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Composition operators on F^+

by

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Abstract. Let $U = \{z: |z| < 1\}$. We denote by F^+ the space of holomorphic functions $f(z) = \sum a_n z^n$ in U for which $\|f\|_c = \sum |a_n| \exp[-c\sqrt{n}] < \infty$ for all $c > 0$. F^+ with the above family of (semi) norms is a Fréchet space containing N^+ as a dense subspace. In the paper we consider composition operators of F^+ into F^+ and H^p, $0 < p < \infty$. We give several necessary and sufficient conditions for a composition operator C_φ to map F^+ into H^p, $0 < p < \infty$. Using these result we show that if C_φ: $F^+ \to H^p$ for some p, then C_φ: $F^+ \to H^q$ for all q, $0 < q < \infty$, and that C_φ is compact. We also give an example of a holomorphic function φ: $U \to U$ such that the composition operator C_φ maps F^+ into H^p for all p, $0 < p < \infty$, but not into H^∞. Some results concerning multiplicative linear functionals and maximal ideals in F^+ are also included.

1. Introduction. Let U denote the unit disc $\{|z| < 1\}$ in $\boldsymbol{C}$. A holomorphic function $f(z)$ in U belongs to the class N of functions of bounded characteristic if

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty.$$

A function $f \in N$ is said *to belong to the class* N^+ if

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta,$$

where $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on $|z| = 1$. For, $f, g \in N^+$, define

$$\varrho(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \log\big(1 + |f(e^{i\theta}) - g(e^{i\theta})|\big)\, d\theta. \tag{1.1}$$

The space N^+ with the metric given by (1.1) is an F-algebra, i.e., an F-space with continuous multiplication [6]. N^+ is neither locally convex nor locally bounded but has sufficiently many continuous linear functionals to separate points in N^+ [6].

The class F^+ consists of those functions $f(z)$ holomorphic in U which satisfy, for any $c > 0$,

$$(1.2)\qquad \|f\|_{F_c} = \int_0^1 \exp\left[\frac{-c}{1-r}\right] M(r,f)\,dr < \infty,$$

where

$$(1.3)\qquad M(r,f) = \max_{|z|=r} |f(z)|.$$

Also, for a function $f(z) = \sum a_n z^n \in F^+$, define

$$(1.4)\qquad \|f\|_c = \sum_{n=0}^{\infty} |a_n| \exp[-c\sqrt{n}] \quad (c > 0).$$

By [7], Theorem 2, the (semi) norms given by (1.2) and (1.4) are equivalent and F^+ with either of the above family of seminorms is a Fréchet space containing N^+ as a dense subspace. If f and g are in F^+, it is clear that for any $c > 0$,

$$(1.5)\qquad \|fg\|_c \leqslant \|f\|_{c/2}\|g\|_{c/2}.$$

Hence F^+ is closed under multiplication. Furthermore, (1.5) also shows that multiplication is continuous in F^+.

In § 2 and § 3 we extend some of our results in [4] to F^+. In § 2 we show that every multiplicative linear functional on F^+ is continuous and is given by point evaluation at some $\lambda \in U$. Consequently, if $\mathscr{M}_\lambda = \{f \in F^+ : f(\lambda) = 0\}$, $\mathscr{M}_\lambda$ is a closed maximal ideal in F^+. In [9] N. Yanagihara has shown that if M is a maximal ideal in F^+, $M = \mathscr{M}_\lambda$ for some $\lambda \in U$ if and only if M is closed in the topology of uniform convergence on compact subsets of U. Using a result of Arens [1], we prove that it suffices to assume that M is closed in F^+. In § 3 we show that every ring homomorphism of F^+ into F^+ is given by a composition operator and that every composition operator is continuous.

In § 4 we consider composition operators of F^+ into H^p, $0 < p < \infty$. We first give several equivalent necessary and sufficient conditions for a composition operator C_φ to map F^+ into H^p. Using these results we show that if the composition operator C_φ maps F^+ into H^p for some p, then $C_\varphi\colon F^+ \to H^q$ for all q, $0 < q < \infty$, and C_φ is a compact operator. We also give a characterization of the compact composition operators of F^+ into F^+. Some of the results are then applied to composition operators on N^+.

