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An extension of a theorem of Rosenthal on operators acting from $l_\infty(\Gamma)$

by

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Abstract. The theorem we prove in this paper, in a somewhat specialized form is as follows: Let Γ be an infinite set, T a continuous linear operator from $l_\infty(\Gamma)$ (or $c_0(\Gamma)$) into a topological vector space E, and suppose that the images by T of the unit vectors lie off some neighbourhood of the origin in E. Then there exists a subset Γ' of Γ with the same cardinality as Γ and such that $T|l_\infty(\Gamma')$ (resp., $T|c_0(\Gamma')$) is an isomorphism (= linear homeomorphism).

For E being a Banach space, this result is due to Rosenthal. For an arbitrary t.v.s. E and the standard c_0 and l_∞ spaces some results of the above form have been recently obtained by Kalton.

H. P. Rosenthal proved in [5] that

(R: Γ) *If an operator $T\colon l_\infty(\Gamma)\to E$, E being a Banach space, is such that $T|c_0(\Gamma)$ is an isomorphism, then there exists $\Gamma'\subset\Gamma$ with* $\operatorname{card}\Gamma' = \operatorname{card}\Gamma$ *such that $T|l_\infty(\Gamma')$ is an isomorphism.*

He established also an analogue of this for operators $T\colon c_0(\Gamma)\to E$ ([5], Theorem 3.4 and Remark 1 following it), and gave numerous interesting applications of those results to Banach space theory.

When $\Gamma = \boldsymbol{N}$, $\boldsymbol{N}$ the set of positive integers, then the results of Rosenthal can be stated in the form:

(R: $\boldsymbol{N}$) *If T is an operator from l_∞, or c_0, into E, then either $T(e_n)\to 0$ or there exists an infinite subset M of $\boldsymbol{N}$ such that $T|l_\infty(M)$* (resp., $T|c_0(M)$) *is an isomorphism.*

Recently, in connection with the theory of the so-called exhaustive operators, N. J. Kalton investigated operators T acting from c_0, or l_∞, to an arbitrary topological vector space E [1]. He obtained an exact analogue of (R: $\boldsymbol{N}$) for $T\colon c_0\to E$ ([1], Theorem 2.3), and proved some special cases of (R: $\boldsymbol{N}$) for $T\colon l_\infty\to E$ ([1], Theorems 3.2, 3.3, 4.3). Kalton conjectured that the statement "If E contains no copy of l_∞ and $T\colon l_\infty\to E$ is a continuous operator, then $T(e_n)\to 0$ (and hence T is exhaustive)" should hold without any further restrictions on E.

In this paper we extend (R: Γ) (and hence also (R: $\boldsymbol{N}$)) to arbitrary topological vector spaces E. Moreover, our Theorem and its proof cover both the cases of $l_\infty(\Gamma)$ and $c_0(\Gamma)$ simultaneously, and actually even more is proved. Our proof of $(R\colon \Gamma)$, though having in common with the proof of the basic Lemma 1.1 in [5] the use of cardinal and ordinal numbers and transfinite induction, seems to be much simpler and clearer. And, after some rather obvious changes, it becomes a considerably simpler proof of that lemma itself; see the Remark at the end of this paper. Note also that the only essential property of $l_\infty(\Gamma)$ we use is that $\|x+y\|\leqslant 1$ whenever $\|x\|\leqslant 1$, $\|y\|\leqslant 1$ and x, y have disjoint supports.

We give also an independent proof of $(R\colon N)$, stated below as Corollary 2, mainly because of its simplicity. (In connection with this, note that the well-known result of Phillips that there is no projection of l_∞ onto c_0 is a trivial consequence of $(R\colon N)$.) The precompactness argument used in the first part of this proof stems directly from [1].

In what follows, Γ denotes an infinite set, $l_\infty(\Gamma)$ the Banach space of all bounded scalar-valued functions on Γ under the sup norm, and X a fixed vector subspace of $l_\infty(\Gamma)$ such that

$$e_\gamma\in X \quad \text{for all } \gamma\in\Gamma,$$

where e_γ is the γth unit vector of $l_\infty(\Gamma)$ (i.e., $e_\gamma(\alpha)=1$ if $\alpha=\gamma$ and 0 otherwise), and

$$x\chi_\Delta\in X \quad \text{whenever } x\in X \text{ and } \Delta\subset\Gamma,$$

where χ_Δ is the characteristic function of Δ.

