STUDIA MATHEMATICA, T. LVII. (1976)

A remark on p-integral and p-absolutely summing operators from l_u into l_v

by

B. CARL (Jena, DDR)

Abstract. Let $\boldsymbol{I}_p$ and $\boldsymbol{P}_p$ be the ideals of p-integral and p-absolutely summing operators, respectively. It is shown that, if $p \neq 2$ and $1 \leqslant p < \infty$, there exist Banach spaces l_u and l_v such that $\boldsymbol{I}_p(l_u, l_v) \neq \boldsymbol{P}_p(l_u, l_v)$.

In this note we shall show that for each exponent $p \geqslant 1$, with $p \neq 2$, the spaces $\boldsymbol{I}_p(l_u, l_v)$ and $\boldsymbol{P}_p(l_u, l_v)$ are distinct provided that $1 < p^* < u < 2 < v < p$ and $1 < p < v < 2 < u < p^*$, respectively.

Thus we have a negative decision over a problem of A. Pietsch [4] and D. J. H. Garling [1]. Furthermore, we get a simple proof for $\boldsymbol{I}_p \neq \boldsymbol{P}_p$ (cf. A. Pełczyński [2]). Finally, we can disprove the conjecture of A. Pietsch [5] (problem 16.1.3) that $\boldsymbol{P}_p(E, l_v) = \boldsymbol{P}_q(E, l_v)$ for arbitrary E and $2 < v \leqslant p < q$.

1. Basic notations. Let l_u denote the Banach space of all u-absolutely summable sequences provided with the norm

$$\|x\|_u = \Big(\sum |\xi_i|^u\Big)^{1/u} \quad \text{if} \quad 1 \leqslant u < \infty,$$

and

$$\|x\|_\infty = \sup |\xi_i| \qquad \text{if} \quad u = \infty,$$

respectively.

Analogously, l_u^n denotes the Banach space of all n-dimensional vectors $(\xi_1, \ldots, \xi_n)$ with the corresponding norm. We refer to A. Perrson/ A. Pietsch [3] or to A. Pietsch [5] for definitions and fundamental properties of the normed ideals $[\boldsymbol{I}_p, \iota_p]$ and $[\boldsymbol{P}_p, \pi_p]$ of p-integral and p-absolutely summing operators, respectively.

2. Limit order of operator ideals. In the sequel, I_n is the identity operator from l_u^n into l_v^n. We define the *limit order* $\Lambda_I(\boldsymbol{A}, u, v)$ of a complete quasi-normed ideal $[\boldsymbol{A}, \boldsymbol{a}]$ as follows (cf. [4]):

$$\Lambda_I(\boldsymbol{A}, u, v) := \inf\{\lambda > 0\colon\ \exists c > 0\ \ \forall n \in \boldsymbol{a}N(I_n\colon\ l_u^n \to l_v^n) \leqslant cn^\lambda\}.$$

The limit orders have been calculated by A. Pietsch [4] for the p-integral and p-absolutely summing operators.

We have, in detail, the following statement for p-integral operators:

THEOREM 1. *Let* $2 < p < \infty$; *then*

$$A_I(I_p, u, v) = \begin{cases} \dfrac{1}{u^*}+\dfrac{1}{v}-\dfrac{1}{2} & \text{if } 1 \leqslant u \leqslant 2,\ 1 \leqslant v \leqslant 2, \\ \dfrac{1}{u^*} & \text{if } 1 \leqslant u \leqslant p^*,\ 2 \leqslant v \leqslant \infty, \\ \dfrac{1}{u^*} & \text{if } p^* \leqslant u \leqslant 2,\ 2 \leqslant v \leqslant u^*, \\ \dfrac{1}{v} & \text{if } p^* \leqslant u \leqslant 2,\ 2 \leqslant u^* \leqslant v \leqslant p, \\ \dfrac{1}{v} & \text{if } 2 \leqslant u \leqslant \infty,\ 1 \leqslant v \leqslant p, \\ \dfrac{1}{p} & \text{if } p^* \leqslant u \leqslant \infty,\ p \leqslant v \leqslant \infty \end{cases}$$

(r^* *is the conjugate exponent of* r, $1/r^*+1/r = 1$).

