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Generalizations of theorems of Fejér and Zygmund on convergence and boundedness of conjugate series*

by

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Abstract. It is shown that the Fejér and Zygmund theorem on the convergence and boundedness of conjugate series remains true for arbitrary translation invariant Banach spaces of distributions on T. Analoga for weak convergence and for certain Banach lattices are given.

1. Introduction. It is the purpose of this note to generalize the following theorem of Fejér and Zygmund ([14]; [13], p. 268) in various directions.

THEOREM A (Fejér and Zygmund). (i) *If f and $\tilde{f}$ are both continuous (i.e. in $C_{2\pi}$) and the Fourier series $S[f]$ converges uniformly, so does the conjugate series $\tilde{S}[f]$. If f and $\tilde{f}$ are both bounded (i.e. in $L^\infty_{2\pi}$) and $S[f]$ has bounded partial sums, so has $\tilde{S}[f]$.*

(ii) *Suppose $\tilde{S}[f]$ is a Fourier series (i.e. in $L_{2\pi}$). If $\int_0^{2\pi}|f-s_n|\,dt$ tends to zero, so does $\int_0^{2\pi}|\tilde{f}-\tilde{s}_n|\,dt$; if $\int_0^{2\pi}|s_n|\,dt$ is bounded, so is $\int_0^{2\pi}|\tilde{s}_n|\,dt$.*

It will be shown that this theorem remains true if the spaces $C_{2\pi}$, $L^\infty_{2\pi}$ and $L_{2\pi}$ are replaced by an arbitrary translation invariant Banach space in $L_{2\pi}$ or even by a translation invariant Banach space of distributions (Th. 3.1.). In Section 4, Theorem 3.1 is reformulated as a statement on homogeneous BK-spaces. Section 5 contains analoga for weak convergence. For example: If $S[f]$ converges weakly in $L_{2\pi}$ and if $\tilde{S}[f]$ is a Fourier series, then this series converges weakly in $L_{2\pi}$ too. Theorem 6.1 shows that if a Fourier series converges uniformly and the arithmetic means of the partial sums of the conjugate series converge in some Banach

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lattice E, then the conjugate series converges in E. Several examples not covered by Zygmund's Theorem will be presented.

For reasons of greater uniformity and flexibility we work mostly with spaces of sequences of Fourier coefficients of distributions or quite arbitrary sequence spaces, rather than with spaces of functions in $L_{2\pi}$.

2. Definitions, notations and preliminary remarks. Let $\boldsymbol{Z}$ be the set of integers and Ω the linear space of complex valued functions on $\boldsymbol{Z}$ (i.e. sequences on $\boldsymbol{Z}$). For $j\in\boldsymbol{Z}$ let δ^j denote the characteristic function of the set $\{j\}$, i.e. $\delta^j_k=0$ if $k\neq j$ and $\delta^j_j=1$. If $x\in\Omega$, then $s_nx=\sum_{k=-n}^{n}x_k\delta^k$ is called the nth *section* of $x=(x_k)$ and $\sigma_nx=\sum_{k=-n}^{n}\big(1-|k|/(n+1)\big)x_k\delta^k$ the nth *Cesàro-section of order one* of x.

Let E be a BK-space (see Zeller [11]), i.e. a Banach space of complex valued sequences x on which the functionals $x\to x_k$ are all continuous. Let $x\in\Omega$ and $s_nx\in E$ for all $n=0,1,\ldots$; then x is said to have: *sectional convergence* or AK (respectively *Cesàro-sectional convergence* or σK) in E, if $x\in E$ and if $\|s_nx-x\|_E\to 0$ $(n\to\infty)$ (resp. $\|\sigma_nx-x\|_E\to 0$ $(n\to\infty)$); *sectional boundedness* or AB (resp. *Cesàro-sectional boundedness* or σB) in E if $\sup_n\|s_nx\|_E<\infty$ (resp. $\sup_n\|\sigma_nx\|_E<\infty$); *functional sectional convergence* or FAK (resp. FσK) in E, if for every $\varphi\in E'$ — the space of linear continuous functionals on E — $\lim_{n\to\infty}\varphi(s_nx)$ (resp. $\lim_{n\to\infty}\varphi(\sigma_nx)$) exists; *weak sectional convergence* or SAK (resp. SσK) in E if $x\in E$ and if for every $\varphi\in E'$, $\lim_{n\to\infty}\varphi(s_nx)=\varphi(x)$ (resp. $\lim_{n\to\infty}\varphi(\sigma_nx)=\varphi(x)$; *sectional density* or AD in E if Φ, the space of $x\in\Omega$ with only finitely many nonzero x_k's, is dense on E. If P is any one of the properties AK, σK, AB, σB, FAK, FσK, SAK, SσK, AD, then E_{P} denotes the subspace of Ω, consisting of those elements which have the property P in E. If P = AK, AB, FAK, SAK then, as is well known, with E also E_{P} is a BK-space under the norm $\|x\|=\sup_n\|s_nx\|_E$. If $E=\sigma$K, σB, FσK, SσK, then E_{P} is a BK-space under the norm $\|x\|=\sup_n\|\sigma_nx\|_E$.