If $\Delta\subset\Gamma$, then $X(\Delta)$ denotes the subspace $\{x\chi_\Delta\colon x\in X\}$ of X, and $B(\Delta)$ the unit ball of $X(\Delta)$; $B=B(\Gamma)$.

The most important examples of such spaces X are $l_\infty(\Gamma)$ itself, its closed subspace $c_0(\Gamma)=\{x\colon \{\gamma\in\Gamma\colon |x(\gamma)|>\varepsilon\}$ is finite for every $\varepsilon>0\}$, and its subspace $c_{00}(\Gamma)=\{x\colon s(x)$ is finite$\}$. Two other examples are the spaces $X=\{x\colon \operatorname{card}s(x)\leqslant\mathfrak{n}\}$ and $X=\{x\colon \operatorname{card}s(x)<\mathfrak{n}\}$, where $\mathfrak{n}$ is a fixed infinite cardinal number $\leqslant\operatorname{card}\Gamma$ and $s(x)=\{\gamma\in\Gamma\colon x(\gamma)\neq 0\}$.

Let us note that the spaces X in these examples have the property that if $\Delta\subset\Gamma$ is such that $\operatorname{card}\Delta=\operatorname{card}\Gamma$, then $X(\Delta)$ is isomorphic to X.

E will denote a Hausdorff topological vector space and $\mathscr{U}$ a base of open balanced neighbourhoods of 0 in E. If $W\subset E$ and $n\in N$, then $W^{(n)}$ will denote the set $W+\ldots+W$ (n copies of W).

THEOREM. *Let $T\colon X\to E$ be a continuous linear operator. Suppose that for some neighbourhood U of 0 in E the set Γ'' of all $\gamma\in\Gamma$ such that*

$$(+)\qquad T(e_\gamma)\notin U$$

is infinite. Then there exists a subset Γ' of Γ'' with $\operatorname{card}\Gamma'=\operatorname{card}\Gamma''$ such that $T|X(\Gamma')$ is an isomorphism.

Proof. There will be no loss of generality if we assume that $\Gamma''=\Gamma$. In our proof we will represent Γ as a set of ordinal numbers. If α is an ordinal number, then P_α will denote the set of all ordinals less than α. Let $\mathfrak{m}=\operatorname{card}\Gamma$, and let μ be the least ordinal number α with $\operatorname{card}P_\alpha=\mathfrak{m}$. We may and will assume that

$$\Gamma=P_\mu.$$

For each $\alpha<\mu$ let $F_\alpha=\{\beta\colon \alpha\leqslant\beta<\mu\}$.

Let $V\in\mathscr{U}$ be such that

$$V+V\subset U;$$

then choose $r\in N$ for which

$$T(B)\subset rV,$$

and finally take $W\in\mathscr{U}$ such that

$$W^{(r)}\subset rV.$$

We shall first prove that

(I) *There is a subset Δ of Γ with $\operatorname{card}\Delta=\mathfrak{m}$ such that for every $\sigma\in\Delta$*

$$(*)\qquad T(e_\sigma+y)\notin W$$

if $y\in B(\Delta\cap F_{\sigma+1})$.

To prove this statement we need the following simple observation:

(1) *If $\sigma<\mu$ and $\Delta_1,\ldots,\Delta_r$ are disjoint subsets of $F_{\sigma+1}$, then there is an i, $1\leqslant i\leqslant r$, such that $(*)$ holds whenever $y\in B(\Delta_i)$.*

For otherwise we could find for each i an element $y_i\in B(\Delta_i)$ such that $T(e_\sigma+y_i)\in W$. Then, writing $y=\sum_{i=1}^{r}y_i$, we would have $y\in B$ and

$$T(re_\sigma+y)\in W^{(r)}\subset rV.$$

Since $T(y)\in rV$, it follows immediately that $T(e_\sigma)\in U$, and this contradicts our assumption $(+)$.

Now let $\mathscr{G}$ be a family of subsets of Γ such that $\operatorname{card}\mathscr{G}>\mathfrak{m}$, $\operatorname{card}G=\mathfrak{m}$ for each $G\in\mathscr{G}$, and $\operatorname{card}(G_1\cap G_2)<\mathfrak{m}$ if $G_1,G_2\in\mathscr{G}$ and $G_1\neq G_2$ (see [6] and [5]).

Fix a $\sigma<\mu$ and consider any r distinct members $G_1,\ldots,G_r$ of $\mathscr{G}$. For sufficiently large τ, $\sigma<\tau<\mu$, the sets $H_j\equiv G_j\cap F_\tau$, $j=1,\ldots,r$, are disjoint, each of them is of cardinality $\mathfrak{m}$ and $\operatorname{card}(G_j\setminus H_j)<\mathfrak{m}$. By (1) there is an i, $1\leqslant i\leqslant r$, such that $(*)$ holds for all $y\in B(H_i)$.

