[3] R. T. Rockafellar, *The elementary vectors of a subspace of* $\boldsymbol{R}^n$, Combinatorial Mathematics and Its Applications, R. C. Bose ansd T. A. Dowling, eds., University of North Carolina Press, 1969, pp. 104–127.
[4] — *Convex Analysis*, Princeton University Press, 1970.

WARSAW UNIVERSITY
INSTITUTE OF MATHEMATICS

Received April 19, 1974, revised version April 9, 1975 (816)

STUDIA MATHEMATICA, T. LVII. (1976)

On the isometries of spaces of Hölder continuous functions

by

A. J. TROMBA* (Santa Cruz, Calif.)

Abstract. Let M be a compact C^r-Riemannian ($1 \leqslant r \leqslant \infty$) or analytic Riemannian m-manifold and let $C^\alpha(M)$ denote the continuous functions which satisfy a Hölder condition with exponent α and with $\lambda^\alpha(M)$ the closure of the C^r-functions in $C^\alpha(M)$. In this paper we show that $\lambda^\alpha(M)$ is isometrically isomorphic to $\lambda^\beta(N)$ iff M is C^r (analytically) isometrically diffeomorphic to N, and $\alpha = \beta$.

1. Introduction. It is a classical result called the *Banach–Stone theorem* that if M and N are two compact Hausdorff spaces, then $C(M)$ is isometrically isomorphic to $C(N)$ iff M and N are homeomorphic. Thus isometries of $C(X)$ determine the topological type of the underlying space completely. This is not true of isomorphism. For example, if M and N are uncountable compact metric spaces, $C(M)$ and $C(N)$ are isomorphic (Milutin [8]).

Let M, N be two compact C^r, $r \geqslant 1$ (analytic) finite dimensional Riemannian manifolds and let $C^\alpha(M)$ and $C^\beta(N)$ denote the spaces of real valued Hölder continuous functions on M and N defined with respect to the Riemannian metrics on M and N (see definition below) with $0 < \alpha, \beta < 1$. Let $\lambda^\alpha(M)$ and $\lambda^\beta(N)$ denote the closure of the C^r functions in $C^\alpha(M)$ and $C^\beta(N)$, respectively. In [1] it is shown that $\lambda^\alpha(M)$ and $\lambda^\beta(N)$ are isomorphic for all M, N, α and β. In this paper we show that $\lambda^\alpha(M)$ ($C^\alpha(M)$) is isometrically isomorphic to $\lambda^\beta(N)$ ($C^\alpha(N)$) iff $\alpha = \beta$ and M is isometrically C^r (analytically) diffeomorphic to N (Theorem 3, § 4 and Remark 2, § 4). This generalizes work of deLeeuw [3] who proved a version of this result when M and N are both the unit intervals.

The author wishes to thank D. K. Elworthy for his thoughtful and stimulating advice.

2. Definitions and preliminaires. Let M be a compact C^r-Riemannian manifold (perhaps with boundary). The Riemannian structure induces a metric ϱ_M on M as follows.

Let I denote the unit interval and let $\sigma\colon I \to M$ be a C^1 mapping (once continuously differentiable). Then $\sigma'(t) \in T_{\sigma(t)}M$, where $\sigma'(t)$ is the

* Research partially supported by NSF grant GP-39060.

derivative of σ at t. The map $t \to \|\sigma'(t)\|_{\sigma(t)} = \sqrt{\langle \sigma'(t), \sigma'(t)\rangle_{\sigma(t)}}$ is continuous. Thus we may take the integral $\int_0^1 \|\sigma'(t)\| dt$. Let $p, q \in M$. Then the metric ϱ_M is defined as $\varrho_M(p, q) = \inf_\sigma \int_0^1 \|\sigma'(t)\| dt$, where the infimum is taken over all C^1 paths σ with $\sigma(0) = p$ and $\sigma(1) = q$. It is well known (e.g., see [9]) that ϱ_M induces the given topology on M. Let $0 < \alpha < 1$. The space $C^\alpha(M)$ is defined to be the set of continuous real valued functions on M with

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{\varrho_M(x, y)^\alpha} < \infty.$$

The norm on $C^\alpha(M)$ is defined by

$$\|f\|_\alpha = \sup_{\substack{p \in M \\ x \neq y}} \left(|f(p)|, \frac{|f(x) - f(y)|}{\varrho_M(x, y)^\alpha} \right). \tag{1}$$

$\| \ \|_\alpha$ gives $C^\alpha(M)$ the structure of a real Banach space. The C^r functions are not dense in $C^\alpha(M)$, and we shall denote their closure by $\lambda^\alpha(M)$. The space $\lambda^\alpha(M)$ can be characterized as the set of functions f in $C^\alpha(M)$ with $|f(x) - f(y)| = o(\varrho_M(x, y)^\alpha)$. In addition in [1] it is shown that $\lambda^\alpha(M)$ is isomorphic to the space c_0 of sequences converging to 0.