For any BK-space E let $E_f=\{x\in\Omega\colon\ \exists\varphi\in E'$ such that $x_k=\varphi(\delta^k)$ if $\delta^k\in E\}$. If $A,B\subset\Omega$, then $(A\to B)=\{x\in\Omega\colon\ xy=(x_ky_k)\in B$ for every $y\in A\}$ is the space of multipliers from A into B. Of particular interest are the cases when B is one of the spaces

$$cs=\Big\{x\in\Omega\colon\ \lim_{n\to\infty}\sum_{k=-n}^{n}x_k\ \text{exists}\Big\},$$

$$\sigma s=\Big\{x\in\Omega\colon\ \lim_{n\to\infty}\sum_{k=-n}^{n}\Big(1-\frac{|k|}{n+1}\Big)x_k\ \text{exists}\Big\},$$

$$bs=\Big\{x\in\Omega\colon\ \sup_n\Big|\sum_{k=-n}^{n}x_k\Big|<\infty\Big\},$$

$$\sigma b=\Big\{x\in\Omega\colon\ \sup_n\Big|\sum_{k=-n}^{n}\Big(1-\frac{|k|}{n+1}\Big)x_k\Big|<\infty\Big\}.$$

If $B=cs,bs,\sigma s,\sigma b$, we use instead of $(A\to B)$ the usual, shorter notations A^β, A^γ, A^σ, $A^{\sigma b}$, respectively.

For $t\in\boldsymbol{T}=\boldsymbol{R}/(2\pi\boldsymbol{Z})$ (real numbers modulo 2π) let $e(t)$ denote the sequence x, where $x_k=e^{ikt}$ $(k\in\boldsymbol{Z})$. A BK-space E is called *translation invariant* if $x\in E$ implies $x\cdot e(t)=(x_ke^{ikt})\in E$ for every $t\in\boldsymbol{T}$ and if in addition for every $x\in E$ and for every $t\in\boldsymbol{T}$, $\|x\cdot e(t)\|_E=\|x\|_E$. A BK-space E is called a *homogeneous* BK-*space* (compare [6], p. 14) if it is a translation invariant BK-space and if translation is continuous in E, i.e. for all $x\in E$ and $t_0\in\boldsymbol{T}$, $\lim_{t\to t_0}\|x\cdot e(t)-x\cdot e(t_0)\|_E=0$. A translation invariant BK-space E is said to have *weakly continuous translation* if for all $\varphi\in E'$ and all $t_0\in\boldsymbol{T}$, $\lim_{t\to t_0}\varphi\big(x\cdot e(t)-x\cdot e(t_0)\big)=0$. We observe here (see Corollary 4.4) that translation is weakly continuous if and only if it is continuous.

If $E\subset\Omega$, then $\tilde{E}=\{\tilde{x}\in\Omega\colon\ \tilde{x}=\{-i(\operatorname{sign}k)x_k\},\ x\in E\}$. Hence if E is a space of sequences of Fourier coefficients, then $\tilde{E}$ denotes the associated space of sequences of coefficients in the conjugate series.