Let $\mathscr{G}_\sigma$ be the subfamily of $\mathscr{G}$ consisting of all those $G\in\mathscr{G}$ for which there is $\tau>\sigma$, $\tau<\mu$, such that $(*)$ holds if $y\in B(G\cap F_\tau)$. It is clear from what was said just before that $\operatorname{card}(\mathscr{G}\setminus\mathscr{G}_\sigma)<r$. Hence the family

$$\bigcap_{\sigma<\mu}\mathscr{G}_\sigma$$

is nonempty, so let H be any of its members. It is obvious that H has the following property:

(2) *For each $\sigma < \mu$ there exists $\tau \in H$, $\tau > \sigma$, such that $T(e_\sigma + y) \notin W$ if $y \in B(H \cap F_\tau)$.*

From this we easily deduce the existence of an increasing transfinite sequence $(\eta(\alpha)\colon \alpha < \mu)$ with terms in H such that

$$T(e_{\eta(\alpha)} + y) \notin W \quad \text{if} \quad y \in B(\{\eta(\gamma)\colon \alpha < \gamma < \mu\}).$$

Indeed, put $\eta(0) = \min H$ and suppose that for some $\alpha < \mu$ we have already determined all the terms $\eta(\gamma)$, where $\gamma < \alpha$, in such a way that $\eta(\gamma) \in H$, $\eta(\gamma_1) < \eta(\gamma_2)$ if $\gamma_1 < \gamma_2 < \alpha$, and $T(e_{\eta(\gamma)} + y) \notin W$ for all $y \in B(H \cap \cap F_{\eta(\gamma+1)})$ provided that $\gamma + 1 < \alpha$.

Then choose τ in H so that $\eta(\gamma) < \tau$ for all $\gamma < \alpha$ and, in case α has the predecessor $\alpha - 1$, so that τ satisfies the condition in (2) for $\sigma = \eta(\alpha - 1)$. Then set $\eta(\alpha) = \tau$. This completes the inductive definition of $(\eta(\alpha)\colon \alpha < \mu)$.

It is evident that the set $\Delta = \{\eta(\alpha)\colon \alpha < \mu\}$ is as required in (I).

Since $\operatorname{card} \Delta = \mathfrak{m}$, to simplify notation we may again identify Δ with P_μ. Under this convention, for each $\alpha < \mu$,

$$T(e_\alpha + y) \notin W \quad \text{if} \quad y \in B(F_{\alpha+1}).$$

Now take $V_1 \in \mathcal{U}$, then choose $s \in \boldsymbol{N}$, and finally $W_1 \in \mathcal{U}$ so that

$$V_1 + V_1 \subset W, \quad T(B) \subset sV_1, \quad W_1^{(s)} \subset sV_1.$$

We are going to prove now that

(II) *There is a subset Γ' of Δ with $\operatorname{card} \Gamma' = \mathfrak{m}$ such that for each $\alpha \in \Gamma'$*

$$(**) \qquad T(x + e_\alpha + y) \notin W_1$$

if $x \in B(\Gamma' \cap P_\alpha)$ and $y \in B(F_{\alpha+1})$.

Similarly as before we easily check that

(3) *If $\alpha < \mu$ and $A_1, \ldots, A_s$ are disjoint subsets of P_α, then there is a k, $1 \leqslant k \leqslant s$, such that $(**)$ holds for $x \in B(A_k)$ and $y \in B(F_{\alpha+1})$.*

Let

$$A_1^\sigma = \{0\}, \quad A_2^\sigma = \{1\}, \ \ldots, \ A_{\sigma+1}^\sigma = \{\sigma\} \quad \text{and} \quad A_i^\sigma = \emptyset \quad \text{for} \quad \sigma + 1 < i \leqslant s,$$

for $\sigma = 0, 1, \ldots, s-1$.

Suppose that the s-tuples

$$A_1^\sigma, \ldots, A_s^\sigma$$

have been already defined for all $\sigma < \tau$, where $s \leqslant \tau < \mu$, in such a way that

(a) $\bigcup_{i=1}^{s} A_i^\sigma = P_{\sigma+1}$ and $A_i^\sigma \cap A_j^\sigma = \emptyset$ if $i \neq j$;

(b) $A_i^{\sigma'} \subset A_i^\sigma$ if $\sigma' < \sigma$, $i = 1, \ldots, s$;

(c) If $\alpha \in A_i^\sigma$, then $(**)$ holds for all $x \in B(A_i^\sigma \cap P_\alpha)$, and $y \in B(F_{\alpha+1})$.