2. Multiplicative linear functionals and maximal ideals. For $\lambda \in U$, define γ_λ on F^+ by

$$(2.1)\qquad \gamma_\lambda(f) = f(\lambda).$$

It is clear that γ_λ is a multiplicative linear functional on F^+. By [7], Theorem 1,

$$(2.2)\qquad |f(\lambda)| \leqslant (8/ce^2)\exp\left[\frac{2c}{1-|\lambda|}\right]\|f\|_{F_c} \quad (c > 0),$$

where $\|\ \|_{F_c}$ is given by (1.2). Consequently, γ_λ is a continuous multiplicative linear functional on F^+ for every $\lambda \in U$. For each $\lambda \in U$, define

$$(2.3)\qquad \mathscr{M}_\lambda = \{f \in F^+ : f(\lambda) = 0\}.$$

Also, let $(z-\lambda)F^+ = \{(z-\lambda)f(z) : f \in F^+\}$. Then $(z-\lambda)F^+ \subset \mathscr{M}_\lambda$. However, if $f \in \mathscr{M}_\lambda$, then one can easily show that $f(z) = (z-\lambda)g(z)$ with $g \in F^+$. Therefore

$$(2.4)\qquad \mathscr{M}_\lambda = (z-\lambda)F^+.$$

Since $\mathscr{M}_\lambda$ is the kernel of a continuous multiplicative linear functional on F^+, $\mathscr{M}_\lambda$ is a closed maximal ideal in F^+.

THEOREM 1. *If γ is a (nontrivial) multiplicative linear functional on F^+, then there exists $\lambda \in U$ such that $\gamma(f) = f(\lambda)$ for every $f \in F^+$ and hence γ is continuous.*

Proof. Let $\lambda = \gamma(z)$. Then $\gamma(z-\lambda) = 0$. If $\lambda \notin U$, then $1/(z-\lambda)$ is in N^+ and hence in F^+. However, $\gamma(f) \neq 0$ for every invertible function f in F^+. Thus $\lambda \in U$. Since $(z-\lambda)F^+ \subset \ker\gamma$, by (2.4), $\mathscr{M}_\lambda \subset \ker\gamma$. But $\mathscr{M}_\lambda$ is a closed maximal ideal in F^+ and therefore $\mathscr{M}_\lambda = \ker\gamma$ and γ is continuous.

It has been shown by example [9] that not every maximal ideal in F^+ is given by an $\mathscr{M}_\lambda$ for some $\lambda \in U$. However, the following result holds.

THEOREM 2. *Let M be a closed maximal ideal in F^+. Then there exists $\lambda \in U$ such that $M = \mathscr{M}_\lambda$.*

Proof. Let $X = F^+/M$. Then in the terminology of Arens [1], X is a complete, metrizable, separable, convex complex topological division algebra. Thus by [1], $X = C$. But then there exists γ a multiplicative linear functional on F^+ with $M = \ker\gamma$. By Theorem 1, $M = \mathscr{M}_\lambda$ for some $\lambda \in U$, which proves the result.

3. Composition operators. Let $\varphi\colon U \to U$ be holomorphic. Define C_φ on F^+ by

$$(3.1)\qquad C_\varphi(f)(z) = f(\varphi(z)), \quad z \in U.$$

Then C_φ is a composition operator of F^+ into $H(U)$, the space of holomorphic functions on U. We first prove that if $\eta\colon F^+ \to F^+$ is a ring homomorphism, then $\eta = C_\varphi$ for some φ. We then prove, by two alternate methods, that if $\varphi\colon U\to U$ is holomorphic, (3.1) defines a continuous ring homomorphism of F^+ into F^+.

THEOREM 3. *If $\eta\colon F^+\to F^+$ is a ring homomorphism, then there exists a holomorphic function $\varphi\colon U\to U$ such that $\eta(f) = C_\varphi(f)$ for all $f\in F^+$. Conversely, if $\varphi\colon U\to U$ is holomorphic, then* (3.1) *defines a continuous ring homomorphism of F^+ into F^+.*

Proof. Let $\eta\colon F^+\to F^+$ be a ring homomorphism and let $\varphi = \eta(z)$, i.e. $\varphi(\xi) = \eta(z)(\xi)$, $\xi\in U$. For $\lambda\in U$, define $\gamma(f) = \eta(f)(\lambda)$, $f\in F^+$. Since γ is a multiplicative linear functional on F^+, by Theorem 1, γ corresponds to point evaluation at some $\beta\in U$. Thus $\beta = \eta(z)(\lambda) = \varphi(\lambda)$. Hence for all $\lambda\in U$, $\varphi(\lambda)\in U$ and for every $f\in F^+$, $\eta(f)(\lambda) = f(\varphi(\lambda))$. Thus $\varphi\colon U\to U$ and $\eta(f) = C_\varphi(f)$.

