+1}H_t(x)$. That is,

(29) $$(-1)^r(\partial/\partial x)^s\hat{h}_t^r(x)>0 \quad \text{for} \quad \mu_0\leqslant x\leqslant\mu_r<t^{-1}<\mu_{r+1},\quad 0\leqslant r\leqslant p+1,\ s = 0,1,\dots$$

There is a minor exception to (29) when $r = p+1$, $\eta = 1$; then $H_t(x) = 1$ and the derivative in (29) is zero for $s\geqslant p+2$. But in all cases we have, by the mean-value theorem for divided differences,

(30) $$\hat{h}_t^r[\mu_0\dots\mu_r] = \frac{1}{r!}\{(\partial/\partial x)^r\hat{h}_t^r(x)\}_{x=\xi} \quad \text{for some } \xi \text{ in } \mu_0<\xi<\mu_r,$$

and since $\hat{k}$ has the sign of $(-1)^{p+1}$, it now follows from (28), (29), (30) that

(31) $$(-1)^{p-r+1}g^{(p+1)}(t)>0 \quad \text{for} \quad \mu_{r+1}^{-1}<t<\mu_r^{-1},\ r = 0,1,\dots,p+1.$$

Now (31) shows that $g^{(p+1)}$ is sectionally of constant (non-zero) sign in each of the intervals $(\mu_{r+1}^{-1},\mu_r^{-1})$ $(r = 0,1,\dots,p+1)$, and hence $g^{(p)}$, being continuous, is strictly monotone in the closure of each such interval. Since, by (22), $g^{(p)}$ vanishes at 0 and μ_0^{-1}, it cannot (by monotonicity) vanish in either $(0, \mu_{p+1}^{-1}]$ or $[\mu_1^{-1}, \mu_0^{-1})$; hence $g^{(p)}$ has at most p distinct zeros in $(0, \mu_0^{-1})$. Let n_i be the number of distinct zeros of $g^{(i)}$ in $(0, \mu_0^{-1})$; then since, by (22), $g^{(i)}$ $(0 \leqslant i \leqslant p-1)$ vanishes at 0 and μ_0^{-1}, it follows by Rolle's theorem that $n_i < n_{i+1}$ for $0 \leqslant i \leqslant p-1$. Thus we now have $0 \leqslant n_0 < n_1 < \ldots < n_p \leqslant p$ and hence $n_i = i$ $(0 \leqslant i \leqslant p)$, which proves (23). Now (24) follows at once from what we have already proved. Since $n_i = i$, the use of Rolle's theorem also shows that, except at the end-points 0 and μ_0^{-1}, $g^{(i)}$ and $g^{(i+1)}$ $(0 \leqslant i \leqslant p-1)$ cannot vanish simultaneously.

LEMMA 7. *Define* $\delta^{\varkappa j}$ $(\varkappa \geqslant 0,\ j = 0, 1, \ldots)$ *by* $T^{\varkappa}\delta^{\varkappa j} = e^j$, *and write* $\Lambda_{jp} = \lambda_{j+p+1}/(\lambda_{j+p+1} - \lambda_j)$. *Then for* $p, j = 0, 1, 2, \ldots$, *we have*

$$\|\delta^{\varkappa_1 j}\|_{\lambda\varkappa} \asymp \Lambda_{jp}^{\eta_1-\eta}, \quad \varkappa = p+\eta,\ \varkappa_1 = p+\eta_1,\ 0 \leqslant \eta_1 < \eta \leqslant 1; \tag{32}$$

$$\|\delta^{pj}\|_{|\lambda\varkappa|} = 2\,\|\delta^{pj}\|_{\lambda\varkappa} \asymp \Lambda_{jp}^{-\eta}, \quad \varkappa = p+\eta,\ 0 < \eta \leqslant 1. \tag{33}$$

Proof. Note first that an FK- (and hence a BK-) space has a unique FK topology. Also the matrix $T^{\varkappa}$ is normal, and hence

$$\delta^{\varkappa_1 j} = (T^{\varkappa_1})^{-1}e^j = (A^{p\eta_1}T^p)^{-1}e^j = (T^p)^{-1}(A^{p\eta_1})^{-1}e^j. \tag{34}$$