The results of this theorem are expressed diagrammatically. In what follows we shall illustrate our results by pairs of diagrams in the unit square with coordinates $1/u$ and $1/v$. In the left-hand diagram we plot

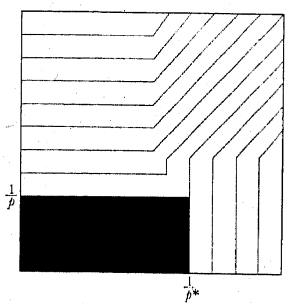

Fig. 1

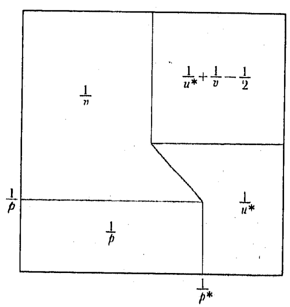

Fig. 2

the level curves of $A_I(I_p, u, v)$. In the right-hand diagram we indicate the algebraic expression for $A_I(I_p, u, v)$.

The limit orders of p-integral and p-absolutely summing operators coincide outside the square $Q = \{(1/u, 1/v)\colon 1 < p^* < u < 2 < v < p\}$.

In order to give an estimate for the limit order of p-absolutely summing operators in the unknown square Q, we need the following lemma.

LEMMA. $A_I(P_p, u, v)$ *is a convex function of* $1/v$.

Proof. The definition of π_p and the relation

$$\|x\|_v \leqslant \|x\|_{v_1}^{1-\theta}\|x\|_{v_2}^{\theta} \quad \text{with} \quad \frac{1}{v} = \frac{1-\theta}{v_1}+\frac{\theta}{v_2} \text{ and } 0 < \theta < 1$$

imply

$$\pi_p(I_n\colon l_u^n \to l_v^n) \leqslant \pi_p^{1-\theta}(I_n\colon l_u^n \to l_{v_1}^n)\,\pi_p^{\theta}(I_n\colon l_u^n \to l_{v_2}^n).$$

Hence, $\pi_p(I_n\colon l_u^n \to l_v^n)$ is a logarithmic convex function of $1/v$. Finally, we get

$$A_I(P_p, u, v) \leqslant (1-\theta)\,A_I(P_p, u, v_1)+\theta A_I(P_p, u, v_2).$$

A consequence of the convexity lemma is

THEOREM 2. *Let* $1 < p^* < u < 2 < v < p$. *Then*

$$\max\left\{\frac{1}{p};\ \frac{1}{u^*}+\frac{1}{v}-\frac{1}{2}\right\} \leqslant A_I(P_p, u, v) \leqslant \frac{1}{u^*}+\frac{\dfrac{1}{u^*}-\dfrac{1}{p}}{\dfrac{1}{2}-\dfrac{1}{p}}\left(\frac{1}{v}-\frac{1}{2}\right).$$

Proof. Using the ideal property and Theorem 1 we have

$$\frac{1}{v} = A_I(P_p, 2, v) \leqslant A_I(Z, 2, u)+A_I(P_p, u, v) \leqslant \frac{1}{u}-\frac{1}{2}+A_I(P_p, u, v)$$

or

$$\frac{1}{u^*}+\frac{1}{v}-\frac{1}{2} \leqslant A_I(P_p, u, v),$$

and, similarly,

$$\frac{1}{p} = A_I(P_p, u, p) \leqslant A_I(P_p, u, v)+A_I(Z, v, p) \leqslant A_I(P_p, u, v)$$

(Z is the class of bounded operators with the operator norm).

By the convexity lemma and Theorem 1 we get

$$A_I(P_p, u, v) \leqslant (1-\theta)\,A_I(P_p, u, 2)+\theta\,A_I(P_p, u, p)$$

$$= \frac{1-\theta}{u^*}+\frac{\theta}{p} \quad \text{for} \quad \frac{1}{v} = \frac{1-\theta}{2}+\frac{\theta}{p}.$$

Combining these relations we obtain the theorem.

