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Freely solvable systems of linear inequalities

by

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Abstract. A system of linear inequalities (in $\boldsymbol{R}^X$) is called *freely solvable* if each of its "partial solutions" can be extended to a solution. A class of such systems is described. It yields a method of solving systems which differs from Farkas' lemma.

Let U and V be topological vector spaces and let C be a convex cone in V. A linear system is given by $Au \leqslant a$ where $A: U \to V$ is a linear operator and $a \in V$. The following statement is a simple consequence of the separation theorem.

FARKAS' LEMMA. *Let $AU + C$ be a closed convex cone. Then one and only one of the following alternatives holds:*

(i) *There exists an element $u \in U$ such that $Au \leqslant a$, i.e., $a - Au \in C$.*

(ii) *There exists an $f \in C^*$ such that $A^*f = 0$ and $f(a) < 0$, where $C^* = \{f \in V^*: f(v) \geqslant 0$, for all $v \in C\}$.*

The aim of this paper is to study linear systems where $U = \boldsymbol{R}^X$, $V = \boldsymbol{R}^T$, $C = \boldsymbol{R}_+^T$, possessing a property which we call *free solvability*. The result obtained is related to Farkas' lemma. The method of step-by-step extension is used, similarly as in the proof of the Hahn–Banach theorem.

Let X be a set. The support of a function $a: X \to R$ is the set $s(a) = \{x \in X: a(x) \neq 0\}$. For any function a on X with a finite support the functional

$$u \to \langle a, u\rangle \overset{\mathrm{df}}{=} \sum_{x \in X} a(x) u(x)$$

is well defined on $\boldsymbol{R}^X$, linear and continuous with respect to the product topology. Moreover, every continuous functional on $\boldsymbol{R}^X$ is of that form. By a *linear inequality over X* we mean the expression

$$\langle a, u\rangle \leqslant \alpha,$$

where α is a real number and a, a real function on X, has a finite support. A function $u: X \to \boldsymbol{R}$ is called a *solution of a system of linear inequalities*

$$M = \{\langle a_t, u\rangle \leqslant \alpha_t: t \in T\}$$

over X if, for every $t \in T$,

$$\sum_{x \in X} a_t(x) u(x) \leqslant a_t.$$

For any $Y \subset X$ we put

$$M_Y = \{\langle a_t, u\rangle \leqslant a_t:\ s(a_t) \subset Y,\ t \in T\}$$

and for every $x \in X$

$$M_x = \{a_t(x) u \leqslant a_t:\ s(a_t) \subset \{x\},\ t \in T\}$$

(it is a system over $\{x\}$).

In particular, if $Y = \emptyset$ then $M_\emptyset$ is the subsystem of M consisting of all inequalities of M with empty supports which are identified with inequalities on real numbers.

A function φ: $Y \to \boldsymbol{R}$ $(\emptyset \neq Y \subset X)$ which is a solution of M_Y is called a *partial solution of* M. A system is called *freely solvable* if it has at least one (partial) solution and any partial solution can be extended to a solution of the system.

Evidently any freely solvable system M has the following *individual extension property*: for any $Y \subset X$ and $x \in X$ every partial solution φ: $Y \to \boldsymbol{R}$ can be extended to a partial solution φ_1: $Y \cup \{x\} \to \boldsymbol{R}$. The converse statement is also true.

PROPOSITION 1. *A system of linear inequalities over X is freely solvable iff it has the individual extension property and there exists at least one partial solution.*

Proof. Let $M = \{\langle a_t, u\rangle \leqslant a_t,\ t \in T\}$ be a system of linear inequalities over X with the individual extension property. Let $\{Y_\nu\}$ be a family of subsets of X, linearly ordered by the inclusion relation. If a function ψ is defined in the set $X_0 = \bigcup_\nu Y_\nu$ and, for every ν, $\psi|_{Y_\nu}$ is a partial solution of M, then ψ is also a partial solution of M. This statement is a consequence of the following evident fact: for every a_t with $s(a_t) \subset X_0$ there exists an index ν such that $s(a_t) \subset Y_\nu$. Thus the assumptions of the Kuratowski–Zorn lemma are satisfied for the family of all partial solutions ordered by the inclusion relation (we identify a function with its graph). Therefore there exists at least one maximal partial solution. This solution, ψ_1 in $X_1 \subset X$, is a partial solution of M and has no extension beyond X_1 which would be a partial solution of M. By the individual extension property, $X_1 = X$. Thus the proposition is proved.