It might be helpful to note (1) is not a "canonical" norm for C^α. One could, for example, define the norm by taking the sum

$$\sup_{p \in M} |f(p)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\varrho_M(x, y)^\alpha}.$$

The main result of this paper depends strongly on the choice of norm for C^α.

By an *isometry* T: $\lambda^\alpha(M) \to \lambda^\beta(N)$ we mean a linear map T with $\|Tf\|_\beta = \|f\|_\alpha$ for all $f \in \lambda^\alpha(M)$.

An *isometry* φ: $M \to N$ is a C^r diffeomorphism (a C^r homeomorphism with C^∞ inverse) with $D\varphi_x$: $T_x M \to T_{\varphi(x)} N$ (the derivative of φ from the tangent space of M at x to the tangent space of N at $\varphi(x)$) a linear isometry. It is not difficult to check that if φ: $M \to N$ is an isometry it must preserve distances; i.e., $\varrho_M(x, y) = \varrho_N(\varphi(x), \varphi(y))$.

Example of an isometry T. Let $\alpha = \beta$ and suppose that N is isometrically diffeomorphic to M with φ: $N \to M$ the isometry. Then T_i: $\lambda^\alpha(M) \to \lambda^\alpha(N)$, $i = 1, 2$ defined by

$$(T_i f)(y) = (-1)^i f(\varphi(y))$$

are isometries of $\lambda^\alpha(M)$ and $\lambda^\alpha(N)$.

We shall show that all isometries of $\lambda^\alpha(M)$ and $\lambda^\alpha(N)$ arise from isometries of the manifolds M and N in the above manner.

Before proceeding with Section 3 we shall state a proposition proved in [5] which we shall need later.

PROPOSITION 1. *The second dual $\lambda^\alpha(M)^{**}$ of $\lambda^\alpha(M)$ is naturally isometrically isomorphic to $C^\alpha(M)$ so that the isometry I: $\lambda^\alpha(M)^{**} \to C^\alpha(M)$ makes the following diagram commute*

$$\begin{tikzcd}
\lambda^\alpha(M) \arrow[r, "\mathrm{inc}"] \arrow[dr, "\mathrm{inc}"'] & \lambda^\alpha(M)^{**} \arrow[d, "I"] \\
 & C^\alpha(M)
\end{tikzcd}$$

where inc denotes the natural inclusions. In addition, identifying $(\lambda^\alpha)^{**}$ with C^α, $f_n \to f$ weak* iff f_n tends to f uniformly and $\|f_n\|_\alpha$ is bounded for all n.

In the following paragraphs we shall often identify $C^\alpha(M)$ with $\lambda^\alpha(M)^{**}$ and weak* convergence with uniform convergence with uniformly bounded C^α norm.

3. Extreme points of the unit sphere of $\lambda^\alpha(M)$. As in the Banach–Stone theorem the main tool will be the characterization of the extreme points of the unit sphere S^* of the dual space $\lambda^\alpha(M)^*$ of $\lambda^\alpha(M)$.

For each $p \in M$ and $(p, q) \in M \times M - \Delta$, where Δ is the diagonal of M, define $\delta_p \in \lambda^\alpha(M)^*$ and $\delta_{p,q} \in \lambda^\alpha(M)^*$ by

$$\delta_p(f) = f(p)$$

and

$$\delta_{p,q}(f) = \frac{f(p) - f(q)}{\varrho_M(p, q)^\alpha}.$$

Clearly, $\|\delta_p\| = 1$. If $\varrho_M(p, q) > 2^{1/\alpha}$, then

$$\|\delta_{p,q}\| = \sup_{\|f\|_\alpha \leqslant 1} \frac{|f(p) - f(q)|}{\varrho_M(p, q)^\alpha} \leqslant \frac{2}{\varrho_M(p, q)^\alpha} < 1.$$

So $\|\delta_{p,q}\| < 1$ if $\varrho_M(p, q) > 2^{1/\alpha}$. We shall show that for $\varrho_M(p, q) \leqslant 2^{1/\alpha}$, $\|\delta_{p,q}\| = 1$, and that the extreme points of the unit ball S^* of $\lambda^\alpha(M)^*$ are the valuations $\pm\delta_p$ for all p and $\delta_{p,q}$ for all p, q with $0 < \varrho_M(p, q) < 2^{1/\alpha}$. Thus if $\varrho_M(p, q) = 2^{1/\alpha}$, $\|\delta_{p,q}\| = 1$ but $\delta_{p,q}$ is *not* an extreme point. To see this, note that for such p, q, $\delta_{p,q} = \frac{1}{2}\delta_p - \frac{1}{2}\delta_q$ which shows that $\delta_{p,q}$ is not extreme.