Then

$$\begin{aligned}
\|\delta^{\varkappa_1 j}\|_{\lambda\varkappa} &\asymp \|\delta^{\varkappa_1 j}\|'_{\lambda\varkappa} && \text{by Lemma 3(a)}\\
&= \|T^{\varkappa}\delta^{\varkappa_1 j}\|_c && \text{by Lemma 2(b)}\\
&= \|A^{p\eta}T^p(T^p)^{-1}(A^{p\eta_1})^{-1}e^j\|_c && \text{by (34)}\\
&= \|De^j\|_c, && D = A^{p\eta}(A^{p\eta_1})^{-1}\\
&= \sup_m |d_{mj}|\\
&= |d_{jj}| && \text{by Lemma 4 and Corollary 3(b)}\\
&= a_{jj}^{p\eta}/a_{jj}^{p\eta_1}\\
&\asymp \Lambda_{jp}^{-\eta}/\Lambda_{jp}^{-\eta_1} && \text{by (15)}\\
&= \Lambda_{jp}^{\eta_1-\eta}.
\end{aligned}$$

By Lemma 2(a) and the notation of Lemma 6,

$$\|\delta^{pj}\|_{|\lambda\varkappa|} = \lim_{\omega\to\infty}|R^{\varkappa}(\delta^{pj}, \omega)| + \int_0^\infty |dR^{\varkappa}(\delta^{pj}, \omega)| = |g(0+)| + \int_0^\infty |dg(\omega^{-1})|.$$

But, by Lemma 6, $g(\omega^{-1})$ increases from 0 at $\omega = \lambda_j$ to a maximum at $\omega = \zeta^{-1}$, and then steadily decreases to 0 as $\omega \to \infty$. Thus $g(0+) = 0$ and the total variation of $g(\omega^{-1})$ is twice its maximum value. That is,

$$\|\delta^{pj}\|_{|\lambda\varkappa|} = 2\sup_\omega |g(\omega^{-1})| = 2\sup_\omega |R^{\varkappa}(\delta^{pj}, \omega)| = 2\,\|\delta^{pj}\|_{\lambda\varkappa}.$$

Since, by (32) with $\eta_1 = 0$, $\|\delta^{pj}\|_{\lambda\varkappa} \asymp \Lambda_{jp}^{-\eta}$, the proof is complete.

4. Applications to Riesz means: main results. The standard limitation theorems for Riesz means are well known. For example ([3], Lemma 3, [1], Lemma 2, [24], Lemma 1):

If $R^{\varkappa}(\omega) = o(1)$, *then, for* $0 \leqslant \mu \leqslant \varkappa$ *and* $\lambda_n \leqslant \omega \leqslant \lambda_{n+1}$,

$$R^{\mu}(\omega) = o(\Lambda_n^{\varkappa-\mu}) \qquad \textit{if } \mu \textit{ is an integer (with } \lambda_n < \omega \textit{ if } \mu = 0), \tag{35}$$

$$R^{\mu}(\omega) = o(\Lambda_{n-1}^{\varkappa-\mu} + \Lambda_n^{\varkappa-\mu}) \ \textit{ if } \mu \textit{ is non-integral.} \tag{36}$$

Hardy and Riesz ([6], footnote p. 37) remark: "It is very curious that the simpler result which holds when μ is intergal should not hold always". We shall see that, on the contrary, many properties of Riesz means experience a change in form as the order μ passes through an integer.

THEOREM 4. *Let* $\varkappa = p+\eta$, *where* p *is a non-negative integer and* $0 < \eta \leqslant 1$; *let* $\lambda_n \nearrow \infty$ *and* $\Lambda_{nr} \equiv \lambda_{n+r+1}/(\lambda_{n+r+1} - \lambda_n)$ $(n, r = 0, 1, 2, \ldots)$; *let* $\{t_n^{\mu}(x)\}$ *denote the* (C^*, λ, μ)-*transform of* $x = \{x_n\}$ *as defined in* (13), *where* x *is the sequence of partial sums of a series* $\sum a_n$; *and let* (τ_{nk}^p), (τ_{nk}^p) *be as given in* (16) *and* (17). *If* $x \in R_{\lambda\varkappa}^0$, *then we have the following limitation theorems, all of them best possible in the sense of Remark* 2:

$$t_n^{p+\eta_1}(x) = o(\Lambda_{np}^{\eta-\eta_1}) \qquad (0 \leqslant \eta_1 < \eta), \tag{37}$$

$$t_n^{p-1}(x) = o\{(\lambda_{n+p} - \lambda_n)^{-1}\max(\lambda_n\Lambda_{n-1,p}^{\eta},\ \lambda_{n+p}\Lambda_{np}^{\eta})\} \qquad (p \geqslant 1), \tag{38}$$

$$x_n = o(\varphi_{n\varkappa}), \qquad \varphi_{n\varkappa} \equiv \max_{n-p\leqslant k\leqslant n}(|\tau_{nk}^p|\Lambda_{kp}^{\eta}), \tag{39}$$

$$a_n = x_n - x_{n-1} = o(\overline{\varphi}_{n\varkappa}), \qquad \overline{\varphi}_{n\varkappa} \equiv \max_{n-p-1\leqslant k\leqslant n}(|\overline{\tau}_{nk}^p|\Lambda_{kp}^{\eta}). \tag{40}$$

Proof. Let $X = R_{\lambda\varkappa}^0$ which, by Lemma 2, is a BK-space.