Two systems of inequalities are called *equivalent* if the sets of their solutions are equal. Evidently if a system M is freely solvable and M_1 is an equivalent system, $M_1 \supset M$, then M_1 is also freely solvable. Let M be a system of linear inequalities and let $\tilde{M}$ be a system consisiting of all inequalities which are consequences of the system M. The system $\tilde{M}$ is equivalent to M. If there exists a freely solvable system equivalent to the system M, then the system $\tilde{M}$ is also freely solvable.

It is not true that every solvable system can be extended to an equivalent and freely solvable one, as the following example shows.

EXAMPLE 1. Let $\boldsymbol{N}$ be the set of natural numbers. Consider the system M over $\boldsymbol{N} \cup \{0\}$:

$$M = \{u(0) + u(n) \leqslant 0:\ n \in \boldsymbol{N}\}.$$

For any finite subset F of $\boldsymbol{N}$, every function u_0: $F \to \boldsymbol{R}$ can be extended to a solution of M; it is enough to put $u(0) = -u(n) = \min_{x \in F}(-u_0(x))$ $(n \in \boldsymbol{N} - F)$. This fact implies that the system $\tilde{M}_{\boldsymbol{N}}$ consists only of trivial inequalities. It is easy to verify that a function u_0: $\boldsymbol{N} \to \boldsymbol{R}$ has an extension to a solution of M iff $\sup_{n \in N} u_0(n) < \infty$. On the other hand, every function u_0: $\boldsymbol{N} \to \boldsymbol{R}$ is a partial solution of $\tilde{M}_{\boldsymbol{N}}$. This means that the system $\tilde{M}$ is not freely solvable.

Now we are going to describe a class of freely solvable systems. First we give a few definitions.

DEFINITION. Let $\lambda_i \geqslant 0$ $(i = 1, \ldots, n)$. The inequality

$$\langle \lambda_1 a_1 + \ldots. + \lambda_n a_n, u\rangle \leqslant \lambda_1 a_1 + \ldots. + \lambda_n a_n$$

is called a *linear combination of inequalities* $\langle a_i, u\rangle \leqslant a_i$ $(i = 1, \ldots, n)$ with coefficients $\lambda_1, \ldots, \lambda_n$. If $\lambda_1 a_1 + \ldots. + \lambda_n a_n \leqslant a$, then the inequality $\langle \lambda_1 a_1 + \ldots. + \lambda_n a_n, u\rangle \leqslant a$ is called an *elementary consequence* of the inequalities $\langle a_i, u\rangle \leqslant a_i$ $(i = 1, \ldots, n)$.

Let $M = \{\langle a_t, u\rangle \leqslant a_t:\ t \in T\}$ be a system of linear inequalities. By M^* we denote the set of all solutions of a system M. A system M_1 is said to be a *consequence of the system* M if $M_1^* \supset M^*$.

A system is called an *elementary consequence of* M if each of its inequalities is an elementary consequence of a finite subsystem of M. Two systems are called *elementarily equivalent* if each of them is an elementary consequence of the other.

We put

$$\operatorname{cone} M = \{\langle \lambda_1 a_{t_1} + \ldots + \lambda_n a_{t_n}, u\rangle \leqslant \lambda_1 a_1 + \ldots + \lambda_n a_n;\quad t_i \in T,\ \lambda_i \geqslant 0,\ i = 1, \ldots, n;\ n = 1, \ldots\}.$$

DEFINITION. Let n be a natural number. A system $M = \{\langle a_t, u\rangle \leqslant a_t,\ t \in T\}$ is called *n-full* if, for any $t_1, \ldots, t_n \in T$ and any non-negative numbers $\lambda_1, \ldots, \lambda_n$, not all of them zero, the inequality

$$\langle \lambda_1 a_{t_1} + \ldots + \lambda_n a_{t_n}, u\rangle \leqslant \lambda_1 a_{t_1} + \ldots\ \lambda_n a_{t_n}$$

is a consequence of the subsystem M_Y where $Y = s(\lambda_1 a_1 + \ldots + \lambda_n a_{t_n})$.

Instead of "2-full" we say "binary full". A system which is n-full for every n is called a *full system*.

PROPOSITION 2. *For any system M the system* cone$M$ *is full.*

Proof. The above is an immediate consequence of the formula cone(cone$M$) = cone$M$.