Our first step will be to show that every extreme point of S^* must be either of the form δ_p or of the form $\delta_{p,q}$.

LEMMA 1. *Let the locally compact space X be the disjoint union of M with $(M\times M)-\Delta$ (Δ = diagonal of $M\times N$). There is an isometry of $\lambda^\alpha(M)$ onto a closed subspace of $C_0(X)$, the continuous functions on X vanishing at infinity.*

Proof. Define a map $\sim$: $\lambda^\alpha(M)\to C_0(X)$ as follows. Given $f\in\lambda^\alpha(M)$, define

$$\tilde f(z) = \begin{cases} f(z) & \text{if } z\in M,\\ \dfrac{f(p)-f(q)}{\varrho_M(p,q)^\alpha} & \text{if } z=(p,q)\in M\times M-\operatorname{diag}(M\times M);\end{cases}$$

$\sim$ is clearly an isometry into.

Let X be a locally compact Hausdorff space. Let $F\subset C_0(X)$ be a closed subspace and denote its dual by F^*. We have

LEMMA 2. *Let Φ be an extreme point of the unit sphere of F^*. Then there exists a $z\in X$ such that*

$$\text{either}\quad \Phi(f) = +f(z)\quad\text{or}\quad \Phi(f) = -f(z)\quad \text{for all } f\in F.$$

Proof. This follows from the Krein–Milman theorem. See Lemma V.8.6 of [4].

COROLLARY. *The extreme points of $S^*\subset\lambda^\alpha(M)^*$ must be of the form $\pm\delta_p$, $\delta_{p,q}$ for $\varrho_M(p,q)<2^{1/\alpha}$.*

Proof. By Lemmas 1 and 2 the extreme points must be of the form $\pm\delta_p$ or $\delta_{p,q}$. But we have already seen that $\delta_{p,q}$ is not an extreme point if $\varrho_M(p,q)^\alpha\geqslant 2$. (In fact, if $\varrho_M(p,q)^\alpha>2$, $\delta_{p,q}\notin S^*$.)

We must now show that the δ_p's and the $\delta_{p,q}$'s given in the above corollary are in fact extreme points. For the next proposition we will need the following standard inequalities

LEMMA 3. *Let τ and η be real positive numbers, and $0<\alpha<1$. Then $|\tau^\alpha-\eta^\alpha|\leqslant|\tau-\eta|^\alpha$ with $|\tau^\alpha-\eta^\alpha|<|\tau-\eta|^\alpha$ if τ and η are non-zero and $\tau\neq\eta$.*

PROPOSITION 2. *Let $p\in M$, and (p,q) be such that $\varrho_M(p,q)\leqslant 2^{1/\alpha}$. Then $\|\delta_p\| = \|\delta_{p,q}\| = 1$.*

Proof. We have already seen that $\|\delta_p\|=1$. We shall show $\|\delta_{p,q}\|=1$. To see this define $g\in C^\alpha(M)$ by

$$g(x) = \tfrac12\varrho_M(x,q)^\alpha-\tfrac12\varrho_M(x,p)^\alpha.$$

Note that

$$\begin{aligned}|g(x)| &= \tfrac12|\varrho_M(x,q)^\alpha-\varrho_M(x,p)^\alpha|\\ &\leqslant \tfrac12|\varrho_M(x,q)-\varrho_M(x,p)|^\alpha\\ &\leqslant \tfrac12\varrho_M(p,q)^\alpha\leqslant 1.\end{aligned}$$

Also that

$$\begin{aligned}(1)\qquad |g(x)-g(y)| &\leqslant \tfrac12|\varrho_M(x,q)^\alpha-\varrho_M(y,p)^\alpha|+\tfrac12|\varrho_M(x,p)^\alpha-\varrho_M(y,p)^\alpha|\\ &\leqslant \tfrac12\varrho_M(x,y)^\alpha+\tfrac12\varrho_M(x,y)^\alpha\\ &= \varrho_M(x,y)^\alpha.\end{aligned}$$

$\|\delta_{p,q}\|\leqslant 1$ by definition. Thus, $\delta_{p,q}(g)=1$ and $\|g\|_\alpha\leqslant 1$ completes the proof for C^α. By Proposition 1 of Section 2, there is a sequence $g_n\in\lambda^\alpha(M)$, $\|g_n\|\leqslant 1$ with $g_n\to g$ weak*. But $\delta_{p,q}(g_n)\to\delta_{p,q}(g)=1$. Thus $\|\delta_{p,q}\|=1$ as was claimed.