Conversely, let $\varphi\colon U\to U$ be holomorphic. We now give two alternate proofs that C_φ given by (3.1) is a continuous ring homomorphism of F^+ into F^+. Proof (i) uses methods of functional analysis and results on composition operators on N^+ given in [4], whereas proof (ii) uses classical analysis and properties of F^+.

(i) Let $\eta = C_{\varphi|N^+}$. By [4], Theorem 1, η is a continuous linear operator of N^+ into N^+. Let $(N^+)^*$ and $(F^+)^*$ denote the spaces of continuous linear functionals on N^+ and F^+, respectively, with the topology of uniform convergence on weakly bounded subsets of N^+ and F^+. For a discussion of these topologies the reader is referred to [3]. By [8], Theorem 3, $(F^+)^* = (N^+)^*$ both set theoretically and topologically. Let $\eta^*\colon (N^+)^*\to(N^+)^*$ given by $\eta^*(\gamma) = \gamma\circ\eta$, $\gamma\in(N^+)^*$, denote the adjoint map of η. Clearly, η^* is a continuous mapping of $(N^+)^*$ into $(N^+)^*$ and hence of $(F^+)^*$ into $(F^+)^*$. Since F^+ is reflexive ([8], p. 35), the second adjoint of η, η^{**}, is a continuous mapping of F^+ into F^+. Let $f\in F^+$. Since N^+ is dense in F^+, there exists a sequence $\{f_n\}$ in N^+ such that $f_n\to f$ in F^+. Therefore $\eta^{**}(f_n)\to\eta^{**}(f)$ and $\eta^{**}(f_n)(z)\to\eta^{**}(f)(z)$ for all $z\in U$. Since $\eta^{**}_{|N^+} = \eta = C_\varphi$,

$$\eta^{**}(f)(z) = \lim_{n\to\infty} f_n(\varphi(z)) = f(\varphi(z))$$

for all $z\in U$. Therefore C_φ is a continuous linear map of F^+ into F^+.

(ii) Let $\varphi\colon U\to U$ be holomorphic. We first show that $C_\varphi(f)\in F^+$ for all $f\in F^+$. Let $\lambda = \varphi(0)$. Without loss of generality we assume that λ is real and nonnegative. For $0<r<1$, let $U_r = \{z\colon |z|<r\}$. By Schwarz's lemma, $\varphi(\overline{U}_r)\subset D_r$ where D_r is given by

$$D_r = \{w\colon |w-\lambda|\leqslant r|1-\lambda w|\}.$$

Let C_r denote the boundary of D_r. Then C_r is a circle with center at $a = \lambda(1-r^2)/(1-r^2\lambda^2)$ and radius $\varrho = r(1-r^2)/(1-r^2\lambda^2)$. Hence $\varphi(U_r) \subset U_{\varrho(r)}$, where

$$\varrho(r) = a+\varrho = (\lambda+r)/(1+r\lambda). \tag{3.2}$$

Clearly, $r<\varrho(r)<1$. Therefore, since $\varphi(\overline{U}_r)\subset\overline{U}_{\varrho(r)}$,

$$M(r, C_\varphi(f))\leqslant M(\varrho(r), f), \tag{3.3}$$

where $M(r,f)$ is given by (1.3). Let $c>0$ and $0<R<1$ be arbitrary. Then by (3.3),

$$\int_0^R \exp\left[\frac{-c}{1-r}\right] M(r, C_\varphi(f))\,dr \leqslant \int_0^R \exp\left[\frac{-c}{1-r}\right] M(\varrho(r), f)\,dr. \tag{3.4}$$

Since $\dfrac{1-\varrho(r)}{1-r} = \dfrac{1-\lambda}{1+r\lambda} > \dfrac{1-\lambda}{1+\lambda}$ and $\varrho'(r)\geqslant\dfrac{1-\lambda}{1+\lambda}$ for all r, $0<r<1$,