DEFINITION. Let X be a set with at least two elements. We say that a set of real functions on X, $A = \{a_t\colon t \in T\}$, *satisfies the symmetry condition at a point* $x \in X$ if, for every $t \in T$ such that the support of a_t is at least a two-element set, there exists a $t_0 \in T$ such that

$$\operatorname{sgn} a_{t_0}(x) = -\operatorname{sgn} a_t(x) \quad \text{and} \quad s(a_{t_0}) \subset s(a_t).$$

We say that the set A *satisfies the finiteness condition at a point* $x \in X$ if the subset $\{t \in T\colon a_t(x) \neq 0\}$ is finite.

If the set A satisfies the symmetry condition at every point of X, then we say that A *satisfies the symmetry condition.* The same applies to the finiteness condition.

If X is a one-element set, then the set of functions, $A = \{a_t\colon X \to \boldsymbol{R}, t \in T\}$, is in fact a set of real numbers. Such a set is said to *satisfy the symmetry condition* if for every $t \in T$ there exists a $t_0 \in T$ such that $\operatorname{sgn} a_{t_0} = -\operatorname{sgn} a_t$, i.e., there are two non-zero elements with opposite signs or all elements of A are zero.

In a natural way we apply the above definitions to systems

$$M = \{\langle a_t, u\rangle \leqslant \alpha_t\colon t \in T\}.$$

Remark. If a system M satisfies the symmetry condition, then the system cone$M$ does not, in general, satisfy this condition. But if for every $x \in X$ the system M_x satisfies the symmetry condition, then the system cone$M$ also satisfies it. If $A = -A$, i.e., if for every $t \in T$ there exists a $t_0 \in T$ such that $a_t = -a_t$, then cone$A$ = $-$cone$A$ and the symmetry condition is satisfied. In this case the corresponding system of inequalities is equivalent to the system of inequalities of the form

$$\alpha' \leqslant \langle a, u\rangle \leqslant \alpha''. \tag{1}$$

each.

LEMMA. *Put $M = \{a_t u \leqslant \alpha_t\colon t \in T\}$, where a_t, α_t are real numbers ($t \in T$). If the system M is binary full and satisfies the symmetry condition or is finite, then M is solvable iff all the inequalities of $M_\varnothing$ are true.*

Proof. Assume that each inequality of M is true. (The necessity is obvious.) Put $T_k = \{t \in T, \operatorname{sgn} a_t = k\}$ for $k = 0, \pm 1$. The symmetry condition implies the following alternative: the sets $T_{\pm 1}$ are both empty or both non-empty. If $T_{\pm 1} = \varnothing$, the lemma is trivial: any real number is a solution of M. Let $T_{\pm 1}$ be non-empty; then a real number u is a solution of M iff

$$\sup_{t \in T_{-1}} \frac{\alpha_t}{a_t} \leqslant u \leqslant \inf_{t \in T_1} \frac{\alpha_t}{a_t}. \tag{2}$$

Evidently, there exists a u satisfying (2) iff for every $t' \in T_1$, $t'' \in T_{-1}$ the inequality

$$\frac{\alpha_{t''}}{a_{t''}} \leqslant \frac{\alpha_{t'}}{a_{t'}}$$

holds, i.e.,

$$0 \leqslant a_{t''}\alpha_{t'} - a_{t'}\alpha_{t''}. \tag{3}$$

Inequality (3) is a linear combination of the inequalities $a_{t'}u \leqslant \alpha_{t'}$ and $a_{t''}u \leqslant \alpha_{t''}$ with coefficients $-a_{t''}$ and $a_{t'}$, respectively. By the binary fullness assumption inequality (3) is a consequence of $M_\varnothing$ and so it is true.

If the symmetry condition is not satisfied, one of the sets $T_{\pm 1}$ is empty and the other is non-empty. Assume that $T_{-1} = \varnothing$ and $T_1 \neq \varnothing$; then T_1 is finite and the set of solution of M is the interval $u \leqslant \min\limits_{t \in T_1} \frac{\alpha_t}{a_t}$. The remaining case is analogous. Thus the lemma is proved.

THEOREM 1. *Let M be a binary full system over X. If for every $x \in X$ the system M satisfies the symmetry or the finiteness condition at x, then M is freely solvable iff for every $x \in X$ the system M_x is solvable.*

Proof. Suppose that M_x are solvable: in particular, that the inequalities of $M_\varnothing$ are true.