DEFINITION. Let X be a locally compact Hausdorff space. A function $\tilde f\in C(X)$ *peaks at* x_1 and x_2 if $|\tilde f(x_1)| = |\tilde f(x_2)| = 1$ and $|\tilde f(y)|<1$ if $x_1\neq y\neq x_2$. A function $\tilde f$ peaks at x if $|\tilde f(x)|=1$ and $|\tilde f(y)|<1$ for all $y\neq x$. A function $f\in C^\alpha(M)$ C^α peaks at a point $p\in M$ if $\tilde f$ (Lemma 1) peaks at p and f C^α peaks at $(p,q)\in M\times M-\Delta$ if $\tilde f$ peaks at (p,q) and (q,p).

LEMMA 4. *Given any $p\in M$ there is a function $f\in\lambda^\alpha(M)$ which C^α peaks at p and given any $(p,q)\in M\times M-\Delta$ with $\varrho_M(p,q)<2^{1/\alpha}$ there is a function $g\in C^\alpha(M)$ which C^α peaks at (p,q).*

Proof. The first part of this lemma is straightforward. The second part is not. Let $g(x) = \frac12\varrho_M(x,q)^\alpha-\frac12\varrho_M(x,p)^\alpha$ as in Proposition 2. Now $|g(x)|\leqslant\frac12\varrho_M(p,q)^\alpha<1$ for all x. Also from Proposition 2 we saw that

$$(2)\qquad |g(x)-g(y)|\leqslant\varrho(x,y)^\alpha\quad\text{if}\quad x\neq y.$$

For equality in (2) we must have equality at every step in inequality (1) of Proposition 2. From Lemma 3 it follows that

$$|\varrho_M(x,q)^\alpha-\varrho_M(y,q)^\alpha| = |\varrho_M(x,q)-\varrho_M(y,q)|^\alpha$$

iff either $\varrho_M(x,q)=\varrho_M(y,q)$ or $\varrho_M(x,q)=0$ or $\varrho_M(y,q)=0$. Also for the next step of inequality (1) $|\varrho_M(x,q)-\varrho_M(y,q)| = \varrho_M(x,y)$ iff either $\varrho_M(x,q)=\varrho_M(x,y)+\varrho_M(y,q)$ or $\varrho_M(y,q)=\varrho_M(x,y)+\varrho_M(x,q)$ which is impossible if $\varrho_M(x,q)=\varrho_M(y,q)$ (we are assuming $x\neq y$). Thus $\varrho_M(x,q)=0$ or $\varrho_M(y,q)=0$. Therefore $x=q$ or $y=q$. Similarly, we get $y=p$ or $x=p$. Consequently, since $x\neq y$, $x=p$, $y=q$ or *vice versa*. Hence

$$\frac{|g(x)-g(y)|}{\varrho_M(x,y)^\alpha}<1\quad\text{if}\quad (x,y)\neq\begin{cases}(p,q),\\ \text{or}\\ (q,p).\end{cases}$$

This shows that g C^α peaks at (p,q).

LEMMA 5. *Let $g\in C^\alpha(M)$ with $g(p)=0$. Then there exists a sequence $g_n\in\lambda^\alpha(M)$, $g_n\to g$ weak (*) such that each g_n vanishes on some neighborhood of p (depending on n). If $g(p)=g(q)=0$, $p\neq q$, then g_n can be chosen to vanish on neighborhoods of p and q.*

Proof. This follows from the fact that λ^α is weak* dense in C^α and that any function $h \in \lambda^\alpha$, $h(p) = 0$ is the limit in norm of functions in λ^α which vanish in a neighborhood of p. (See [9].) The proof for two points is analogous. It is not too hard to prove the lemma directly by working in local coordinates.

LEMMA 6. *Suppose $g \in C^\alpha(M)$ with $\delta_{p,q}(g) = 0$. Then there exists a sequence of $g_n \in \lambda^\alpha(M)$, $g_n \to g$ weak* so that for each n, $\delta_{p',q'}(g_n) = 0 = \delta_{q',p'}(g)$ for all (p', q') and (q', p') in some neighborhoods W_n and V_n of (p, q) and (q, p), respectively.*

Proof. Consider the function $f(x) = g(x) - g(p)$. Clearly, $f(p) = 0$, $f(q) = 0$. So f may be approximated by a sequence f_n weak* with f_n vanishing in some neighborhoods of p and q. Let $g_n = f_n + g(p)$.