Let

$$C_i = \bigcup_{\sigma < \tau} A_i^\sigma, \quad i = 1, \ldots, s.$$

Then $\bigcup_{i=1}^{s} C_i = P_\tau$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. By (3) there is a k, $1 \leqslant k \leqslant s$, such that

$$T(x + e_\tau + y) \notin W_1$$

if $x \in B(C_k)$ and $y \in B(F_{\tau+1})$. Then we define

$$A_i^\tau = \begin{cases} C_i & \text{for} \quad i \neq k, \\ C_k \cup \{\tau\} & \text{for} \quad i = k; \ i = 1, \ldots, s. \end{cases}$$

This completes the inductive definition of a transfinite sequence of s-tuples $A_1^\sigma, \ldots, A_s^\sigma$ $(\sigma < \mu)$ such that (a), (b), (c) hold for all $\sigma < \mu$.

Let

$$\Delta_i = \bigcup_{\sigma < \mu} A_i^\sigma, \quad i = 1, \ldots, s.$$

Then $\Delta = \Delta_1 \cup \ldots \cup \Delta_s$, hence for some k, $1 \leqslant k \leqslant s$, $\operatorname{card} \Delta_k = \mathfrak{m}$. Write $\Gamma' = \Delta_k$. Then, for each $\alpha \in \Gamma'$, $(**)$ holds for all $x \in B(\Gamma' \cap P_\alpha)$ and $y \in B(\Gamma' \cap F_{\alpha+1})$. Hence, since W_1 is balanced, if $z \in B(\Gamma')$ is such that $|z(\alpha)| = 1$ for some $\alpha \in \Gamma'$, then $T(z) \notin W_1$. It follows easily that $T(z) \notin W_1$ for each $z \in X(\Gamma')$ with $\|z\|_\infty = 1$. This implies that $T | X(\Gamma')$ is an isomorphism.

Corollary 1. *Let $T\colon l_\infty(\Gamma) \to E$ be a continuous linear operator such that $T | c_0(\Gamma)$ is an isomorphism. Then there exists a subset Γ' of Γ with $\operatorname{card} \Gamma' = \operatorname{card} \Gamma$ such that $T | l_\infty(\Gamma')$ is an isomorphism.*

The next corollary is a particular case $(\Gamma = \boldsymbol{N})$ of the Theorem. However, since its independent proof though somewhat similar is considerably simpler than the proof of the Theorem, we find it worth presenting here.

Corollary 2. *Let X be either l_∞ or c_0, and let $T\colon X \to E$ be a continuous linear operator. Then exactly one of the following two possibilities holds:*

(i) $T(e_n) \to 0$,

or

(ii) *there exists an infinite subset M of $\boldsymbol{N}$ such that $T | X(M)$ is an isomorphism.*

Proof. Suppose (i) does not hold. Then $F = \{T(e_n)\colon n \in \boldsymbol{N}\}$ is not precompact.

In fact, if (i) is false, there is $U \in \mathcal{U}$ such that

$$T(e_n) \notin U$$

for infinitely many n, and we may suppose that for all $n \in \boldsymbol{N}$. Let V, r, W be chosen as in the proof of the Theorem. Then, assuming that F is precompact, we can find a finite subset F_0 of F such that

$$F \subset F_0 + W.$$

Hence for some m the set $T(e_m)+W$ contains infinitely many members of F, say $T(e_{n_1}), T(e_{n_2}), \ldots (n_1 < n_2 < \ldots)$. Then

$$T\Big(\sum_{i=1}^{r} e_{n_i}\Big) \in rT(e_m) + W^{(r)} \subset rT(e_m) + rV$$

and hence

$$rT(e_m) \in T(B) + rV \subset rU,$$

so that $T(e_m) \in U$. A contradiction, for we have assumed that $F \cap U = \emptyset$.

Now, since F is not precompact, there is $U \in \mathscr{U}$ (possibly different from the U we had above) such that for any compact set K in E the set $K+U$ does not contain F. In particular, for any $n \geqslant 0$ there is $m > n$ such that $T(e_m) \notin T(B_n) + U$, where $B_n = B(\{1, \ldots, n\})$ for $n \geqslant 1$ and $= \{0\}$ for $n = 0$.