To prove the solvability of M it is enough, by Proposition 1, to prove the individual extension property of M. Let a function $u_0\colon Y \to \boldsymbol{R}$ be a partial solution of the system $M = \{\langle a_t, u\rangle \leqslant \alpha_t\colon t \in T\}$ and $x \in X - Y$. We have to show that the following system over the one-point set $\{x\}$

$$a_t(x)\, u(x) \leqslant \alpha_t - \sum_{y \in Y - \{x\}} a_t(y)\, u_0(y), \qquad t \in T', \tag{4}$$

where

$$T' = \{t \in T\colon s(a_t) \subset Y \cup \{x\}\},$$

has a solution. It is easy to deduce from our assumptions that system (4) is binary full and each of its inequalities with empty support is true. If system (4) satisfies the symmetry condition, then by the lemma it is solvable. If it does not satisfy this condition, then the numbers $a_t(x)$, $t \in T'$, are all non-negative or all non-positive. Put $T'' = \{t \in T'\colon x \in s(a_t) \neq \{x\}\}$ and $T_0 = \{t \in T'\colon x \notin s(a_t)\}$. The system $M_x - M_\varnothing$ consisting of inequalities (4) corresponding to $t \in T - (T_0 \cup T'')$ is solvable. Assume that $a_t(x) \geqslant 0$ ($t \in T'$); then the set of solution of $M_x - M_\varnothing$ is an interval $\{u \leqslant \alpha\}$ and the set T'' is finite (by the finiteness condition assumption). Therefore system (4) is solvable and the set of its solution is an interval $\{u \leqslant \beta\}$.

COROLLARY 1. *If a system M over X is binary full and satisfies the symmetry of the finiteness condition at every point $x \in X$, and each system M_x satisfies these assumptions, then M is freely solvable iff each inequality of $M_\varnothing$ is true.*

Taking into account the remark after Proposition 2, we immediately get the statement.

COROLLARY 2. *Let M be a system over X. If for every $x \in X$ the system M_x satisfies the symmetry condition, then system M is solvable iff each inequality of* cone$M$ *with empty support is true.*

An analogous statement can be formulated for systems consisting of inequalities of the type (1).

As an illustration of Theorem 1 consider the following example.

EXAMPLE 2. Let X be a set with at least two elements. Denote by $\tilde{X}$ the set of all two-element subsets of X. A *pseudometric* is a function $d\colon \tilde{X} \to \boldsymbol{R}$ which satisfies the conditions

(a) $d(x, y) \geqslant 0$, (b) $d(x, y) \leqslant d(x, z) + d(z, y)$,

where x, y, z are any points of X, $x \neq y \neq z \neq x$. The following system of inequalities is elementarily equivalent to the previous system:

(5) (a) $-d(x, y) \leqslant 0$,

(b) $d(x_1, x_n) - d(x_1, x_2) - \ldots - d(x_{n-1}, x_n) \leqslant 0$,

where $x_1, \ldots, x_n \in X$, $x_i \neq x_j$ for $i \neq j$.

Using the well-known properties of a cyclic path (in graph theory sense), we easily check the binary fullness of system (5). Subsystem (5) (b) satisfies the symmetry condition; subsystem (5) (a) is solvable. Thus all the assumptions of Theorem 1 are satisfied and system (5) is freely solvable. In this way we have obtained a characterization of functions in a set $G \subset \tilde{X}$ which can be extended to a pseudometric in X. Namely, a function $d_0\colon G \to [0, \infty)$ $(G \subset \tilde{X})$ can be extended to a pseudometric in X iff for every cyclic path $\{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ in G the following inequality is satisfied:

$$d_0(x_1, x_n) \leqslant d_0(x_1, x_2) + \ldots + d_0(x_{n-1}, x_n).$$

Such a statement is not valid for metrics (unless X is a finite set).

Now we are going to discuss the free solvability of finite systems.

Let L be a linear subspace of $\boldsymbol{R}^n$. An elementary vector of L is defined as a non-zero vector of L whose support is minimal, i.e., does not properly contain the support of any other non-zero vector of L. Any subspace L has only finitely many elementary vectors, up to scalar multiplies ([1], [3], [4]).

Denote by Λ the set of all elementary vectors of L. We have span $\Lambda = L$ and moreover

(6) $$\boldsymbol{R}_+^n \cap L = \operatorname{cone}(\Lambda \cap \boldsymbol{R}_+^n),$$

where $\boldsymbol{R}_+$ is the set of non-negative numbers.

The proof of formula (6) is of a combinatorial character.