THEOREM 1. *$\pm\delta_p$ and $\delta_{p,q}$, $\varrho_M(p, q) < 2^{1/\alpha}$ are extreme points of S^*.*

Proof. (a) Suppose $\delta_p = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$, $\varphi_i \in S^*$. Let $f \in \lambda^\alpha(M)$ with $f(p) = 1$ and $\|f\|_\alpha = 1$. Then $\delta_p(f) = 1 = \frac{1}{2}\varphi_1(f) + \frac{1}{2}\varphi_2(f)$. But $|\varphi_i(f)| \leqslant \|f\|_\alpha = 1$. Therefore $\varphi_i(f) = 1$ if $f(p) = 1$.

To show that $\varphi_i = \delta_p$ it suffices to show that $\varphi_i(g) = 0$ whenever $g(p) = 0$, $g \in \lambda^\alpha$. Suppose $g(p) = 0$, in fact, suppose g vanishes in a neighborhood U of p. Pick an f in $\lambda^\alpha(M)$ with $f(p) = 1$, $\|f\|_\alpha = 1$ and which C^α peaks at p. Then there is a $\lambda > 0$ so small that $\lambda \cdot \sup |\tilde{g}(x)| < 1 - \sup\limits_{x \in -U} |\tilde{f}(x)|$. Therefore $\|\lambda g - f\|_\alpha = 1$ and $(\lambda g + f)(p) = 1$, and so

$$\varphi_i(\lambda g + f) = 1 = \lambda\varphi_i(g) + \varphi_i(f) = \lambda\varphi_i(g) + 1.$$

Hence $\varphi_i(g) = 0$. If $g \in \lambda^\alpha$ does not vanish in a neighborhood of p, we can approximate g by $g_n \to g$ weak*, $\|g_n\| \leqslant 1$, with each g_n vanishing in a neighborhood of p. In fact, since $g \in \lambda^\alpha$, we can find such a sequence g_n with $\|g_n - g\|_\alpha \to 0$. From above it follows that $\varphi_i(g_n) = 0$ for all n and therefore that $\varphi_i(g) = 0$. This shows that δ_p (and hence $-\delta_p$) is extreme.

(b) Suppose $\delta_{p,q} = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$, $\varphi_i \in S^*$. Again it suffices to show that $\varphi_i(g) = 0$ whenever $\delta_{p,q}(g) = 0$. By Lemma 4 there is a function $f \in C^\alpha(M)$ which C^α peaks at (p, q), say $\delta_{p,q}(f) = 1$. Now technically $\varphi_i(f)$ is not defined since $f \in C^\alpha(M)$. But since we are identifying $C^\alpha(M)$ with $\lambda^\alpha(M)^{**}$, we can take this to be $\hat{f}(\varphi_i)$ where $\hat{f}$ is the image of the natural isometry Λ: $C^\alpha(M) \to \lambda^\alpha(M)^{**}$. From the fact that $1 = \hat{f}(\delta_{p,q}) = \delta_{p,q}(f) = \frac{1}{2}\varphi_1(f) + \frac{1}{2}\varphi_2(f)$ it follows that $\varphi_i(f) = 1$, $i = 1, 2$. Suppose first that $\delta_{p,q}(g) = 0$ and that $\delta_{p',q'}(g) = \delta_{q',p'}(g) = 0$ for all (p', q') and (q', p') in neighborhoods W of (p, q) and V of (q, p), respectively. Let λ be so small that

$$\lambda \sup_{x \in X - (U \cup V)} |\tilde{g}(x)| < 1 - \sup_{x \in X - (U \cup V)} |\tilde{f}(x)|.$$

From this we may conclude that $\|\lambda g + f\|_\alpha = 1$ with $\varphi_i(\lambda g + f) = 1 = \lambda\varphi_i(g) + \varphi_i(f)$. Thus $\varphi_i(g) = 0$.

In the case that we only have $\delta_{p,q}(g) = 0$ we can apply Lemma 6 to obtain a sequence $g_n \to g$ weak (*) with $\varphi_i(g_n) = 0$ for all n. Therefore $\varphi_i(g) = 0$ and $\varphi_i = \delta_{p,q}$.

We conclude Section 3 with a proposition which we have essentially proved in this section.