Without loss of generality we may assume that

$$(*') \qquad T(e_{n+1}) \notin T(B_n) + U, \quad n = 0, 1, \ldots$$

Let V, r, W be chosen as in the proof of the Theorem. We shall define inductively a decreasing sequence (A_i) of infinite subsets of N such that the sequence

$$m_i = \inf A_i$$

is strictly increasing and

$$(**') \qquad T(x + e_{m_i} + y) \notin W$$

whenever $x \in B(\{m_1, \ldots, m_{i-1}\})$ (or $x = 0$ if $i = 1$) and $y \in B(A_{i+1})$; $i = 1, 2, \ldots$

Set $A_1 = N$ and suppose that the infinite subsets $A_1 \supset \ldots \supset A_k$ of N have been already chosen so that $1 = m_1 < \ldots < m_k$ and $(**')$ holds for $i = 1, \ldots, k-1$; $k \geqslant 2$. Then consider any decomposition of $A_k \setminus \{n_k\}$ into r infinite and mutually disjoint sets $C_1, \ldots, C_r$. As in the proof of the Theorem we easily deduce from $(*')$ that there is j, $1 \leqslant j \leqslant r$, such that $T(x + e_{m_k} + y) \notin W$ if $x \in B(\{m_1, \ldots, m_{k-1}\})$ and $y \in B(C_j)$. Then we define $A_{k+1} = C_j$. The set $M = \{m_1, m_2, \ldots\}$ is as required in Corollary 2.

Remark. The proof of the Theorem can easily be modified so that it will become a considerably simplified proof of Rosenthal's basic Lemma 1.1 in [5]. We shall briefly indicate how this can be done. Without loss of generality we can formulate this lemma as follows:

(L) *Let $\{m_a\colon a \in \Gamma\}$ be a family of finitely additive positive measures on the power set of Γ such that* $\sup\{m_a(\Gamma)\colon a \in \Gamma\} = a < \infty$. *Then, for all* $\varepsilon > 0$, *there exists $\Gamma' \subset \Gamma$ with* $\operatorname{card}\Gamma' = \operatorname{card}\Gamma = \mathfrak{m}$ *such that*

$$m_a(\Gamma' \setminus \{a\}) < \varepsilon \quad \textit{for all } a \in \Gamma'.$$

Choose $r \in N$ so that $a < r(\varepsilon/2)$. Then observe that if $a \in \Gamma$ and $A_1, \ldots, A_r$ are disjoint subsets of Γ, there is i such that $m_a(A_i) < \varepsilon/2$. Given $\sigma < \mu$, consider the subfamily $\mathscr{G}_\sigma$ of $\mathscr{G}$ consisting of all those $G \in \mathscr{G}$ for which there is $\tau > \sigma$ such that $m_\sigma(G \cap F_\tau) < \varepsilon/2$. Then $\operatorname{card}(\mathscr{G} \setminus \mathscr{G}_\sigma) < r$. Take any H in $\bigcup_{\sigma<\mu} \mathscr{G}_\sigma$. Then, via a transfinite sequence with terms in H, we obtain a subset Δ of H with $\operatorname{card}\Delta = \mathfrak{m}$ such that $m_a(\Delta \cap F_{a+1}) < \varepsilon/2$ for each $a \in \Delta$. We identify Δ with P_μ so that we have $m_a(F_{a+1}) < \varepsilon/2$ for each $a < \mu$. Then we continue quite similarly as in the part (II) of the proof of the Theorem (take $s = r$). (In particular, condition (c) should be replaced by the following one: If $a \in A_i^\sigma$, then $m_a(A_i^\sigma \cap P_a) < \varepsilon/2$.) This gives us $\Gamma' \subset \Delta$ with $\operatorname{card}\Gamma' = \mathfrak{m}$ such that $m_a(\Gamma' \cap P_a) < \varepsilon/2$ for all $a \in \Gamma'$, and Γ' is as required in (L).

Added May 4, 1975. The referee has kindly pointed out a recent paper by J. Kupka [2], where a much simpler proof of Rosenthal's Lemma is given.

Let us also mention that a variation of the argument used here has been recently applied by I. Labuda ([3], [4]) to obtain some interesting results on finitely additive vector measures and some relevant classes of topological vector spaces.

Added in proof. On combining the techniques of this paper with those of [2], the author has found in the meantime an extremely simple proof of the Theorem, see *Un théorème sur les opérateurs de $l_\infty(\Gamma)$*, C. R. Acad. Sc. Paris, Sér. A, 281 (1975), pp. 967–969.

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