If $L_{2\pi}$, $C_{2\pi}$, $L^\infty_{2\pi}$ and $M_{2\pi}$ are the classical Banach spaces of 2π-periodic, complex valued functions f on $\boldsymbol{T}$ which are respectively Lebesgue integrable, continuous, essentially bounded, respectively bounded (Borel) measures on $\boldsymbol{T}$, then the associated spaces $\hat{L}$, $\hat{C}$, $\widehat{L^\infty}$ and $\hat{M}$ of sequences of Fourier coefficients (respectively Fourier–Stieltjes coefficients) $x=\hat{f}$ are translation invariant BK-spaces under the norm $\|x\|=\|f\|$. Furthermore $\hat{L}=\hat{L}_{\sigma\mathrm{K}}$, $\hat{C}=\hat{C}_{\sigma\mathrm{K}}$, $\widehat{L^\infty}=\widehat{L^\infty_{\sigma\mathrm{B}}}$, $\hat{M}=\hat{M}_{\sigma\mathrm{B}}$ ([14], pp. 144, 136, 134, 137). The reader is reminded on the following facts on multipliers:

$$(\widehat{L^\infty}\to\hat{C})=\widehat{L^\infty}^{\sigma}=\hat{L}=\hat{L}_{\mathrm{F}\sigma\mathrm{K}}.$$

For the first equation see [14], p. 177; for the second equation see [14], p. 158. The third equation follows from the second and the fact that for any BK-space E containing Φ,

$$E_{\mathrm{F}\sigma\mathrm{K}}=(E_f)^\sigma\quad\text{and}\quad(\hat{L})_f=\widehat{L^\infty}.$$

$$(\widehat{L^\infty}\to\hat{C}_{\mathrm{AK}})=(\widehat{L^\infty})^\beta=\hat{L}_{\mathrm{FAK}}.$$

For the first equation see [5] and [3], p. 379. The second equation is parallel to the equation $(\widehat{L^{\infty}})^{\delta} = \hat{L}_{\mathrm{F}\sigma\mathrm{K}}$. Furthermore, we have

$$\hat{C}_{\mathrm{F}\sigma\mathrm{K}} = \hat{M}^{\sigma} = \Big\{x \in \Omega\colon \lim_{n\to\infty} \sum_{k=-n}^{n} \Big(1-\frac{|k|}{n+1}\Big) x_k e^{ikt} \text{ exists boundedly on } \boldsymbol{T}\Big\},$$

$$\hat{C}_{\mathrm{FAK}} = \hat{M}^{\delta} = \Big\{x \in \Omega\colon \lim_{n\to\infty} \sum_{k=-n}^{n} x_k e^{ikt} \text{ exists boundedly on } \boldsymbol{T}\Big\}.$$

The first equations follow from $\hat{C}_f = \hat{M}$ and the second equations are proven in [3], pp. 374 and 375. We have also $(\hat{M}\to\hat{C}_{\mathrm{AK}}) = \hat{C}_{\mathrm{AK}}$ (see [2]).

3. The generalization of Zygmund's Theorem. With the notations introduced in Section 2, Zygmund's Theorem (Th. A) can be written also in the following form:

THEOREM A′. (i) $\hat{C}_{\mathrm{AK}} \cap \tilde{\hat{C}} \subset \tilde{\hat{C}}_{\mathrm{AK}}$; $\hat{L}^{\infty}_{\mathrm{AB}} \cap \tilde{\hat{L}}^{\infty} \subset \tilde{\hat{L}}^{\infty}_{\mathrm{AB}}$.

(ii) $\hat{L}_{\mathrm{AK}} \cap \tilde{\hat{L}} \subset \tilde{\hat{L}}_{\mathrm{AK}}$; $\hat{L}_{\mathrm{AB}} \cap \tilde{\hat{L}} \subset \tilde{\hat{L}}_{\mathrm{AB}}$.

With respect to the last inclusion we note that $\hat{L}_{\mathrm{AB}} \not\subset \hat{L}$ ([9] and [6]) but $\hat{L}_{\mathrm{AB}} \cap \tilde{\hat{L}} \subset \hat{L}$ by the F. and M. Riesz Theorem on the absolute continuity of analytic measures ([14], p. 285).

Evidently the following statement generalizes Theorem A′ and hence Zygmund's Theorem:

3.1. THEOREM. *Let E be a translation invariant* BK-*space. Then*
(i) $E_{\mathrm{AK}} \cap \tilde{E}_{\sigma\mathrm{K}} \subset \tilde{E}_{\mathrm{AK}}$; (ii) $E_{\mathrm{AB}} \cap \tilde{E}_{\sigma\mathrm{B}} \subset \tilde{E}_{\mathrm{AB}}$.