PROPOSITION 3. *We have $\|\delta_p - \delta_q\| = \varrho_M(p, q)^\alpha \|\delta_{p,q}\|$. Therefore if $\varrho_M(p, q) \leqslant 2^{1/\alpha}$,*

$$\|\delta_p - \delta_q\| = \varrho_M(p, q)^\alpha.$$

4. Isometries of Hölder spaces. Suppose T: $\lambda^\alpha(M) \to \lambda^\beta(N)$ is an isometry. Then T^*: $\lambda^\beta(N)^* \to \lambda^\alpha(M)^*$ is also an isometry and so takes extreme points of the unit sphere of $\lambda^\beta(N)^*$ to extreme points of the unit sphere of $\lambda^\alpha(M)^*$. Denote the extreme points of the former by Σ_N and of the latter by Σ_M. From the last section we know that $\Sigma_N = \{\pm\delta_x, \delta_{x,y}, 0 < \varrho_N(x, y) < 2^{1/\beta}\}$ and $\Sigma_M = \{\pm\delta_p, \delta_{p,q}, 0 < \varrho_M(x, y) < 2^{1/\alpha}\}$. Now $T^*\Sigma_N = \Sigma_M$ and so $T^*\delta_x$ is an extreme point of the type $\pm\delta_p$ (call this an *extreme point of the first type*) or of the type $\delta_{p,q}$ (call this an *extreme point of the second type*). We shall show that $T^*\delta_x = \pm\delta_p$ for some p. Moreover, T^* establishes a bijective correspondence between extreme points of the first type and a bijective correspondence between extreme points of the second type.

LEMMA 7. *$f \in \lambda^\alpha(M)$ is constant iff the set of real numbers $\{\Phi(f)$: $\Phi \in \Sigma_M\}$ consists only of at most three points.*

Proof. Straightforward.

LEMMA 8. *An isometry T: $\lambda^\alpha(M) \to \lambda^\beta(N)$ takes constant functions to constant functions; that is, T induces a bijection of the constant functions of M with the constant functions of N.*

Proof. Let $f \in \lambda^\alpha(M)$ be constant. By Lemma 7, Tf is constant iff $\{\Phi(Tf)$: $\Phi \in \Sigma_N\}$ consists of at most three points. But this set is equal to $\{(T^*\Phi)(f)$: $\Phi \in \Sigma_N\} = \{\tilde{\Phi}(f)$: $\tilde{\Phi} \in \Sigma_M\}$. By assumption the set on the right has at most three elements. Consequently, Tf is also constant if f is.

LEMMA 9. *$T^*\delta_{x,y} = \pm\delta_{p,q}$ for some (p, q).*

Proof. It suffices to show that $(T^*\delta_{x,y})(f) = 0$ for all constant f. But $(T^*\delta_{x,y})(f) = \delta_{x,y}(Tf) = 0$ for all constant f by Lemma 8.

Thus, T^* takes extreme points of type two bijectively onto extreme points of type two. Since T^* takes Σ_N bijectively onto Σ_M it must also take extreme points of type one bijectively onto extreme points of type one. So we get

PROPOSITION 4. *For each $x \in N$ there is a unique $p \in M$ with $T^*\delta_x = a(x)\delta_p$, where a: $N \to \{-1, 1\}$ is continuous.*

Proof. The first part has already been done. To see that a is continuous let $g \equiv 1$ be the constant function on M. Then $a(x) = \delta_x(Tg) = (Tg)(x)$.

For each $x \in N$ define $\Gamma(x) \in M$ by $T^*\delta_x = a(x)\,\delta_{\Gamma(x)}$. Since T is bijective, it follows that Γ is bijective. Since M and N are compact, it follows as in the Banach–Stone theorem that Γ is a homeomorphism. From this we can conclude, using a standard result in topology, that $\dim M = \dim N$.

PROPOSITION 5. *Given $z \in N$; if $x, y \in N$ are sufficiently close to z, then*

$$\varrho_N(x, y)^\beta = \varrho_M\big(\Gamma(x), \Gamma(y)\big)^a.$$

Proof. Since Γ is a homeomorphism, we can pick x and y close enough to z so that they are in the same component of N and $\varrho_N(x, y)^\beta < 2$ and $\varrho_M\big(\Gamma(x), \Gamma(y)\big)^a < 2$. Using Proposition 3 of Section III we have

$$\|T^*\delta_x - T^*\delta_y\| = \|\delta_x - \delta_y\| = \varrho_N(x, y)^\beta.$$

But

$$\|T^*\delta_x - T^*\delta_y\| = \|a(x)\,\delta_{\Gamma(x)} - a(y)\,\delta_{\Gamma(y)}\|.$$

Since x and y are in the same component of N and a is continuous, this is equal to $\|\delta_{\Gamma(x)} - \delta_{\Gamma(y)}\| = \varrho_M\big(\Gamma(x), \Gamma(y)\big)^a$, which concludes the proof of Proposition 5.