(3.5)
$$\int_0^R \exp\left[\frac{-c}{1-r}\right] M(\varrho(r), f)\,dr \leqslant \frac{1+\lambda}{1-\lambda}\int_0^R \exp\left[\frac{-c_1}{1-\varrho(r)}\right] M(\varrho(r), f)\,\varrho'(r)\,dr,$$

where $c_1 = c(1-\lambda)/(1+\lambda)$. However,

$$\begin{aligned}(3.6)\qquad &\int_0^R \exp\left[\frac{-c_1}{1-\varrho(r)}\right] M(\varrho(r), f)\,\varrho'(r)\,dr \\ &\qquad = \int_\lambda^{\varrho(R)} \exp\left[\frac{-c_1}{1-\varrho}\right] M(\varrho, f)\,d\varrho \leqslant \|f\|_{c_1}.\end{aligned}$$

Hence, since (3.4)–(3.6) are valid for all R, $0<R<1$,

$$\|C_\varphi(f)\|_c \leqslant \left(\frac{1+\lambda}{1-\lambda}\right)\|f\|_{c_1}, \tag{3.7}$$

where $c_1 = c\left(\dfrac{1-\lambda}{1+\lambda}\right)$. Therefore, $C_\varphi(f)\in F^+$ and $C_\varphi\colon F^+\to F^+$ is continuous.

4. Compact composition operators. A function f holomorphic in U is said *to belong to the Hardy space H^p*, $0<p<\infty$, if

$$\|f\|_p = \sup_{0<r<1}\left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right]^{1/p} < \infty.$$

If $f \in H^p$, then $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. is in L^p and

$$\|f\|_p = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right]^{1/p}.$$

For the above and other properties of the spaces H^p the reader is referred to [2].

The following lemma will be used throughout this section to establish continuity of the composition operators.

Lemma 1. *Suppose that X is an F-space and Γ is a linear map of X into F^+ (N^+ or H^p) which is continuous in the topology of uniform convergence on compact sets; then Γ is continuous in the usual topology on F^+ (N^+ or H^p).*

Proof. For $0 < r < 1$ define ψ_r by $\psi_r(f) = f_r$, $f \in F^+$ (N^+ or H^p), where $f_r(z) = f(rz)$. Set $\Gamma_r = \psi_r \circ \Gamma$. Clearly, Γ_r is a continuous mapping from X into F^+ (N^+ or H^p). Furthermore, for each $x \in X$, $\Gamma_r(x) \to \Gamma(x)$ as $r \to 1$. Hence as a consequence of the uniform boundedness principle ([5], Theorem 2.8), Γ is continuous.

Theorem 4. *Let φ: $U \to U$ be holomorphic and let $0 < p < \infty$ be arbitrary. Then the following are necessary and sufficient that C_φ given by (3.1) is a continuous composition operator of F^+ into H^p.*

(a) C_φ: $F^+ \to H^p$.

(b) *If* $f(z) = \sum_{n=0}^{n} a_n z^n \in F^+$, *then* $a_n\varphi^n \to 0$ *in* H^p.

(c) *If* $\lambda_n \geq 0$ *and* $\lambda_n \to 0$, *then* $\exp[\lambda_n\sqrt{n}]\varphi^n \to 0$ *in* H^p.

(d) *There exists* $\lambda > 0$ *such that* $\exp[\lambda\sqrt{n}]\varphi^n \to 0$ *in* H^p.

(d′) *There exists* $\lambda > 0$ *such that* $\sum_{n=0}^{\infty} \exp[\lambda\sqrt{n}]|\varphi(e^{i\theta})|^n$ *is in* $L^p[0, 2\pi]$.

(e) $\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{i\theta})|^{pn} d\theta = O(\exp[-\lambda\sqrt{n}])$ *for some* $\lambda > 0$.

(e′) $\limsup_{n\to\infty} \|\varphi^n\|_p^{1/\sqrt{n}} < 1$.

Proof. (a)$\Rightarrow$(b). If C_φ: $F^+ \to H^p$ for some p, $0 < p < \infty$, then by Lemma 1 C_φ is continuous. Suppose $f(z) = \sum a_n z^n \in F^+$. Then for every $c > 0$, $\|f\|_c$ given by (1.4) is finite and thus $\|a_n z^n\|_c \to 0$ for every $c > 0$. Therefore $C_\varphi(a_n z^n) = a_n \varphi^n \to 0$ in H^p.

(b)$\Rightarrow$(c). Suppose $\lambda_n \geq 0$ and $\lambda_n \to 0$. Consider $f(z) = \sum a_n z^n$, where $a_n = \exp[\lambda_n\sqrt{n}]$. Since $a_n = O(\exp[o(\sqrt{n})])$, by [7], Theorem 1, $f \in F^+$, from which (c) follows.