The main argument for the proof is contained in the following lemma:

3.2 LEMMA. *Let E be a translation invariant* BK-*space, then*

(i) $E_{\mathrm{AK}} \subset \tilde{E}_{\Delta 0} \equiv \{x \in \Omega\colon \lim_{n\to\infty} \|\sigma_n(\tilde{x}) - s_n(\tilde{x})\|_E = 0\}$,

(ii) $E_{\mathrm{AB}} \subset \tilde{E}_{\Delta\mathrm{B}} \equiv \{x \in \Omega\colon \sup_n \|\sigma_n(\tilde{x}) - s_n(\tilde{x})\|_E < \infty\}$.

Proof. The first half of the proof is used for both, (i) and (ii): Let E be a translation invariant BK-space and let X be E_{AK} or E_{AB}. Let $x \in X$. Then $(s_n x)\cdot e(t) = s_n\big(x\cdot e(t)\big) \in X$ for every $n = 0, 1, \ldots$ and for every $t \in \boldsymbol{T}$. Let $e'(t) = (ik\, e^{ikt})$ be the "differentiated sequence" of $e(t)$. Then for $T_n(t) = s_n\big(x\cdot e(t)\big)$ and $T_n'(t) = s_n\big(x\cdot e'(t)\big)$ the following equation holds for every $n = 0, 1, \ldots$ and every $t \in \boldsymbol{T}$:

$$(1)\qquad T_n'(t) = \sum_{k=-n}^{n} x_k ik\, e^{ikt}\delta^k = \frac{2n}{\pi}\sum_{k=-n}^{n} x_k \int_0^{2\pi} e^{ik(t+u)} \sin nu\, K_{n-1}(u)\,du\,\delta^k$$
$$= \frac{2n}{\pi}\int_0^{2\pi} T_n(t+u)\sin nu\, K_{n-1}(u)\,du$$

([14], p. 118). Here K_{n-1} denotes the $(n-1)$th Fejér-kernel and the last integral has to be considered as a vector valued Riemann–Stieltjes integral with $\sin nu\, K_{n-1}(u)\,du$ as integrator ([4], p. 62). Evidently, (1) implies the generalized Bernstein inequality

$$(2)\qquad \|T_n'(t)\|_E \leqslant \frac{2n}{\pi}\int_0^{2\pi} \|T_n(t)\|_E K_{n-1}(u)\,du = 2n\|T_n(t)\|_E,$$

observing the translation invariance of the norm in E, the positivity of the Fejér-kernel and $\|K_n\|_L = \pi$ for every $n = 0, 1, \ldots$

Now let $n > n_0 \geqslant 0$ and $\tau_n = T_n - T_{n_0}$. Then

$$(3)\qquad \sigma_n\big(\tilde{x}\cdot e(t)\big) - s_n\big(\tilde{x}\cdot e(t)\big) = \frac{T_n'(t)}{n+1} = \frac{\tau_n'(t)}{n+1} + \frac{T_{n_0}'(t)}{n+1}.$$

(i) Let $x \in E_{\mathrm{AK}}$ and $\varepsilon > 0$ be given. Then there exists an n_0 such that $\|\tau_n(t)\|_E < \varepsilon/2$ for all $n > n_0$. This implies by (2) (if $T_n(t)$ in (2) is replaced by $\tau_n(t)$) that also

$$\Big\|\frac{\tau_n'(t)}{n+1}\Big\|_E < \varepsilon \quad \text{for all } n \geqslant n_0.$$

Since for some $n' > n_0$ and for all $n > n' > n_0$ also $\|T_{n_0}'(t)/(n+1)\|_E < \varepsilon$, it follows by (3) that for $n > n' > n_0$

$$\big\|\sigma_n\big(\tilde{x}\cdot e(t)\big) - s_n\big(\tilde{x}\cdot e(t)\big)\big\|_E < 2\varepsilon.$$

This implies by the translation invariance of the norm in E that $x \in \tilde{E}_{\Delta 0}$.

(ii) This can be proved correspondingly as (i) using now that $x \in E_{\mathrm{AB}}$ implies $\sup_n \|\tau_n(t)\|_E < \infty$.