(a) By Lemma 5A, X has basis $\{\delta^{\varkappa_1 j}\}$ and representation (21). Now the choice $B = I$ in Theorem 3, and use of Lemma 7 (32), gives, with $\varkappa_1 = p+\eta_1$,

$$u_n(x) = t_n^{\varkappa_1}(x) = o(\|\delta^{\varkappa_1 n}\|_{\lambda\varkappa}^{-1}) = o(\Lambda_{np}^{\eta-\eta_1}).$$

(b) As is easily verified from the definitions in § 3 (or see [22], (28)),

$$(\lambda_{n+p} - \lambda_n)t_n^{p-1} = \lambda_{n+p}t_n^p - \lambda_n t_{n-1}^p. \tag{41}$$

Choose in Theorem 3 the basis $\{\delta^{pj}\}$ (so $f_j(x) = t_j^p(x)$, by Lemma 5), and $b_{nn} = \lambda_{n+p}$, $b_{n,n-1} = -\lambda_n$, $b_{nk} = 0$ otherwise. Then, by Theorem 3 and (32) $(\eta_1 = 0)$,

$$\begin{aligned}
u_n(x) = (\lambda_{n+p} - \lambda_n)t_n^{p-1}(x) &= o\{\max(\lambda_n\|\delta^{p,n-1}\|_{\lambda\varkappa}^{-1},\ \lambda_{n+p}\|\delta^{pn}\|_{\lambda\varkappa}^{-1})\}\\
&= o\{\max(\lambda_n\Lambda_{n-1,p}^{\eta},\ \lambda_{n+p}\Lambda_{np}^{\eta})\}.
\end{aligned}$$

(c) Since $(T^p)^{-1} = (\tau_{nk}^p)$ and $T^px = \{t_n^p(x)\}$, we have $x_n = \sum_{k=n-p}^{n} \tau_{nk}^p t_k^p(x)$.

Choose in Theorem 3 the basis $\{\delta^{pj}\}$ and $B = (\tau^p_{nk})$, and we get

$$u_n(x) = x_n = o\{\max_{n-p\leqslant k\leqslant n} (|\tau^p_{nk}|\cdot\|\delta^{pk}\|^{-1}_{\lambda\varkappa})\}$$

which, by (32), gives (39).

(d) If $a_n = x_n - x_{n-1}$, then $\overline{T}^p a = T^p x$ and so $a_n = \sum_{k=n-p-1}^{n} \bar{\tau}^p_{nk} t^p_k(x)$. The choice $B = (\bar{\tau}^p_{nk})$ in Theorem 3 now gives the required result (40). In every case, the use of Theorem 3 gives the best-possible order conditions.

Remark 5. It is proved in [4], Lemma 7(a)(i) that if $x\in C^{*0}_{\lambda\varkappa}$ $(\varkappa = p+\eta)$ then

$$(42)\qquad t^p_n(x) = o(A^\eta_{np}).$$

But since the matrix $A^{p\eta}$ which transforms $\{t^p_n\}$ into $\{t^{p+\eta}_n\}$ satisfies the hypotheses of Corollary 3(b), and $a^{p\eta}_{nn} \asymp A^\eta_{np}$, by (15), we deduce that (42) is a best-possible limitation theorem in $C^{*0}_{\lambda\varkappa}$ (and hence in $R^0_{\lambda\varkappa}$, by Lemma 3(a)). It is easy to deduce from (41) and (42), by induction (see [4], Lemma 7 or [22], Theorem 3) that if $x\in C^{*0}_{\lambda\varkappa}$ $(=R^0_{\lambda\varkappa})$ then

$$(43)\qquad t^r_n(x) = o(A^{p-r+\eta}_{nr}),\qquad r = 0,1,\ldots,p,$$

but this is not best-possible for $r<p$ except in certain special cases.