(c)$\Rightarrow$(d). We first show that there exists $\lambda > 0$ such that $\limsup_{n\to\infty} \exp[\lambda\sqrt{n}]\|\varphi^n\|_p < \infty$. Suppose such a λ does not exist. Let $\varepsilon_j > 0$, $\varepsilon_j \downarrow 0$ be arbitrary. Then for each j, there exists n_j, $n_j > n_{j-1}$, such that

$$\exp[\varepsilon_j\sqrt{n_j}]\|\varphi^{n_j}\|_p \geq 1. \tag{4.1}$$

Define $\lambda_m \geq 0$ as follows:

$$\lambda_m = \begin{cases} \varepsilon_j & \text{if } m = n_j, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\lambda_m \to 0$ and by (c), $\exp[\lambda_m\sqrt{m}]\varphi^m \to 0$ in H^p, which is a contradiction of (4.1). Hence there exists $\lambda > 0$ such that $\limsup_{n\to\infty} \exp[\lambda\sqrt{n}]\|\varphi^n\|_p$ is finite. Choose $c > 0$ such that $0 < c < \lambda$. Then

$$\|\exp[c\sqrt{n}]\varphi^n\|_p = \exp[-(\lambda - c)\sqrt{n}]\|\exp[\lambda\sqrt{n}]\varphi^n\|_p$$

which converges to zero since $\lambda - c > 0$ and $\|\exp[\lambda\sqrt{n}]\varphi^n\|_p$ is bounded.

The proof that (d) is equivalent to (d′) is obvious.

(d)$\Rightarrow$(e). Choose $c > 0$ such that $\exp[c\sqrt{n}]\varphi^n \to 0$ in H^p. Hence there exists I such that for all $n \geq I$, $\|\exp[c\sqrt{n}]\varphi^n\|_p \leq 1$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^{pn} dt \leq \exp[-pc\sqrt{n}]$$

for all $n \geq I$, form which (e) follows with $\lambda = pc$.

The proof that (e) is equivalent to (e′) is also straightforward and is omitted.

(e)$\Rightarrow$(a). Assume (e) holds. Then there exists a constant M, $\lambda > 0$ and I such that

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^{pn} dt \leq M\exp[-\lambda\sqrt{n}] \tag{4.2}$$

for all $n \geq I$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^+$. Since H^p is complete, it suffices to show that $\sum_{j=0}^{m} a_j\varphi^j$ is a Cauchy sequence in H^p. Since $f \in F^+$, by [7], Lemma 1, there exists a sequence $\{\lambda_n\}$, $\lambda_n \downarrow 0$ and a constant $A > 0$ such that

$$|a_n| \leq A\exp[\lambda_n\sqrt{n}] \tag{4.3}$$

for all n. Let $0 < r < 1$ be arbitrary. If $0 < p < 1$, then by (4.2), (4.3) and the inequality $|a+b|^p \leqslant |a|^p + |b|^p$,

$$(4.4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{j=n}^{m} a_j \varphi(re^{it})^j\Big|^p dt \leqslant A^p M \sum_{j=n}^{m} \exp[(p\lambda_j - \lambda)\sqrt{j}]$$

for all $n \geqslant I$. Since (4.4) is valid for all r, $0 < r < 1$,

$$(4.5)\qquad \Big\|\sum_{j=n}^{m} a_j \varphi^j\Big\|_p^p \leqslant A^p M \sum_{j=n}^{m} \exp[(p\lambda_j - \lambda)\sqrt{j}].$$

If $1 \leqslant p < \infty$, then as above one obtains

$$(4.6)\qquad \Big\|\sum_{j=n}^{m} a_j \varphi^j\Big\|_p \leqslant A M^{1/p} \sum_{j=n}^{m} \exp[(\lambda_j - \lambda/p)\sqrt{j}].$$

Since $\sum_{j=0}^{\infty} \exp[(p\lambda_j - \lambda)\sqrt{j}]$ and $\sum_{j=0}^{\infty} \exp[(\lambda_j - \lambda/p)\sqrt{j}]$ converge, (4.5) and (4.6) show that $\sum_{j=n}^{m} a_j\varphi^j$ is a Cauchy sequence in H^p. Hence $C_\varphi f = \sum_{n=0}^{\infty} a_n \varphi^n \in H^p$, which proves (a).