Before we show that $a = \beta$ we need a lemma.

LEMMA 10. *Let $\Gamma\colon U \to \boldsymbol{R}^m$ be a map, where $U \subset \boldsymbol{R}^n$ is open, which satisfies a Hölder condition with exponent $\gamma > 1$. That is, $\|\Gamma(x) - \Gamma(y)\| \leqslant \|x - y\|^\gamma$. Then Γ is constant.*

Proof. We shall show that all partial derivatives of Γ exist and are zero. Let $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$. Let $v_i = (0, 0, \ldots, 1, 0, \ldots, 0)$, where 1 is in the ith place. Then

$$\|\Gamma(x + tv_i) - \Gamma(x)\| \leqslant \mathrm{Const}\,|t|^\gamma.$$

So

$$\left\|\frac{\|\Gamma(x + tv_i) - \Gamma(x)}{t}\right\| \leqslant \mathrm{Const}\,|t|^{\gamma - 1}.$$

Therefore as $t \to 0$,

$$\lim_{t \to 0}\left\|\frac{\Gamma(x + tv_i) - \Gamma(x)}{t}\right\| = 0$$

which implies that $\partial\Gamma/\partial x_i(x) = 0$ for all i and for all x and Γ is constant.

THEOREM 2. *If $T\colon \lambda^a(M) \to \lambda^\beta(N)$ is an isometry, then $a = \beta$.*

Proof. We can assume without loss of generality that $a < \beta$. By Proposition 5 if x and y are sufficiently close to z

$$\varrho_N(x, y)^\beta = \varrho_M\big(\Gamma(x), \Gamma(y)\big)^a$$

or

$$(3) \qquad \varrho_M\big(\Gamma(x), \Gamma(y)\big) = \varrho_N(x, y)^\gamma, \qquad \gamma = \beta/a.$$

Use coordinate charts to pass to $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$, respectively, and note that locally Riemannian and Euclidean distances are metrically equivalent. Then (3) says that Γ satisfies locally a Hölder condition with exponent $\gamma > 1$. Thus (at least locally) by Lemma 10 Γ is constant. But this contradicts the fact that Γ is a homeomorphism, which means that $a = \beta$.

COROLLARY. *If $T\colon \lambda^a(M) \to \lambda^\beta(N)$ is an isometry and x and y are sufficiently close, then*

$$\varrho_M\big(\Gamma(x), \Gamma(y)\big) = \varrho_N(x, y).$$

Therefore Γ is locally distance preserving.

THEOREM 3. *If $T\colon \lambda^a(M) \to \lambda^\beta(N)$ is an isometry, then $a = \beta$ and M and N are C^r (analytically) isometrically diffeomorphic.*

Proof. By Theorem 2, $a = \beta$. Since Γ is locally distance preserving, it follows from the Steenrod–Myers theorem ([7]) that Γ is a C^r (analytic) local isometry; i.e., Γ is C^r (analytic) and $\|D\Gamma_x(v_x)\|_{\Gamma(x)} = \|v_x\|_x$ for all $v_x \in T_xN$. From the fact that Γ is a homeomorphism it follows that Γ is also a diffeomorphism. Thus Γ is a global isometry.

COROLLARY. *Γ is globally distance preserving; i.e., $\varrho_M\big(\Gamma(x), \Gamma(y)\big) = \varrho_N(x, y)$.*

Remarks and generalizations.

1. Although we proved this theorem in the case of real valued functions, there is essentially no additional difficulty in proving the same result for complex valued functions $\lambda^a_C(M)$. The extreme points of the unit sphere S^* of $\lambda^a_C(M)$ are functionals of the form δ_p, $\delta_{p,q}$, $\varrho_M(p, q)^a < 2$, with $\delta_p(f) = \lambda$, and

$$\delta_{p,q}(f) = \lambda\left(\frac{f(p) - f(q)}{\varrho_M(p, q)^a}\right), \qquad \text{where} \quad |\lambda| = 1.$$

2. For reasons motivated primarily by partial differential equations on manifolds the author is more interested in the spaces λ^a than C^a. As expected the main result of this paper holds for the C^a spaces, namely $C^a(M)$ isometrically isomorphic to $C^\beta(N)$ implies that $a = \beta$ and M is C^r (analytically) isometrically diffeomorphic to N. The extreme points of the unit sphere S^* of $C^a(M)^*$ will be of the form $\pm\delta_p$ and δ_ξ, where ξ lies in the Stone–Čech compactification of $M \times M - \Delta$. The proof that isometries carry extreme points of one type to the same type is the same as before.