If $x\in R^0_{|\lambda\varkappa|}$, then $x\in R^0_{\lambda\varkappa}$ and the limitation results (37), (38), (39), (40) follow from Theorem 4. However, in applying Theorem 3 to determine whether these results are best-possible in $R^0_{|\lambda\varkappa|}$, the only Schauder basis which we currently have for $R^0_{|\lambda\varkappa|}$ is $\{\delta^{pj}\}$ (Lemma 5). Nevertheless, this allows us to deduce the best-possible nature of all the results of Theorem 4 except the fractional case $0<\eta_1<\eta$ of (37); the proofs are identical to Theorem 4 except that $\|\delta^{pj}\|_{|\lambda\varkappa|}$ replaces $\|\delta^{pj}\|_{\lambda\varkappa}$ — but Lemma 7 (33) then ensures that the actual order conditions remain the same. Thus we have:

THEOREM 5. *With the notation of Theorem 4, let $x\in R^0_{|\lambda\varkappa|}$. Then* (42), (38), (39), (40) *hold, and these order conditions are all best-possible in $R^0_{|\lambda\varkappa|}$.*

Remark 6. Although (37), (38), (42) give order conditions for generalized Cesàro means rather than Riesz means, we can deduce results for the latter, at least for integer means or when the order of magnitude is increasing. Thus from [22], Corollary 4, and the Riesz convexity theorem [5], Theorem 1.71, we have, for $\lambda_n<\omega\leqslant\lambda_{n+1}$, $q<\mu<\varkappa$, $q = 0,1,2,\ldots$,

$$(44)\qquad t^q_n = o(\pi_n),\ (\pi_n>0)\ \Rightarrow\ R^q(\omega) = o(\max_{n-q\leqslant k\leqslant n}\pi_k);$$

$$(45)\qquad [R^\varkappa(\omega) = o(1) \text{ and } R^q(\omega) = o(\varrho_n)\ (0<\varrho_n\nearrow)] \Rightarrow$$
$$\Rightarrow R^\mu(\omega) = o(\varrho_n^{(\varkappa-\mu)/(\varkappa-q)}).$$

A simple example using (42) and (44) is, for $\lambda_n<\omega\leqslant\lambda_{n+1}$,

$$R^{p+\eta}(\omega) = o(1)\ \Rightarrow\ t^p_n = o(A^\eta_{np})\ \Rightarrow\ R^p(\omega) = o(\max_{n-p\leqslant k\leqslant n} A^\eta_{kp}).$$

The standard limitation theorem (35), which yields $R^p(\omega) = o(A^\eta_n)$, is weaker because (i) $\max_{n-p\leqslant k\leqslant n} A_{kp}\leqslant A_n$, and (ii) we may easily construct a sequence $\{\lambda_n\}$ for which $A_n\neq O(\max_{n-p\leqslant k\leqslant n} A_{kp})$ (e.g. take $p = 1$, $\lambda_{2n} = n$, $\lambda_{2n+1} = n+\delta_n$, $0<\delta_n<1$, $\delta_n\to 0$).

Remark 7. Notice particularly that (16) gives a simple explicit formula for $\bar{\tau}^p_{nk}$, so that (40) specifies the best-possible order condition on the terms a_n of an $(R,\lambda,\varkappa)$-summable series $\sum a_n$ explicitly in terms of λ and $\varkappa$. Two special cases of Theorems 4 and 5 deserve mention:

(i) The case $p = 0$. Observe, from (39) and (40) with $p = 0$, that

$$(46)\qquad \varphi_{n\varkappa} = A^\varkappa_n,\qquad \bar{\varphi}_{n\varkappa} = \max(A^\varkappa_n, A^\varkappa_{n-1}),\qquad 0<\varkappa\leqslant 1;$$

thus best order conditions in both $R^0_{\lambda\varkappa}$ and $R^0_{|\lambda\varkappa|}$, $0<\varkappa\leqslant 1$, are

$$(47)\qquad x_n = o(A^\varkappa_n)\quad\text{and}\quad a_n = x_n - x_{n-1} = o\{\max(A^\varkappa_n, A^\varkappa_{n-1})\}.$$

(ii) The case $A_{n-1} = O(A_n)$. Here (16), (17) and [23], pp. 416–417, show that the expressions for $\varphi_{n\varkappa}$ and $\bar{\varphi}_{n\varkappa}$ in (39) and (40) satisfy

$$(48)\qquad \varphi_{n\varkappa}\asymp\bar{\varphi}_{n\varkappa}\asymp A^\varkappa_n\quad\text{when}\quad A_{n-1} = O(A_n),\ \varkappa>0.$$

It follows that the standard limitation theorem $x_n = o(A^\varkappa_n)$ (put $\mu = 0$ in (35)), together with $a_n = x_n - x_{n-1} = o(A^\varkappa_n)$, are best-possible in both $R^0_{\lambda\varkappa}$ and $R^0_{|\lambda\varkappa|}$ for every $\varkappa>0$, when $A_{n-1} = O(A_n)$. However, without some restriction on λ when $\varkappa>1$, the result $x_n = o(A^\varkappa_n)$ is certainly weaker than that given by Theorem 4. For example, when $1<\varkappa\leqslant 2$, the best limitation theorem is

$$x\in R^0_{\lambda\varkappa}\ (1<\varkappa\leqslant 2)\Rightarrow x_n = o(\varphi_{n\varkappa}),$$
$$\varphi_{n\varkappa} = (\lambda_{n+1}-\lambda_n)^{-1}\max(\lambda_n A^{\varkappa-1}_{n-1,1}, \lambda_{n+1}A^{\varkappa-1}_{n1}),$$

and the example at the end of Remark 6 shows that $A^\varkappa_n\neq O(\varphi_{n\varkappa})$ in this case.