COROLLARY. *If C_φ: $F^+ \to H^p$ for some $p > 0$, then C_φ: $F^+ \to H^q$ for all q, $0 < q < \infty$.*

Proof. Suppose C_φ: $F^+ \to H^p$ for some $p > 0$. If $p = \infty$, the result is obvious. Assume $p < \infty$. Clearly, one need only consider H^q for $p < q < \infty$. It is clear that if C_φ: $F^+ \to H^p$, $p > 0$, then $|\varphi(e^{it})| < 1$ a.e. Hence, for any $q > p$,

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^{qn} dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^{pn} dt$$

for all n. Hence by Theorem 4 (e), C_φ: $F^+ \to H^q$.

Remark. As a consequence of the above corollary one might conjecture that if C_φ: $F^+ \to H^p$ for some $p > 0$, then C_φ: $F^+ \to H^\infty$, the space of bounded holomorphic functions on U. One can easily show that C_φ: $F^+ \to H^\infty$ if and only if $\sup_{z \in U} |\varphi(z)| < 1$. In the following example we construct a function φ such that C_φ: $F^+ \to H^1$ but $\sup_{z\in U} |\varphi(z)| = 1$.

EXAMPLE. Let h: $U \to U$ be holomorphic and such that h is continuous on $\overline{U}$, $h(1) = 1$ and $h^n \to 0$ in H^1, e.g., take $h(z) = (1+z)/2$. Let $c > 0$ and $\alpha > 0$ be arbitrary. Choose $\beta_n > 0$ with $\sum \beta_n = 1$. Since $\beta_1 < 1$, $\exp[c\sqrt{n}]\beta_1^n \to 0$. Therefore, $\exp[c\sqrt{n}](\beta_1|h|^k)^n$ is bounded for all n, k and

$$\lim_{n\to\infty} \exp[c\sqrt{n}] \int_0^{2\pi} (\beta_1|h|^k)^n = 0$$

independent of k. Therefore there exists n_1 such that

$$\sup_n \exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} (\beta_1|h|^{n_1})^n < \alpha.$$

Since $\beta_1 + \beta_2 < 1$, $\exp[c\sqrt{n}](\beta_1+\beta_2)^n \to 0$. Therefore $\exp[c\sqrt{n}](\beta_1|h|^{n_1} + \beta_2|h|^k)^n$ is bounded for all n, k and

$$\lim_{n\to\infty} \exp[c\sqrt{n}] \int_0^{2\pi} (\beta_1|h|^{n_1} + \beta_2|h|^k)^n = 0$$

independent of k. Furthermore, since

$$\lim_{k\to\infty} \exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} (\beta_1|h|^{n_1} + \beta_2|h|^k)^n = \exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} (\beta_1|h|^{n_1})^n < \alpha,$$

there exists $n_2 > n_1$ such that

$$\sup_n \exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} (\beta_1|h|^{n_1} + \beta_2|h|^{n_2})^n < \alpha.$$

Continuing inductively there exists a sequence n_k, $n_{k+1} > n_k$, $k = 1, 2, \ldots$, such that

$$(4.7)\qquad \sup_n \exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} (\beta_1|h|^{n_1} + \ldots + \beta_k|h|^{n_k})^n < \alpha.$$

Let $\varphi = \sum_{k=1}^{\infty} \beta_k h^{n_k}$. Clearly, φ is holomorphic in U, continuous on $\overline{U}$, φ: $U \to U$ and $\varphi(1) = 1$. Furthermore, by (4.5) and Fatou's lemma

$$\exp[c\sqrt{n}] \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{it})|^n dt \leqslant \alpha$$

for all n. Hence by Theorem 4 (e), C_φ: $F^+ \to H^1$. Since $\sup_{z\in U} |\varphi(z)| = 1$, C_φ cannot map F^+ to H^∞.

Let X and Y be topological vector spaces and let Γ be a linear map of X into Y. As in [3], we say that Γ is *compact* if there exists a neighborhood V of 0 in X such that $\Gamma(U)$ has compact closure in Y.