THEOREM 2. *Let $M = \{\langle a_i, u\rangle \leqslant \alpha_i\colon i = 1, \ldots, m\}$ be a finite system of linear inequalities over $\{1, \ldots, n\}$. For each set $\sigma \subset \{1, \ldots, n\}$ let Λ_σ be the set of elementary vectors $\lambda = (\lambda_1, \ldots, \lambda_m)$ of the subspace*

$$L_\sigma = \{\lambda \in \boldsymbol{R}^m\colon s(\lambda_1 a_1 + \ldots + \lambda_m a_m) \subset \sigma\}.$$

Then the system

(7) $$Q = \{\langle \lambda_1 a_1 + \ldots + \lambda_m a_m, u\rangle \leqslant \lambda_1 \alpha_1 + \ldots + \lambda_m \alpha_m,\ \lambda_i \geqslant 0,\ \lambda \in \bigcup_\sigma \Lambda_\sigma\}$$

is full and finite up to scalar multiples.

Proof. For every σ the set Λ_σ is finite up to scalar multiples, and so system (7) has the same property.

Let $\langle a, u\rangle \leqslant \alpha$ be an inequality of cone$M$. Put $s(a) = \sigma$. Taking in (6) $L = L_\sigma$, we have $\boldsymbol{R}_+^m \cap L_\sigma = \operatorname{cone}(\Lambda_\sigma \cap \boldsymbol{R}_+^m)$. By this formula the inequality $\langle a, u\rangle \leqslant \alpha$ is an elementary consequence of Q_σ. So the system Q is full.

Applying Corollary 1, we immediately obtain the statement.

COROLLARY 3. *If M is a finite system of linear inequalities, then there exists a finite system elementarily equivalent to M and full. A finite system M is solvable iff every inequality of* cone M *(or system (7)) with empty support is true.*

This means, in particular, that we have independently proved Farkas' lemma for finite systems. (Compare proof of Th. 22.6 in [4]. See also [2].)

Various generalizations of combinatorial character as well as for topological vector spaces will be presented elsewhere.

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On the isometries of spaces of Hölder continuous functions

by

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Abstract. Let M be a compact C^r-Riemannian ($1 \leqslant r \leqslant \infty$) or analytic Riemannian m-manifold and let $C^\alpha(M)$ denote the continuous functions which satisfy a Hölder condition with exponent α and with $\lambda^\alpha(M)$ the closure of the C^r-functions in $C^\alpha(M)$. In this paper we show that $\lambda^\alpha(M)$ is isometrically isomorphic to $\lambda^\beta(N)$ iff M is C^r (analytically) isometrically diffeomorphic to N, and $\alpha = \beta$.

1. Introduction. It is a classical result called the *Banach–Stone theorem* that if M and N are two compact Hausdorff spaces, then $C(M)$ is isometrically isomorphic to $C(N)$ iff M and N are homeomorphic. Thus isometries of $C(X)$ determine the topological type of the underlying space completely. This is not true of isomorphism. For example, if M and N are uncountable compact metric spaces, $C(M)$ and $C(N)$ are isomorphic (Milutin [8]).

Let M, N be two compact C^r, $r \geqslant 1$ (analytic) finite dimensional Riemannian manifolds and let $C^\alpha(M)$ and $C^\beta(N)$ denote the spaces of real valued Hölder continuous functions on M and N defined with respect to the Riemannian metrics on M and N (see definition below) with $0 < \alpha, \beta < 1$. Let $\lambda^\alpha(M)$ and $\lambda^\beta(N)$ denote the closure of the C^r functions in $C^\alpha(M)$ and $C^\beta(N)$, respectively. In [1] it is shown that $\lambda^\alpha(M)$ and $\lambda^\beta(N)$ are isomorphic for all M, N, α and β. In this paper we show that $\lambda^\alpha(M)$ ($C^\alpha(M)$) is isometrically isomorphic to $\lambda^\beta(N)$ ($C^\alpha(N)$) iff $\alpha = \beta$ and M is isometrically C^r (analytically) diffeomorphic to N (Theorem 3, § 4 and Remark 2, § 4). This generalizes work of deLeeuw [3] who proved a version of this result when M and N are both the unit intervals.

The author wishes to thank D. K. Elworthy for his thoughtful and stimulating advice.

2. Definitions and preliminaires. Let M be a compact C^r-Riemannian manifold (perhaps with boundary). The Riemannian structure induces a metric ϱ_M on M as follows.

Let I denote the unit interval and let $\sigma\colon I \to M$ be a C^1 mapping (once continuously differentiable). Then $\sigma'(t) \in T_{\sigma(t)}M$, where $\sigma'(t)$ is the

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