COROLLARY 4. *Let $\varkappa>0$ and $\varphi_{n\varkappa}$, $\bar{\varphi}_{n\varkappa}$ be defined as in* (39) *and* (40).

(a) *In order that $b_n x_n = o(1)$ whenever $x\in R^0_{|\lambda\varkappa|}$ it is necessary that $b_n = O(1/\varphi_{n\varkappa})$ as $n\to\infty$.*

(b) *In order that $b_n a_n = o(1)$ whenever $x\in R^0_{|\lambda\varkappa|}$, where $a_n = x_n - x_{n-1}$, it is necessary that $b_n = O(1/\bar{\varphi}_{n\varkappa})$ as $n\to\infty$.*

Proof. (a) Suppose that $\theta_n\equiv b_n\varphi_{n\varkappa}\neq O(1)$. Then, by Theorem 5, $\exists x^*\in R^0_{|\lambda\varkappa|}$ such that $\theta_n x^*_n\neq o(\varphi_{n\varkappa})$; that is, $b_n x^*_n\neq o(1)$. (b) The proof is similar.

Remark 8. Since the convergence of $\sum b_n x_n$ implies that $b_n x_n = o(1)$, Corollary 4 gives a necessary condition for convergence factors in any space containing $R^0_{|\lambda\varkappa|}$.

Corollary 5. *Let $\varkappa > 0$, and if $\varkappa > 1$ assume $\Lambda_{n-1} = O(\Lambda_n)$. Given any $\theta_n \neq O(1)$, there is a sequence $\{x^*_n\}$, and a series $\sum a^*_n$, both summable $|R, \lambda, \varkappa|_0$, but such that $\theta_n x^*_n \neq o(\Lambda^\varkappa_n)$ and $\theta_n a^*_n \neq o(\Lambda^\varkappa_n)$.*

Proof. Apply Corollary 4 and the two special cases in Remark 7. When $0 < \varkappa \leqslant 1$ the condition $\Lambda_{n-1} = O(\Lambda_n)$ is not required for the conclusion since, by (46), we then have $\varphi_{n\varkappa} = \Lambda^\varkappa_n$ and $\bar{\varphi}_{n\varkappa} \geqslant \Lambda^\varkappa_n$.

Corollary 5 has been proved by Jakimovski and Tzimbalario [10], Lemma 3, and includes earlier results of Russell [23], Theorem 2, for the larger space $R^0_{\lambda\varkappa}$, and of Jurkat [11], Satz 4, where a heavier restriction on λ is also used. In all these previous proofs, it is shown that $\{x^*_n\}$ can also be taken as the sequence of partial sums of $\sum a^*_n$.

Our next application concerns the following theorem [1] of Borwein and Cass [2], which we shall extend, together with its analogue for absolute summability, to fractional orders of summability.

Theorem A. *Let p be a non-negative integer and $\beta > p$. Then in order that $(R, \lambda, \beta) \sim (R, \lambda, p)$ it is necessary and sufficient that $\liminf_{n\to\infty} (\lambda_{n+p+1}/\lambda_n) > 1$.*

The condition $\liminf_{n\to\infty} (\lambda_{n+p+1}/\lambda_n) > 1$ can also be written $\Lambda_{np} = O(1)$; and for completeness the statement and proof of the sufficiency part of Theorem A is given as Theorem 6(a).

Theorem 6. (a) *Let p be a non-negative integer and $\Lambda_{np} = O(1)$ $(n \to \infty)$. Then*

$$R_{\lambda\alpha} = R_{\lambda p} \quad \textit{for every } \alpha \geqslant p.$$

(b) *Let $\beta > \varkappa \geqslant 0$ and $R_{\lambda\beta} = R_{\lambda\varkappa}$. Then*

$$\Lambda_{np} = O(1) \ (n \to \infty), \quad \textit{where} \quad p = [\varkappa].$$

Proof. (a) Note that $\Lambda_{nr} \searrow$ as $r \nearrow$; hence if $\Lambda_{np} = O(1)$ then $\Lambda_{nr} = O(1)$ for each $r \geqslant p$. Rewriting (41) in the form

$$t^r_n = \Lambda_{nr}(t^{r+1}_n - t^{r+1}_{n-1}) + t^{r+1}_{n-1} \tag{49}$$

we see that if $\Lambda_{nr} = O(1)$ and $\{t^{r+1}_n\} \in c$ then $\{t^r_n\} \in c$. It follows that $C^*_{\lambda,r+1} \subseteq C^*_{\lambda r}$ for $r \geqslant p$ and, by (19) and Lemma 3(a), $R_{\lambda\alpha} = R_{\lambda p}$ for every $\alpha \geqslant p$.