THEOREM 5. *If C_φ is a composition operator of F^+ into H^p, $0 < p < \infty$, then C_φ is compact.*

Proof. Suppose C_φ: $F^+ \to H^p$ and assume $1 \leqslant p < \infty$. By Theorem 4(d) there exists $\lambda > 0$ such that $\exp[\lambda\sqrt{n}]\varphi^n \to 0$ in H^p as $n \to \infty$. Choose c such that $0 < c < \lambda$ and let

$$U = \{f \in F^+ : \|f\|_c < 1\}.$$

Let $\varepsilon > 0$ be arbitrary. Choose an integer K such that

$$\|\exp[\lambda\sqrt{n}]\varphi^n\|_p < 1 \tag{4.8}$$

for all $n \geqslant K$ and

$$\sum_{n=K+1}^{\infty} \exp[(c-\lambda)\sqrt{n}] < \varepsilon. \tag{4.9}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in U$ be arbitrary. Since $\|f\|_c < 1$, $|a_n| < \exp[c\sqrt{n}]$ for all n. By (4.8) and (4.9)

$$\Big\|C_\varphi f - \sum_{n=0}^{K} a_n\varphi^n\Big\|_p \leqslant \sum_{n=K+1}^{\infty} \|a_n\varphi^n\|_p \tag{4.10}$$

$$\leqslant \sum_{n=K+1}^{\infty} \exp[(c-\lambda)\sqrt{n}]\|\exp[\lambda\sqrt{n}]\varphi^n\|_p < \varepsilon.$$

Since (4.10) is valid for all $f \in U$, it follows that $C_\varphi(U)$ is totally bounded and hence has compact closure since H^p is complete.

If $0 < p < 1$, then by the Corollary to Theorem 4, C_φ: $F^+ \to H^1$. Therefore, there exists a neighborhood U of 0 in F^+ such that $C_\varphi(U)$ has compact closure in H^1. Since the identity map of H^1 into H^p is continuous, $C_\varphi(U)$ has compact closure in H^p.

THEOREM 6. *Let X be a Banach space and let Γ be any continuous linear map from F^+ or N^+ to X. Then Γ is compact.*

Proof. Suppose that Γ: $F^+ \to X$ is continuous. As in the proof of Theorem 4 (d), there exists $\lambda > 0$ such that $\exp[\lambda\sqrt{n}]\Gamma(z^n) \to 0$ in X. Using the method of proof of Theorem 5, one can show that there exists an open neighborhood U of 0 in F^+ such that $\Gamma(U)$ is totally bounded in X and hence has compact closure.

Suppose Γ: $N^+ \to X$ is continuous. Then as in the proof of Theorem 3, Γ^{**} (the second adjoint of Γ) is a continuous linear map of F^+ into X^{**} (the second dual space of X) and hence compact. Therefore there exists a neighborhood U of 0 in X such that $\Gamma^{**}(U)$ is totally bounded in X^{**}. Let $V = U \cap N^+$. Since the topology in F^+ defined by the family of seminorms (1.4) is weaker than the topology in N^+ defined by the metric (1.1) ([7], Theorem 4), V is open in N^+. Since $\Gamma^{**}_{|N^+} = \Gamma$, $\Gamma(V)$ is totally bounded in X. Therefore, Γ is compact.

We conclude by giving necessary and sufficient conditions for a composition operator C_φ of F^+ into F^+ to be compact.

THEOREM 7. *C_φ: $F^+ \to F^+$ is compact if and only if there exists $\lambda > 0$ such that $\exp[\lambda\sqrt{n}]\varphi^n \to 0$ in F^+.*

Proof. Suppose C_φ: $F^+ \to F^+$ is compact. Then there exists $c > 0$ such that if $E = \{f \in F^+ : \|f\|_c \leqslant 1\}$ then $C_\varphi(E)$ has compact closure in F^+. Consider the sequence $f_n(z) = \exp[c\sqrt{n}]z^n \in E$. Then $C_\varphi(f_n) = \exp[c\sqrt{n}]\varphi^n \in C_\varphi(E)$. Since $\overline{C_\varphi(E)}$ is compact it is bounded in every norm on F^+ given by (1.4). Thus if $a_n \to 0$, $a_n C_\varphi(f_n) \to 0$ in F^+. Choose $0 < \lambda < c$ and let $a_n = \exp[-(c-\lambda)\sqrt{n}]$. Since $a_n \to 0$, $a_n C_\varphi(f_n) = \exp[\lambda\sqrt{n}]\varphi^n \to 0$ in F^+.

Conversely, suppose there exists $\lambda > 0$ such that $\exp[\lambda\sqrt{n}]\varphi^n \to 0$ in F^+. Since the topology of F^+ is determined by countably many of the norms $\|\ \|_c$ given by (1.4), F^+ is metrizable. Hence to show that C_φ is compact, it suffices to show that there exists a neighborhood U of 0 in F^+ such that $C_\varphi(U)$ is totally bounded for each norm $\|\ \|_c$.