Proof of Theorem 3.1. Let E be a translation invariant BK-space Evidently, $E_{\mathrm{AK}} = E_{\sigma\mathrm{K}} \cap E_{\Delta 0}$ and $\tilde{E}_{\mathrm{AK}} = \tilde{E}_{\sigma\mathrm{K}} \cap \tilde{E}_{\Delta 0}$. Since by Lemma 3.2, $E_{\mathrm{AK}} \subset \tilde{E}_{\Delta 0}$, it follows that $E_{\mathrm{AK}} \cap \tilde{E}_{\sigma\mathrm{K}} \subset \tilde{E}_{\Delta 0} \cap \tilde{E}_{\sigma\mathrm{K}} = \tilde{E}_{\mathrm{AK}}$, hence (i) is proved.

Correspondingly we have $E_{\mathrm{AB}} = E_{\sigma\mathrm{B}} \cap E_{\Delta\mathrm{B}}$. Since by Lemma 3.2, $E_{\mathrm{AB}} \subset \tilde{E}_{\Delta\mathrm{B}}$, we obtain $E_{\mathrm{AB}} \cap \tilde{E}_{\sigma\mathrm{B}} \subset \tilde{E}_{\Delta\mathrm{B}} \cap \tilde{E}_{\sigma\mathrm{B}} = \tilde{E}_{\mathrm{AB}}$.

4. Homogeneous BK-spaces. G. E. Šilov [8] introduced the concept of a homogeneous space of functions. Homogeneous Banach spaces of functions on T are considered in Katznelson [7], p. 14. We gave the definition of a homogeneous BK-space in Section 2. The following simple proposition allows another formulation of Theorem 3.1.

4.1. PROPOSITION. (i) *If E is a translation invariant* BK-*space, then* $E \subset E_{\sigma\mathrm{B}}$.

(ii) *E is a translation invariant* BK-*space and $E = E_{\sigma K}$ if and only if E is a homogeneous* BK-*space.*

Proof. (i) Let E be a translation invariant BK-space, K_n the nth Fejér-kernel, $x \in E$ and $n = 0, 1, \ldots$ Then

$$\|\sigma_n x\|_E = \left\|\frac{1}{\pi}\int_0^{2\pi} K_n(t)\, x\cdot e(-t)\, dt\right\|_E \leqslant \frac{1}{\pi}\int_0^{2\pi} K_n(t)\, dt\, \|x\cdot e(-t)\|_E = \|x\|_E.$$

Hence $E \subset E_{\sigma B}$.

(ii) Let E be a translation invariant BK-space and $E = E_{\sigma K}$, $x \in E$ and $\varepsilon > 0$. Then there exists a positive integer n_0 such that for all $n > n_0$ and for all $t, t_0 \in \boldsymbol{T}$:

$$\begin{aligned}\|x\cdot e(t) - x\cdot e(t_0)\|_E &= \|x\cdot e(t-t_0) - x\|_E \\ &\leqslant \|x\cdot e(t-t_0) - \sigma_n(x\cdot e(t-t_0))\|_E + \|\sigma_n(x\cdot e(t-t_0)) - \sigma_n(x)\|_E + \|\sigma_n x - x\|_E \\ &< \varepsilon + \max_{|k|\leqslant n}\|\sigma^k\|_E \sum_{k=-n}^{n} |x_k|\, |e^{ik(t-t_0)} - 1|.\end{aligned}$$

Evidently, the last expression converges to zero if $t \to t_0$. Hence E is a homogeneous BK-space.

Conversely: Let E be a homogeneous BK-space and $x \in E$. Then g, where $g(t) = x\cdot e(t)$, is a continuous E-valued function on T. Furthermore,

$$\sigma_n x = \frac{1}{\pi}\int_0^{2\pi} K_n(t)\, g(-t)\, dt.$$

As in [7], p. 10, one shows that $\lim_{n\to\infty} \sigma_n x = x$ in E. Thus $E = E_{\sigma K}$.

The last proposition and 3.1 together imply:

4.2. Theorem. (i) *If E is a homogeneous* BK-*space, then $E_{AK} \cap \tilde{E} \subset \tilde{E}_{AK}$.*

(ii) *If E is a translation invariant* BK-*space, then $E_{AB} \cap \tilde{E} \subset \tilde{E}_{AB}$.*

4.3. Proposition. *Let E be a translation invariant* BK-*space. Then $x \in E$ has weakly continuous translation if and only if x has σK.*

Proof. Let E be a translation invariant BK-space and let x have weakly continuous translation. Then for every $\varphi \in E'$ by Fejér's Theorem

$$\varphi(\sigma_n x) = \frac{1}{\pi}\int_0^{2\pi} \varphi\big(x\cdot e(-t)\big) K_n(t)\, dt \to \varphi(x) \quad (n\to\infty).$$