(b) Let $\beta > \varkappa = p + \eta_1$, where $p = [\varkappa]$ and $0 \leqslant \eta_1 < 1$, and take the equivalent hypothesis $R^0_{\lambda\beta} = R^0_{\lambda\varkappa}$. Then, by (19), $R^0_{\lambda,p+\eta_1} = R^0_{\lambda,p+\eta_0} = R^0_{\lambda,p+\eta}$ for some η_0, η with $0 \leqslant \eta_1 < \eta_0 < \eta < 1$ and $p + \eta \leqslant \beta$. Suppose that $\Lambda_{np} \neq O(1)$. Then choose $\theta_n = \Lambda^{\eta-\eta_0}_{np} \neq O(1)$ (since $\eta - \eta_0 > 0$), and it

(1) The case $\beta = p+1$ (which includes the sufficiency part of theorem A) was given by B. I. Korenblyum [Dokl. Akad. Nauk SSSR 81 (1951) pp. 725–727; Theorem 2].

follows from Theorem 4 (or Remark 5) that $\exists\, x^* \in R^0_{\lambda,p+\eta}$ such that $t^p_n(x^*) \neq o(\Lambda^\eta_{np} \Lambda^{\eta_0-\eta}_{np})$, i.e., $t^p_n(x^*) \neq o(\Lambda^{\eta_0}_{np})$. But since also $x^* \in R^0_{\lambda,p+\eta_0}$, Theorem 4 (or Remark 5) gives $t^p_n(x^*) = o(\Lambda^{\eta_0}_{np})$, a contradiction. Hence $\Lambda_{np} = O(1)$.

Theorem 7. (a) *Let p be a non-negative integer and $\Lambda_{np} = O(1)$ $(n \to \infty)$ Then*

$$R_{|\lambda\alpha|} = R_{|\lambda p|} \quad \textit{for every } \alpha \geqslant p.$$

(b) *Let $\beta > \varkappa \geqslant 0$ and $R_{|\lambda\beta|} = R_{|\lambda\varkappa|}$. Then*

$$\Lambda_{np} = O(1) \ (n \to \infty), \quad \textit{where} \quad p = [\varkappa].$$

Proof. (a) If $\Lambda_{np} = O(1)$, then $\Lambda_{nr} = O(1)$ for each $r \geqslant p$. Now, from (49),

$$t^r_n - t^r_{n+1} = \Lambda_{n+1,r}(t^{r+1}_n - t^{r+1}_{n+1}) - (\Lambda_{nr} - 1)(t^{r+1}_{n-1} - t^{r+1}_n); \tag{50}$$

hence if $\Lambda_{nr} = O(1)$ and $\{t^{r+1}_n\} \in v$ then $\{t^r_n\} \in v$. If follows that $C^*_{|\lambda,r+1|} \subseteq C^*_{|\lambda r|}$ for $r \geqslant p$ and, by (19) and Lemma 3(b), $R_{|\lambda\alpha|} = R_{|\lambda p|}$ for every $\alpha \geqslant p$.

(b) The proof follows in the same way as that of Theorem 6(b), but using absolute summability fields and appealing to Theorem 5 instead of Theorem 4.

Remark 9. (i) Theorems 6 and 7 show the remarkable result that if all the Riesz means (or absolute Riesz means) are equivalent from order $\varkappa$ onwards, then they are in fact equivalent down to $[\varkappa]$; that is, the equivalences can never stop at a fractional value of $\varkappa$.

(ii) As a corollary of Theorem 7 (take $p = \varkappa = 0$) we see that

$$\textit{for any } \beta > 0, \quad R_{|\lambda\beta|} = v \Leftrightarrow \Lambda_n = O(1). \tag{51}$$

The $\Leftarrow$ implication in (51) may be deduced, for instance, from Ratti [21], Theorem 1 and Remark on p. 1006; the $\Rightarrow$ implication in (51) was conjectured by Maddox [16], p. 263, and proved by Jakimovski and Tzimbalario [10], Theorem 7.