Choose λ', $0 < \lambda' < \lambda$, and let

$$U = \{f \in F^+ : \|f\|_{\lambda'} < 1\}.$$

Then if $f(z) = \sum a_n z^n \in U$, $|a_n| < \exp[\lambda'\sqrt{n}]$.

Let $\varepsilon > 0$ and $c > 0$ be arbitrary. Since $\|\exp[\lambda\sqrt{n}]\varphi^n\|_c \to 0$ and $\sum \exp[-(\lambda-\lambda')\sqrt{n}] < \infty$, there exists an integer K such that

$$\exp[\lambda\sqrt{n}]\|\varphi^n\|_c < 1 \tag{4.11}$$

for all $n \geqslant K$ and

$$\sum_{n=K+1}^{\infty} \exp[-(\lambda-\lambda')\sqrt{n}] < \varepsilon. \tag{4.12}$$

Therefore, for any $f(z) = \sum a_n z^n \in U$,

$$\Big\|C_\varphi(f) - \sum_{n=0}^{k} a_n\varphi^n\Big\|_c \leqslant \sum_{n=K+1}^{\infty} |a_n|\,\|\varphi^n\|_c \tag{4.13}$$

$$= \sum_{n=K+1}^{\infty} (|a_n|\exp[-\lambda\sqrt{n}])(\exp[\lambda'\sqrt{n}]\|\varphi^n\|_c)$$

$$= \sum_{n=K+1}^{\infty} \exp[-(\lambda-\lambda')\sqrt{n}] < \varepsilon.$$

By inequality (4.13) it follows that $C_\varphi(U)$ is totally bounded with respect to $\|\ \|_c$. But $c > 0$ was arbitrary. Hence $C_\varphi(U)$ is totally bounded for all $c > 0$ and therefore has compact closure.

References

[1] R. Arens, *Linear topological division algebras*, Bull. Amer. Math Soc. 53 (1947), pp. 623–630.
[2] P. L. Durer, *Theory of H^p spaces*, Academic Press, New York 1970.
[3] R. E. Edwards, *Functional analysis*, Holt, Rinehart and Winston, New York 1965.
[4] J. W. Roberts and M. Stoll, *Prime and principal ideals in the algebra N^+*, Archiv der Mathematik (to appear).
[5] W. Rudin, *Functional Analysis*, Mc-Graw Hill, New York 1973.
[6] N. Yanagihara, *Multipliers and linear functionals for the class N^+*, Trans. Amer. Math. Soc. 180 (1973), pp. 449–461.
[7] — *The containing Fréchet space for the class N^+*, Duke Math. Jour. 40 (1973), pp. 93–103.
[8] — *The second dual space for the space N^+*, Proc. Japan Acad. 49 (1973), pp. 33–36.
[9] — *Generators and maximal ideals in some algebras of holomorphic functions*, Tôhoku Math. Jour. (to appear).

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On power series in the operators s^a*

by

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Abstract. Necessary and sufficient conditions for convergence, in the field of Mikusiński operators, of the series $S = \sum_{n=0}^{\infty} \gamma_n s^{an}$ and the uniqueness of this representation are given. Here a is real positive, γ_n are complex and s is the differentiation operator.

This extends a result of T. K. Boehme when $a = 1$. It is also shown that J. Wloka's sufficient condition for convergence is also a necessary one.

1. Introduction. In the field of Mikusiński operators M the convergence class is defined. But nobody has investigated the conditions for convergence or divergence of series in operators in general case.

The special class of power series in the operator s^a

$$\sum_{n=0}^{\infty} \gamma_n s^{an} \lambda^n, \tag{1.1}$$

where γ_n and λ are complex numbers, a real and positive, s the differentiation operator, has an important role in the operational calculus and its applications.

In the case $a = 1$ we know one sufficient condition for the convergence of the series (1.1) and one for its divergence [4]. We know also generalization of these results to the case $a > 0$ [6]. J. Wloka [8] found a sufficient condition for the convergence of the series (1.1), in case $a = 1$, too. In the mentioned paper he asked the question: "Is this condition also a necessary condition?". Recently, T. K. Boehme [1] gave a necessary and sufficient condition for the convergence of the series (1.1) in the case $a = 1$. Our aim is to enlarge the result of Boehme to the case $a > 0$. We prove two propositions, both containing sufficient and necessary conditions for the convergence of the series (1.1). We give also the answer to the question of J. Wloka and a proposition about the uniqueness of the development of an element of M in a series of the form (1.1) for a fixed $a > 0$.

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