3. A study of the Banach algebra structure of $C^a(M)$ and $\lambda^a(N)$, including a classification of the ideals in this algebra, has been carried out by Sherbert [11].

References

[1] R. Bonic, J. Frampton and A. Tromba, *Λ-manifolds*, J. Functional Analysis 3 (1969), pp. 310–320.

[2] Z. Ciesielski, *On the isomorphism of the spaces H^α and m*, Bull. Acad. Polon. Sci. 8, No. 4 (1960), pp. 217–222.

[3] K. deLeeuw, *Banach spaces of Lipschitz functions*, Studia Math. 21 (1961), pp. 55–56.

[4] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York 1958.

[5] J. Frampton and A. Tromba, *Spaces of Hölder continuous functions*, J. Functional Analysis 10 (1972), pp. 336–345.

[6] G. Glaeser, *Étude de quelques algèbres Tayloriennes*, J. Analyse Math. 6 (1958), pp. 1–125.

[7] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York 1967.

[8] A. Milutin, *On spaces of continuous functions*, Dissertation, Moscow State University, 1952.

[9] R. Palais, *Lusternik Schnirelmann category theory on Banach manifolds*, Topology 5 (1966), pp. 115–132.

[10] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), pp. 209–228.

[11] D. R. Sherbert, *The structure of ideas and point derivations in Banach algebras of Lipschitz functions*, Transactions AMS, Vol. 111 (1964), pp. 240–272.

UNIVERSITY OF CALIFORNIA
SANTA CRUZ

Received June 29, 1974 (856)

STUDIA MATHEMATICA, T. LVII, (1976)

An extension of a theorem of Rosenthal on operators acting from $l_\infty(\Gamma)$

by

L. DREWNOWSKI (Poznań)

Abstract. The theorem we prove in this paper, in a somewhat specialized form is as follows: Let Γ be an infinite set, T a continuous linear operator from $l_\infty(\Gamma)$ (or $c_0(\Gamma)$) into a topological vector space E, and suppose that the images by T of the unit vectors lie off some neighbourhood of the origin in E. Then there exists a subset Γ' of Γ with the same cardinality as Γ and such that $T|l_\infty(\Gamma')$ (resp., $T|c_0(\Gamma')$) is an isomorphism (= linear homeomorphism).

For E being a Banach space, this result is due to Rosenthal. For an arbitrary t.v.s. E and the standard c_0 and l_∞ spaces some results of the above form have been recently obtained by Kalton.

H. P. Rosenthal proved in [5] that

(R: Γ) *If an operator* T: $l_\infty(\Gamma)\to E$, *E being a Banach space, is such that* $T|c_0(\Gamma)$ *is an isomorphism, then there exists* $\Gamma'\subset\Gamma$ *with* $\operatorname{card}\Gamma' = \operatorname{card}\Gamma$ *such that* $T|l_\infty(\Gamma')$ *is an isomorphism.*

He established also an analogue of this for operators T: $c_0(\Gamma)\to E$ ([5], Theorem 3.4 and Remark 1 following it), and gave numerous interesting applications of those results to Banach space theory.

When $\Gamma = \boldsymbol{N}$, $\boldsymbol{N}$ the set of positive integers, then the results of Rosenthal can be stated in the form:

(R: $\boldsymbol{N}$) *If T is an operator from l_∞, or c_0, into E, then either $T(e_n)\to 0$ or there exists an infinite subset M of $\boldsymbol{N}$ such that* $T|l_\infty(M)$ (resp., $T|c_0(M)$) *is an isomorphism.*

Recently, in connection with the theory of the so-called exhaustive operators, N. J. Kalton investigated operators T acting from c_0, or l_∞, to an arbitrary topological vector space E [1]. He obtained an exact analogue of (R: $\boldsymbol{N}$) for T: $c_0\to E$ ([1], Theorem 2.3), and proved some special cases of (R: $\boldsymbol{N}$) for T: $l_\infty\to E$ ([1], Theorems 3.2, 3.3, 4.3). Kalton conjectured that the statement "If E contains no copy of l_∞ and T: $l_\infty\to E$ is a continuous operator, then $T(e_n)\to 0$ (and hence T is exhaustive)" should hold without any further restrictions on E.

In this paper we extend (R: Γ) (and hence also (R: $\boldsymbol{N}$)) to arbitrary topological vector spaces E. Moreover, our Theorem and its proof cover