A consequence of Theorems 1 and 2 is

THEOREM 3. *Let* $1 < p^* < u < 2 < v < p$. *Thne*

$$\Lambda_I(\boldsymbol{I}_p, u, v) - \Lambda_I(\boldsymbol{P}_p, u, v) \geqslant g_p(u, v)$$

$$= \begin{cases} \dfrac{\dfrac{1}{2} - \dfrac{1}{v}}{\dfrac{1}{2} - \dfrac{1}{p}} \left(\dfrac{1}{u^*} - \dfrac{1}{p}\right) & \text{if } 2 < v \leqslant u^* < p, \\[2ex] \dfrac{\dfrac{1}{2} - \dfrac{1}{u^*}}{\dfrac{1}{2} - \dfrac{1}{p}} \left(\dfrac{1}{v} - \dfrac{1}{p}\right) & \text{if } 2 < u^* \leqslant v < p. \end{cases}$$

Illustrating the preceding results in the following diagram, the functions $\Lambda_I(\boldsymbol{I}_p, u, v)$ and $\Lambda_I(\boldsymbol{P}_p, u, v)$ are plottered in dependence on $1/v$ while u and p are constant. The bold line shows the graph of $\Lambda_I(\boldsymbol{I}_p, u, v)$, which coincides with $\Lambda_I(\boldsymbol{P}_p, u, v)$ in the intervals $1 \leqslant v \leqslant 2$ and $p \leqslant v \leqslant \infty$. The graph of $\Lambda_I(\boldsymbol{P}_p, u, v)$ for $2 < v < p$ is contained in the indicated region of the parallelogram, generated by the vertices $\left(\frac{1}{p}, \frac{1}{p}\right)$, $\left(\frac{1}{u^*}, \frac{1}{u^*}\right)$, $\left(\frac{1}{2}, \frac{1}{u^*}\right)$ and $\left(-\frac{1}{u^*} + \frac{1}{p} + \frac{1}{2}, \frac{1}{p}\right)$.

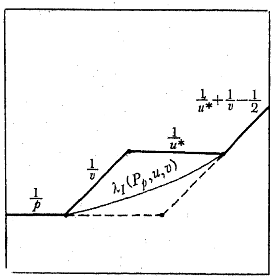

Fig. 3

The function $g_p(u, v)$ takes its maximum at $\frac{1}{v} = \frac{1}{u^*} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{p}\right)$. At this point we get the following estimate for the distance of these two limit orders:

$$\Lambda_I(\boldsymbol{I}_p, v^*, v) - \Lambda_I(\boldsymbol{P}_p, v^*, v) \geqslant \frac{1}{4}\left(\frac{1}{2} - \frac{1}{p}\right).$$

The $1/v$-coordinate of the top left-hand vertex and the bottom right-hand vertex of the parallelogram are the same. We conjecture that even the identity

$$\Lambda_I(\boldsymbol{I}_p, v^*, v) - \Lambda_I(\boldsymbol{P}_p, v^*, v) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$$

holds.

As a corollary of Theorem 3 we get

THEOREM 4. *Let* $1 < p < v < 2 < u < p^*$ *or* $1 < p^* < u < 2 < v < p$. *Then*

$$\boldsymbol{I}_p(l_u, l_v) \subsetneqq \boldsymbol{P}_p(l_u, l_v).$$

Proof. By using the duality theorem of A. Perrson/A. Pietsch (cf. [3]) the first part of this theorem can be deduced from the second one.

Now, let $1 < p^* < u < 2 < v < p$. We suppose $\boldsymbol{P}_p(l_u, l_v) \subset \boldsymbol{I}_p(l_u, l_v)$. Then by the closed-graph theorem there exists a constant $c > 0$ such that for every operator $T \in \boldsymbol{P}_p(l_u, l_v)$ the inequality

$$\iota_p(T) \leqslant c\pi_p(T)$$

is valid.

Using Theorem 3 we obtain

$$\Lambda_I(\boldsymbol{I}_p, u, v) > \Lambda_I(\boldsymbol{P}_p, u, v).$$

Define the operator $P_n\colon l_u \to l_v$ by

$$P_n(\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots) := (\xi_1, \ldots, \xi_n, 0, 0, \ldots),$$

then

$$\pi_p(P_n\colon l_u \to l_v) = \pi_p(I_n\colon l_u^n \to l_v^n) \text{ and } \iota_p(P_n\colon l_u \to l_v) = \iota_p(I_n\colon l_u^n \to l_v^n).$$

Consequently, there does not exist such a constant c.

