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The property of weak type (p, p) for the Hardy–Littlewood maximal operator and derivation of integrals

by

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Abstract. Connections between differentiation of integrals of functions in L^p spaces and the property of weak type (p, p) for the Hardy–Littlewood maximal operator are established, being $1 \leqslant p < \infty$.

§ 1. In this paper the two following properties of a differentiation basis $\mathfrak{R}$ in $\boldsymbol{R}^n$ are considered, being $1 \leqslant p < \infty$,

(a) $\mathfrak{R}$ *differentiates* $\int f$ *for every* $f \in L^p(\boldsymbol{R}^n)$.

(b) *The Hardy–Littlewood maximal operator associated with* $\mathfrak{R}$ *is of weak type* (p, p).

We will prove that these properties are equivalent when $\mathfrak{R}$ is a special basis invariant by translations, and when $\mathfrak{R}$ is a general basis homothecy invariant.

The case $p = 1$ and $\mathfrak{R}$ homothecy invariant was proved by M. de Guzmán and G. V. Welland [3], using a lemma of A. P. Calderón, and they proposed the generalization. A. M. Bruckner [1] shows also the interest of the problem.

This result is contained in my doctoral thesis at Madrid University, being M. de Guzmán my thesis adviser, to whom I wish to thank for his help.

§ 2. A differentiation basis $\mathfrak{R}$ for a subset A of $\boldsymbol{R}^n$ is defined giving for every $x \in A$ a collection $\mathfrak{R}(x)$ of open bounded sets such that there is at least one sequence $\{R_k\}$ in $\mathfrak{R}(x)$ which verifies $R_k \to x$ (i.e., for every neighborhood U of x is $R_k \subset U$ for k greater than some k_0).

Given a locally Lebesgue integrable function $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ we define the *upper derivative* $\overline{D}(\int f, x)$ of $\int f$ with respect to $\mathfrak{R}$ at the point x by

$$\overline{D}\left(\int f, x\right) = \sup\left\{\limsup_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f(y)\, dy\right\},$$

where the sup is taken over all the sequences $\{R_k\} \subset \mathfrak{R}(x)$ such that $R_k \to x$. In a similar way is defined the *lower derivative* $\underline{D}(\int f, x)$, setting inf(liminf) above. We say that $\mathfrak{R}$ *differentiates* $\int f$ if $\bar{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$ almost everywhere.

A differentiation basis $\mathfrak{R}$ in $\boldsymbol{R}^n$ is *homothecy invariant* if, for every homothecy transformation h, $R \in \mathfrak{R}(x)$ implies that $hR \in \mathfrak{R}(hx)$. In a similar way is defined t at $\mathfrak{R}$ is *invariant by translations*. Every translation will be considered as a special homothecy transformation.

The Hardy–Littlewood maximal operator M associated with $\mathfrak{R}$ is defined by

$$Mf(x) = \sup_{R \in \mathfrak{R}(x)} \frac{1}{|R|} \int_R |f(y)|\, dy$$

for any f locally integrable. When $\mathfrak{R}$ is invariant by translations, the set $\{x\colon Mf(x) > \lambda\}$ is open for every λ, and so Mf is measurable. In the following we will consider always bases which are invariant by translations.

The operator M is of *weak type* (p, p), $1 \leqslant p < \infty$, if there exists $c > 0$ such that for every $f \in L^p(\boldsymbol{R}^n)$ and $\lambda > 0$ is verified

$$|\{x\colon Mf(x) > \lambda\}| \leqslant c \int_{\boldsymbol{R}^n} \left|\frac{f(y)}{\lambda}\right|^p dy.$$

We will use the following Sawyer's version [2] of a theorem of E. M. Stein [4]:

Let $(X, \mathscr{A}, m)$ be a space of finite measure, and $1 \leqslant p < \infty$. We consider a sequence $\{T_k\}$ of positive linear operators continuous in measure defined in $L^p(X)$, and suppose there exists a family F of transformations in X preserving the measure and commuting with every T_k such that, given $\varrho > 1$ and two subsets A, B of X with positive measure, there exists $t \in F$ which verifies

$$m(X)\, m(A \cap t^{-1}B) \leqslant \varrho m(A)\, m(B).$$

Then the following conditions are equivalent:

(a) *For every $f \in L^p(X)$, $T^*f(x)$ is finite* a.e. *in x, being $T^*f(x) = \sup\{|T_k f(x)|\colon k = 1, 2, \ldots\}$.*

(b) *There exists $c > 0$ such that*

$$m\{x\colon T^*f(x) > \lambda\} \leqslant c \int_X \left|\frac{f(y)}{\lambda}\right|^p dy$$

for arbitrary $f \in L^p(X)$ and $\lambda > 0$.

§ 3. We can state now the following theorem.

THEOREM I. *Let $\{R_k\}$ be a sequence of open bounded sets in $\boldsymbol{R}^n$ with positive Lebesgue measure such that $R_k \to 0$ (the zero of $\boldsymbol{R}^n$). We consider the differentiation basis $\mathfrak{R}(x) = \{x + R_k\colon k = 1, 2, \ldots\}$ in $\boldsymbol{R}^n$, and the associated Hardy–Littlewood maximal operator M. The two following conditions are equivalent, being $1 \leqslant p < \infty$,*

(a) *$\mathfrak{R}$ differentiates $\int f$ for every $f \in L^p(\boldsymbol{R}^n)$.*

(b) *M is of weak type (p, p).*

Proof. It is necessary only to prove (a)$\Rightarrow$(b), because the proof of (b)$\Rightarrow$(a) is easy.

We can suppose diameter $R_k < 1$ for every k. Let X be the unit interval $[0, 1]^n$. For every $j \in Z^n$, $j + X$ will be identified with X and so, if diameter $E < 1$, there is a natural bijection between E and a subset E' of X. For $x \in X$ and every k we write $T_k(x) = (x + R_k)'$, and consider the differentiation basis $\mathfrak{F}$ in X defined by $\mathfrak{F}(x) = \{T_k(x)\colon k = 1, 2, \ldots\}$ and the operator T_k in $L^p(X)$ such that

$$T_k f(x) = \frac{1}{|R_k|} \int_{T_k(x)} f(y)\, dy.$$

It can be observed that the bases $\mathfrak{R}$ and $\mathfrak{F}$ have the same differentiation properties in the interior points of X, because in such an x we have $T_k(x) = x + R_k$ for large k. This means that $T^*f(x) < \infty$ a.e. in X, for every $f \in L^p(X)$, being $T^*f(x) = \sup\{|T_k f(x)|\colon k = 1, 2, \ldots\}$. Now the Stein theorem is used to obtain the existence of a number $c > 0$ such that

$$|\{x \in X\colon T^*f(x) > \lambda\}| \leqslant c \int_X \left|\frac{f(y)}{\lambda}\right|^p dy$$

for every $f \in L^p(X)$ and $\lambda > 0$. It is easy to prove that all the conditions to apply this theorem are satisfied. The required transformations in X are the translations of X considered as a torus. Every T_k is a positive linear operator which is continuous in $L^p(X)$ and commutes with the translations in X. Furthermore, given A, B in X with positive measure, if we suppose $|A \cap tB| > \varrho |A|\,|B|$ for every translation t, since $|A \cap tB| = (\chi_A * \chi_{-B})(t)$, we have

$$\int_X (\chi_A * \chi_{-B})(t)\, dt > \varrho |A|\,|B|$$

and so one obtains a contradiction, because by the Fubini theorem the first member above is $|A|\,|B|$.

Now for each $u_i \in \{0, 1\}^n$ we consider the interval $X_i = u_i + X$, and put $X_{ij} = 2j + X_i$ for $j \in Z^n$. It is possible to do in every X_{ij} the same that we did in X, and the corresponding operator is denoted by T^*_{ij}. The constant obtained to apply the Stein theorem does not depend on i, j.