(iii) For $|R^*, \lambda, \beta|$ summability, it follows from (51) that

$$\textit{for any } \beta > 0, \quad R^*_{|\lambda\beta|} = v \Rightarrow \Lambda_n = O(1) \tag{52}$$

(Maddox [16], Theorem 2);

in the opposite direction, it is known that

$$\textit{for } 0 < \beta \leqslant 1, \quad \Lambda_n = O(1) \Rightarrow R^*_{|\lambda\beta|} = v \quad \text{(Körle [14], Satz 2)}. \tag{53}$$

(iv) By trivial modifications in the method of proof of Theorems 6(b) and 7(b), it is possible to combine these two results into the following more general statement:

if $R_{|\lambda\beta|} \subseteq R_{\lambda\varkappa}$ for some $\beta > \varkappa \geqslant 0$, then $\Lambda_{np} = O(1)$, where $p = [\varkappa]$.

Our final applications of the results of § 2 allow us to write down, in improved form, and without restrictions on the type λ, complete necessary and sufficient conditions for an arbitrary method to include ordinary or absolute Riesz summability. We take the liberty of using c_B to denote the set of all B-limitable sequences, $B=(b_{\varrho\nu})$, even when ϱ may be a continuous parameter.

THEOREM 8. *Let p be a non-negative integer, $0<\eta\leqslant 1$, $\varkappa=p+\eta$; $\lambda_n\nearrow\infty$; $B=(b_{\varrho\nu})$; and let $(\tau_{\nu k}^p)$, given by* (16) *and* (17), *be the inverse of the (C^*,λ,p) sequence-to-sequence matrix T^p; let $\varphi_{\nu\varkappa}\equiv\max_{\nu-p\leqslant k\leqslant\nu}(|\tau_{\nu k}^p|\Lambda_{kp}^\eta)$, $\nu\geqslant p$. In order that $R_{\lambda\varkappa}\subseteq c_B$ it is necessary and sufficient that*

$$\exists\lim_{\varrho} b_{\varrho\nu}\equiv\beta_\nu\quad(\nu=0,1,\ldots),\qquad \exists\lim_{\varrho}\sum_\nu b_{\varrho\nu}\equiv\beta, \tag{54}$$

$$|b_{\varrho\nu}|\leqslant H_\varrho/\varphi_{\nu\varkappa}\quad \textit{for each } \varrho \textit{ and for all } \nu\geqslant p, \textit{ for some } H_\varrho<\infty, \tag{55}$$

and that a family of functions $\{g_\varrho\}$ exists, defined in $[\lambda_0,\infty)$, such that

$$b_{\varrho\nu}=\Delta_\nu\int_{\lambda_\nu}^{\infty}(\omega-\lambda_\nu)^\varkappa dg_\varrho(\omega),\qquad \int_{\lambda_0}^{\infty}\omega^\varkappa|dg_\varrho(\omega)|\equiv M_\varrho\leqslant M<\infty. \tag{56}$$

Condition (55) *may be omitted if either* (i) *B is row-finite* (*since* (55) *is then trivially satisfied*), *or if* (ii) $0<\varkappa\leqslant 1$ (*since* (56) *implies* (55) *in this case*).

Proof. For $0<\varkappa\leqslant 1$ see Russell [25], Theorem 1 and Remark (i). For $\varkappa>1$ the theorem is given by Jakimovski and Tzimbalario ([9], Theorem 1) (see also [8], Theorem 1 for $1<\varkappa\leqslant 2$) with condition (55) replaced by

$$\lim_{n\to\infty}u_n(x)=0\quad \text{for each } \varrho \text{ and for each } x\in R_{\lambda\varkappa}^0, \tag{57}$$

where

$$u_n(x)=\sum_{k=n}^{n+p-1}d_{nk}t_k^p(x),\qquad d_{nk}=\sum_{j=k}^{n+p-1}b_{\varrho j}\tau_{jk}^p.$$

However, since $R_{\lambda\varkappa}\subseteq c_B$ implies that $\sum_\nu b_{\varrho\nu}x_\nu$ must converge for each ϱ whenever $x\in R_{\lambda\varkappa}^0$, it follows from Corollary 4(a) that (55) is necessary. Conversely, if $\varkappa>1$ and (55) holds then, by Lemma 7 and the definitions of d_{nk} and $\varphi_{j\varkappa}$, we have, for $n\leqslant k\leqslant n+p-1$, $n=0,1,2,\ldots$,

$$|d_{nk}|\leqslant p\cdot\max_{k\leqslant j\leqslant n+p-1}|b_{\varrho j}\tau_{jk}^p|\leqslant p\cdot\max_{k\leqslant j\leqslant n+p}|b_{\varrho j}\tau_{jk}^p|\leqslant pH_\varrho\Lambda_{kp}^{-\eta}\leqslant K_\varrho\|\delta^{pk}\|_{\lambda\varkappa}$$

and hence, by Theorem 3 and Lemma 5, for each $x\in R_{\lambda\varkappa}^0$,

$$u_n(x)=o\{\max_{n\leqslant k\leqslant n+p-1}(|d_{nk}|\cdot\|\delta^{pk}\|_{\lambda\varkappa}^{-1})\}=o(K_\varrho)\quad \text{as}\quad n\to\infty.$$

Thus (57) holds, and the sufficiency part now follows from [9], Theorem 1.