From the above proof it follows that a conjecture of D. J. H. Garling [1] is false. We have

THEOREM 5. *Let* $1 < p < v < 2 < u < p^*$ *or* $1 < p^* < u < 2 < v < p$. *Then there exist diagonal operators D from l_u into l_v such that*

$$D \in \boldsymbol{P}_p(l_u, l_v) \quad \text{and} \quad D \notin \boldsymbol{I}_p(l_u, l_v).$$

Finally, we may disprove the conjectures 16.1.3 and 16.1.4 of [5]. Actually, as a further corollary of Theorems 1 and 2 we get

THEOREM 6. *Let* $2 < v \leqslant p < u^* < q$. *Then*

$$\boldsymbol{P}_p(l_u, l_v) \subsetneqq \boldsymbol{P}_q(l_u, l_v).$$

Proof. We suppose $\boldsymbol{P}_p(l_u, l_v) = \boldsymbol{P}_q(l_u, l_v)$. Then by the closed-graph theorem there is a constant $c > 0$ such that for every operator $T \in \boldsymbol{P}_q(l_u, l_v)$ the inequality

$$\boldsymbol{\pi}_p(T) \leqslant c\boldsymbol{\pi}_q(T)$$

holds. Theorems 1 and 2 imply the relations

$$\Lambda_I(\boldsymbol{P}_p, u, v) = \frac{1}{u^*} \quad \text{and} \quad \Lambda_I(\boldsymbol{P}_q, u, v) < \frac{1}{u^*}.$$

Therefore, similarly as in Theorem 4, we get a contradiction.

I want to thank Professor A. Pietsch for his suggestions.

References

[1] D. J. H. Garling, *Diagonal mappings between sequence spaces*, Studia Math. 51 (1974), pp. 129–138.

[2] A. Pełczyński, *p-integral operators commuting with group representations and examples of quasi-p-integral operators which are not p-integral*, ibid. 33 (1969), pp. 63–70.

[3] A. Persson and A. Pietsch, *p-nukleare und p-integrale Abbildungen in Banachräumen*, ibid. 33 (1969), pp. 19–62.

[4] A. Pietsch, *Absolutely p-summing operators in L_r-spaces I, II*, Sim. Goulaouic–Schwartz, Paris 1970/71.

[5] — *Theorie der Operatorenideale (Zusammenfassung)*, Jena 1972.

Received March 4, 1975 (965)

STUDIA MATHEMATICA, T. LVII. (1976)

Neutrices and the product of distributions

by

B. FISHER (Leicester)

Abstract. The product of two distributions f and g is defined to be the neutrix limit of the sequence $\{f_n g_n\}$, provided this limit exists, where

$$f_n = f * \delta_n, \quad g_n = g * \delta_n,$$

$\{\delta_n\}$ is a delta-sequence with support contained in the interval $(-\alpha_n, \alpha_n)$ and the negligible functions of the neutrix N are linear sums of the functions α_n^λ with $\lambda < 0$, $\alpha_n^\lambda \ln^p \alpha_n$ with $\lambda \leqslant 0$ and $p = 1, 2, \ldots$ and all functions $f(\alpha_n)$ for which $\lim\limits_{n\to 0} f(\alpha_n) = 0$. It is proved that

$$(x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-) = \frac{(-1)^k \Gamma(-k-\lambda)\Gamma(k+1+\lambda)}{2\Gamma(-\lambda)\Gamma(r+\lambda)} B(r+\lambda, p; -\lambda, q)\,\delta^{(r-1)}(x),$$

for $-k-1 < \lambda < -k$, $k = 1, 2, \ldots, r-2$, $r = 2, 3, \ldots$, and $p, q = 0, 1, 2, \ldots$, where

$$B(\lambda, p; \mu, q) = \int_0^1 v^{\lambda-1} \ln^p v\,(1-v)^{\mu-1} \ln^q (1-v)\,dv,$$

$$x_+^r\, \delta^{(r+p)}(x) = \frac{(-1)^r (r+p)!}{2p!} \delta^{(p)}(x),$$

for $r, p = 0, 1, 2, \ldots$ and

$$\delta^{(r)}(x)\,\delta^{(p)}(x) = 0,$$

for $r, p = 0, 1, 2, \ldots$

1. Introduction. J. G. van der Corput developed his neutrix calculus having noticed that, in his study of the asymptotic behaviour of integrals, functions of a certain type could be neglected. This idea was also used by J. Hadamard, see [4], when he defined the finite part of an integral by neglecting powers of $x - a$.

A neutrix N is defined, see [1], as a commutative additive group of functions $\nu(\xi)$ defined at each element ξ of a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The functions in N are called *negligible functions*.

Now let N' be set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with