Thus x has SσK. Since E has σB (Prop. 4.1), it follows that x has σK. In fact: $E_{S\sigma K}$ has AD since $\varphi \in E'$ and $\varphi(\sigma^k) = 0$ for every $k \in \boldsymbol{Z}$ implies $\varphi(E_{S\sigma K}) = 0$ ([10], p. 109). Thus $E_{S\sigma K} \subset (E_{\sigma B})_{AD} = E_{\sigma K}$, where the last

equation can be proved as in the case $(E_{AB})_{AD} = E_{AK}$ ([12], p. 59 Satz 3.3 and p. 70).

Conversely, if x has σK then by 4.1 (ii) x has continuous and hence also weak continuous translation.

4.4. Corollary. *In a translation invariant* BK-*space weak continuity of translation is the same as continuity of translation.*

5. Extensions to classes of multipliers. In this section simple extensions of Theorem 3.1 to classes of multipliers are considered.

5.1. Proposition. *Let $S \subset \Omega$ and let E be a translation invariant* BK-*space. Then $(S \to E_{AK}) \cap (S \to \tilde{E}_{\sigma K}) \subset (S \to \tilde{E}_{AK})$.*

Proof. Evidently, $(S \to E_{AK}) \cap (S \to \tilde{E}_{\sigma K}) = (S \to E_{AK} \cap \tilde{E}_{\sigma K})$. Hence the statement follows by 3.1.

5.2. Example. If $S = \widehat{L^\infty}$ and $E = \hat{C}$, then by 5.1

$$(\widehat{L^\infty} \to \hat{C}_{AK}) \cap (\widehat{L^\infty} \to \tilde{\hat{C}}_{\sigma K}) \subset (\widehat{L^\infty} \to \tilde{\hat{C}}_{AK}).$$

This implies by the facts on multipliers listed at the end of Section 2, that

$$(\widehat{L^\infty})^\beta \cap \tilde{\hat{L}} \subset (\widetilde{\widehat{L^\infty}})^\beta,$$

or in words: If $S[f]$ is a Fourier series which is weakly convergent in $L_{2\pi}$ and if $\tilde{S}[f]$ is a Fourier series (by the F. and M. Riesz Theorem it is actually enough to assume that $\tilde{S}[f]$ is a Fourier–Stieltjes series), then this series is weakly convergent in $L_{2\pi}$ too.

5.3. Proposition. *Let $E \subset \Omega$ be a* BK-*space containing Φ and*

$$B = \left\{x \in \Omega\colon \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=-n}^{n} kx_k = 0\right\}.$$

Then $(E_f \to B) \cap \tilde{E}_{F\sigma K} = \tilde{E}_{FAK}$.

Proof. $cs = \sigma s \cap \tilde{B}$. Thus $(E_f \to B) \cap \tilde{E}_{F\sigma K} = (E_f \to B) \cap (E_f \to \widetilde{\sigma s}) = E_{FAK}$.

5.4. Corollary. *Let $E \subset \Omega$ be a* BK-*space containing Φ; then*

$$(E_f \to \hat{C}_{AK}) \cap \tilde{E}_{F\sigma K} \subset \tilde{E}_{FAK}.$$

Proof. By 3.2,

$$\hat{C}_{AK} \subset \tilde{\hat{C}}_{\Delta 0} = \left\{x \in \Omega\colon \limsup_{n\to\infty}\sup_{t\in T} \left|\frac{1}{n+1}\sum_{k=-n}^{n} kx_k e^{ikt}\right| = 0\right\}.$$

5.5. Example. Let $E = \hat{C}$. Then $(\hat{C}_f \to \hat{C}_{AK}) = (\hat{M} \to \hat{C}_{AK}) = \hat{C}_{AK}$ [2]; furthermore, $\hat{C}_{F\sigma K} = \hat{M}^\sigma$ and $\hat{C}_{FAK} = \hat{M}^\beta$ (see end of Section 2). Thus, by 5.4, $\hat{C}_{AK} \cap \tilde{\hat{M}}^\sigma \subset \tilde{\hat{M}}^\beta$. In words: If $S[f]$ is a uniformly convergent

Fourier series and $[\tilde{S}f]$ has boundedly convergent Cesàro-partial sums, then $\tilde{S}[f]$ is boundedly convergent.