THEOREM 9. *Let p be a non-negative integer, $0<\eta\leqslant 1$, $\varkappa=p+\eta$; $\lambda_n\nearrow\infty$; $B=(b_{\varrho\nu})$. In order that $R_{|\lambda\varkappa|}\subseteq c_B$ it is necessary and sufficient*

that (54) *and* (55) *hold, and that a family of functions $\{g_\varrho\}$ exists, defined on $[\lambda_0,\infty)$, such that*

$$b_{\varrho\nu}=\int_{\lambda_\nu}^{\infty}g_\varrho(\omega)\frac{d}{d\omega}[\Delta_\nu(1-\lambda_\nu/\omega)^\varkappa]d\omega,\quad \operatorname*{ess\,sup}_{\omega\geqslant 0}|g_\varrho(\omega)|\equiv M_\varrho\leqslant M<\infty, \tag{58}$$

where $\Delta_\nu(1-\lambda_\nu/\omega)^\varkappa$ is to be interpreted as $(1-\lambda_\nu/\omega)^\varkappa$ for $\lambda_\nu\leqslant\omega\leqslant\lambda_{\nu+1}$. Condition (55) *may be omitted if either B is row-finite, or if $0<\varkappa\leqslant 1$.*

Proof. This theorem is given by Jakimovski and Tzimbalario ([10], Theorem 1) with (55) replaced by the condition that the limit in (57) should hold for each $x\in R_{|\lambda\varkappa|}^0$. Since we know, by Lemma 5, that $\{\delta^{pk}\}$ is also a basis in $R_{|\lambda\varkappa|}^0$, and, by Lemma 7, $\|\delta^{pk}\|_{|\lambda\varkappa|}\asymp\Lambda_{kp}^{-\eta}$, the proof follows in the same way as for Theorem 8.

Remark 10. (i) If in Theorems 8 and 9 we require the B-limit to be the same as the $(R,\lambda,\varkappa)$-limit, we must put $\beta_\nu=0$ $(\nu=0,1,\ldots)$, $\beta=1$ in (54).

(ii) If $\Lambda_{n-1}=O(\Lambda_n)$, then (48) shows that (55) is equivalent to

$$|b_{\varrho\nu}|\leqslant H_\varrho\Lambda_\nu^{-\varkappa}.$$

Thus the only effect on Theorems 8 and 9 of a restriction on λ is to simplify condition (55). Compare with [25], Remark (i), [9], Theorems 2 and 3, [10], Theorems 2 and 3, and for an alternative form of Theorem 8 in the case where $\varkappa$ is an integer, see [26], Theorem 1.

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On the moduli of convexity and smoothness

by

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Abstract. In the paper the moduli of convexity and smoothness of general Banach spaces and products thereof are discussed. An attempt is made to give precise estimates where only qualitative results have been known. (E.g. it is proved that the moduli of $l_2(X)$ are equivalent to the corresponding ones of X.) The problem how far the modulus of convexity can be improved by a suitable renorming is studied for spaces with local unconditional structure.

In this paper we are concerned with general properties of the moduli of convexity and smoothness of Banach spaces and certain products thereof. Our purpose was to obtain some estimates, useful in the isomorphic theory of Banach spaces, in a precise form and with no redundant assumptions on the spaces involved. Renorming problems are considered only in the case of the existence of local unconditional structure, which may be regarded as elementary (cf. [5], [24]). Our terminology tends to be consistent with [16].

Section I is of an introductory nature. The main results are Propositions 3 and 10 and Corollary 11. The first two of them seem to have been implicit in the literature, but their role has not been recognized. For the sake of completeness, short proofs of some known results are also given.

The main result of Section II is that the moduli of convexity and smoothness of $l_2(X)$ are essentially the same as those of X. This completes the results of [7]. The method used to estimate $\varrho_{l_2(X)}$ can easily be adapted to the case of Orlicz spaces of vector valued functions, $L_M(X)$. The formulae obtained are analogous to those found in [18], where the case $X = \boldsymbol{R}$ is discussed. The results of Section I allow us to show that the latter formulae are the best possible. The corresponding results for the moduli of convexity are obtained by duality, with the use of some formulae for the Legendre transform.

In Section III we investigate the uniform convexifiability of a space E with an unconditional basis. The dual results are not formulated, their

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