Let f be a non-negative function in $L^p(\boldsymbol{R}^n)$; f_i and f_{ij} are the corresponding restrictions to $\bigcup_j X_{ij}$ and X_{ij}, respectively. It is easy to prove that $Mf_{ij}(x) \leqslant T^*_{ij}f_{ij}(x)$ a.e. in X_{ij}, and so we have

$$|\{x \in \boldsymbol{R}^n: Mf(x) > \lambda\}| \leqslant \sum_i \Big|\Big\{x \in \bigcup_j X_{ij}: Mf_i(x) > \lambda 2^{-n}\Big\}\Big| \leqslant \sum_{i,j} |\{x \in X_{ij}: T^*_{ij}f_{ij}(x) > \lambda 2^{-n}\}| \leqslant 2^{np} c \int_{\boldsymbol{R}^n} \left(\frac{f(y)}{\lambda}\right)^p dy.$$

§ 4. For general homothecy invariant bases we have an analogous theorem.

THEOREM II. *Let $\mathfrak{R}$ be a homothecy invariant differentiation basis in $\boldsymbol{R}^n$, and M the associated Hardy–Littlewood maximal operator. The two following conditions are equivalent, being $1 \leqslant p < \infty$,*

(a) *$\mathfrak{R}$ differentiates $\int f$ for every $f \in L^p(\boldsymbol{R}^n)$.*

(b) *M is of weak type (p, p).*

Proof. As before we only have to prove (a) $\Rightarrow$ (b). Because $\mathfrak{R}$ is homothecy invariant, it is easy to prove that, M_δ being the maximal operator associated with the basis $\mathfrak{R}_\delta(x) = \{R \in \mathfrak{R}(x)\colon$ diameter $R < \delta\}$, M is of weak type (p, p) if and only if M_δ is so for some $\delta > 0$.

In order to prove (a) $\Rightarrow$ (b) we will prove that otherwise we obtain a contradiction with Theorem I. We suppose that (a) and not (b) are true. Then the maximal operator M_k associated with $\mathfrak{R}_{1/k}$ is not of weak type (p, p) for every k, and we can choose one sequence $\{f_k\}$ of non-negative functions in $L^p(\boldsymbol{R}^n)$ and one sequence $\{\lambda_k\}$ of positive numbers such that, being $E_k = \{x \in \boldsymbol{R}^n: M_k f_k(x) > \lambda_k\}$,

$$|E_k| > k\lambda_k^{-p} \|f_k\|_p^p$$

holds. If g_k verifies $\lambda_k g_k = kf_k$, we have $E_k = \{x: M_k g_k(x) > k\}$ and $|E_k| > k^{1-p}\|g_k\|_p^p$. We take a compact $F_k \subset E_k$ such that $|F_k|$ is also greater than $k^{1-p}\|g_k\|_p^p$. Given $x \in F_k$, there exists $R \in \mathfrak{R}(0)$ such that the diameter $R < k^{-1}$ and

$$\frac{1}{|R|} \int_{x+R} g_k(y)\,dy > k.$$

There is also a sphere $B(x)$ with center in x such that for every z in $B(X)$

$$\frac{1}{|R|} \int_{z+R} g_k(y)\,dy > k$$

holds, with the same R as before. We select a finite number of such spheres to cover F_k. Let $R_{k1}, R_{k2}, \ldots, R_{kh_k}$ be the members of $\mathfrak{R}_{1/k}(0)$ associated with them.

Now we consider the sequence R_j defined by

$$R_{11}, \ldots, R_{1h_1}, R_{21}, \ldots, R_{2h_2}, R_{31}, \ldots$$

It is clear that $R_j \to 0$, and so we can define a differentiation basis $\mathfrak{R}'$ in $\boldsymbol{R}^n$ setting $\mathfrak{R}'(x) = \{x + R_j: j = 1, 2, \ldots\}$. By Theorem I the maximal operator M' associated with $\mathfrak{R}'$ is of weak type (p, p). But this is impossible, because given $c > 0$ we have for $k > c$

$$|\{x \in \boldsymbol{R}^n: M'g_k(x) > k\}| \geqslant |F_k| > ck^{-p}\|g_k\|_p^p.$$

References

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