6. Conjugate series in some Banach lattices.

6.1. THEOREM. *Let $E \subset L_{2\pi}$ be a Banach space which is also a Banach lattice under the partial ordering $\succeq$ defined by: $f, g \in E$ $f \succeq g \Leftrightarrow f(t) \geqslant g(t)$ for every $t \in \boldsymbol{T}$. Let the associated space $\hat{E}$ of sequences $\hat{f}$ of Fourier coefficients of $f \in E$ be a translation invariant* BK*-space under the norm $\|\hat{f}\|_{\hat{E}} = \|f\|_E$. Then*

(i) $\hat{C}_{AK} \cap \hat{\tilde{E}}_{\sigma K} \subset \hat{\tilde{E}}_{AK}$; (ii) $\hat{C}_{AB} \cap \hat{\tilde{E}}_{\sigma B} \subset \hat{\tilde{E}}_{AB}$.

Proof. (i) Let E fulfil the hypotheses and let $\hat{E}_{\sigma K} \neq \{0\}$ (without loss of generality). Let us assume also first that the constant functions belong to E. Then by formula (1) in the proof of Lemma 3.2

$$T_n'(t) = \frac{2n}{\pi}\int_0^{2\pi} T_n(t+u)\sin nu K_{n-1}(u)\,du$$

$$= -\frac{2n}{\pi}\int_0^{2\pi} T_n(u)\sin n(t-u) K_{n-1}(t-u)\,du$$

and

$$(1) \qquad \|T_n'(t)\|_{\hat{E}} = \Big\|\sum_{k=-n}^{n} ikx_k e^{ikt}\Big\|_E$$

$$= \frac{2n}{\pi}\Big\|\int_0^{2\pi}\sum_{k=-n}^{n} x_k e^{iku}\sin n(t-u) K_{n-1}(t-u)\,du\Big\|_E$$

$$\leqslant \frac{2n}{\pi}\Big\|\Big\|\sum_{k=-n}^{n} x_k e^{iku}\Big\|_{L^\infty}\int_0^{2\pi}|\sin n(t-u)|\, K_{n-1}(t-u)\,du\Big\|_E$$

$$\leqslant 2n\Big\|\sum_{k=-n}^{n} x_k e^{iku}\Big\|_{L^\infty}\|\delta^0\|_{\hat{E}} = 2n\|T_n\|_{\widehat{L^\infty}}\cdot M,$$

where $M = \|\delta^0\|_{\hat{E}}$.

Hence if $x \in \hat{C}_{AK} \cap \hat{\tilde{E}}_{\sigma K}$ then for given $\varepsilon > 0$ there exists an n_0 such that if $n > n_0 \geqslant 0$ and $\tau_n = T_n - T_{n_0}$ we have $\|\tau_n\|_{\widehat{L^\infty}} < \varepsilon/2$. Hence (1) implies for $n > n_0$ that $\|\tau_n'/(n+1)\|_{\hat{E}} < \varepsilon$. As in the proof of Lemma 3.2 it follows that $\|\sigma_n(\tilde{x}) - s_n(\tilde{x})\|_{\hat{E}} < 2\varepsilon$ if $n > n_0$. Thus $x \in \hat{\tilde{E}}_{AK}$. If the constant functions do not belong to E, then by adjoining the constant functions to E we obtain the space $E_\lambda = E + [\lambda]$, where $[\lambda]$ is the linear space of all constant functions. Then with E also E_λ fulfils the hypotheses of the

theorem if we define the norm in E_λ by $\|f\|_{E_\lambda} = \|f-\lambda\|_E + |\lambda|$, where $f \in E_\lambda$, λ a constant function and $(f-\lambda) \in E$. Then the theorem holds for E_λ and hence, since $\hat{\tilde{E}}_{\sigma K} = (\hat{\tilde{E}}_\lambda)_{\sigma K}$, $\hat{\tilde{E}}_{AK} = (\hat{\tilde{E}}_\lambda)_{AK}$, also for E.

(ii) This can be proved correspondingly.

6.2. Remark. The theorem coincides with a special case of theorem 3.1 if $\hat{E} = \hat{C}$. In view of Theorem 3.1 it is of no additional interest if $\hat{C} \underset{\neq}{\subset} \hat{E}$.

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