Proof. We suppose $\boldsymbol{P}_p(l_u, l_v) = \boldsymbol{P}_q(l_u, l_v)$. Then by the closed-graph theorem there is a constant $c > 0$ such that for every operator $T \in \boldsymbol{P}_q(l_u, l_v)$ the inequality

$$\boldsymbol{\pi}_p(T) \leqslant c\boldsymbol{\pi}_q(T)$$

holds. Theorems 1 and 2 imply the relations

$$\Lambda_I(\boldsymbol{P}_p, u, v) = \frac{1}{u^*} \quad \text{and} \quad \Lambda_I(\boldsymbol{P}_q, u, v) < \frac{1}{u^*}.$$

Therefore, similarly as in Theorem 4, we get a contradiction.

I want to thank Professor A. Pietsch for his suggestions.

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Received March 4, 1975 (965)

STUDIA MATHEMATICA, T. LVII. (1976)

Neutrices and the product of distributions

by

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Abstract. The product of two distributions f and g is defined to be the neutrix limit of the sequence $\{f_n g_n\}$, provided this limit exists, where

$$f_n = f * \delta_n, \quad g_n = g * \delta_n,$$

$\{\delta_n\}$ is a delta-sequence with support contained in the interval $(-\alpha_n, \alpha_n)$ and the negligible functions of the neutrix N are linear sums of the functions α_n^λ with $\lambda < 0$, $\alpha_n^\lambda \ln^p \alpha_n$ with $\lambda \leqslant 0$ and $p = 1, 2, \ldots$ and all functions $f(\alpha_n)$ for which $\lim\limits_{n\to 0} f(\alpha_n) = 0$. It is proved that

$$(x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-) = \frac{(-1)^k \Gamma(-k-\lambda)\Gamma(k+1+\lambda)}{2\Gamma(-\lambda)\Gamma(r+\lambda)} B(r+\lambda, p; -\lambda, q)\, \delta^{(r-1)}(x),$$

for $-k-1 < \lambda < -k$, $k = 1, 2, \ldots, r-2$, $r = 2, 3, \ldots$, and $p, q = 0, 1, 2, \ldots$, where

$$B(\lambda, p; \mu, q) = \int\limits_0^1 v^{\lambda-1} \ln^p v\, (1-v)^{\mu-1} \ln^q (1-v)\, dv,$$

$$x_+^r\, \delta^{(r+p)}(x) = \frac{(-1)^r (r+p)!}{2p!}\, \delta^{(p)}(x),$$

for $r, p = 0, 1, 2, \ldots$ and

$$\delta^{(r)}(x)\, \delta^{(p)}(x) = 0,$$

for $r, p = 0, 1, 2, \ldots$

1. Introduction. J. G. van der Corput developed his neutrix calculus having noticed that, in his study of the asymptotic behaviour of integrals, functions of a certain type could be neglected. This idea was also used by J. Hadamard, see [4], when he defined the finite part of an integral by neglecting powers of $x-a$.

A neutrix N is defined, see [1], as a commutative additive group of functions $\nu(\xi)$ defined at each element ξ of a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi) = \gamma$ for all ξ in N', then $\gamma = 0$. The functions in N are called *negligible functions*.

Now let N' be set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi)-\beta$ is negligible in N, then β is called the *neutrix limit* or N-*limit* of f as ξ tends to b, and we write

$$\operatorname*{N-lim}_{\xi\to b} f(\xi) = \beta.$$

This limit is of course unique if it exists.

As an example of how neutrices can be used to define distributions, let us consider the distribution x_+^λ. When $\lambda > -1$, this is an ordinary summable function defined by

$$x_+^\lambda = \begin{cases} x^\lambda, & \text{for} \quad x > 0, \\ 0, & \text{for} \quad x \leqslant 0. \end{cases}$$

For other values of λ, $\lambda \neq -1, -2, \ldots$, x_+^λ is defined inductively by the equation

$$(\lambda+1)(x_+^\lambda, \varphi) = -(x_+^{\lambda+1}, \varphi'),$$

for arbitrary test function φ in the space K of infinitely differentiable test functions with compact support. It follows that if $-r-1 < \lambda < -r$,

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx.$$

Now let us consider the integral

$$\int_{a_n}^\infty x^\lambda \varphi(x)\,dx,$$

where $-r-1 < \lambda < -r$, $0 < a_n \to 0$ and φ is an arbitrary test function. We can rewrite this integral as

$$\int_{a_n}^\infty x^\lambda \varphi(x)\,dx = \int_{a_n}^\infty x^\lambda \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx - \sum_{s=0}^{r-1} \frac{a_n^{\lambda+s+1}}{s!(\lambda+s+1)}\varphi^{(s)}(0).$$

We will now let N be the neutrix having domain $N' = \{a_1, a_2, \ldots, a_n, \ldots\}$ and range N'' the real numbers. The negligible functions of N will be linear sums of the function a_n^λ with $\lambda < 0$ and all functions $f(a_n)$ for which $\lim_{n\to\infty} f(a_n) = 0$. We will consider the N-limit of functions f or N' as n tends to infinity.

We notice that

$$\lim_{n\to\infty} \int_{a_n}^\infty x^\lambda \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx = \int_0^\infty x^\lambda \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx$$

and that the function

$$\sum_{s=0}^{r-1} \frac{a_n^{\lambda+s+1}}{s!(\lambda+s+1)}\varphi^{(s)}(0)$$

is negligible in N. It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_{a_n}^\infty x^\lambda \varphi(x)\,dx = \int_0^\infty x^\lambda \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx$$

and so for $\lambda \neq -1, -2, \ldots$, we can write

$$(x_+^\lambda, \varphi) = \operatorname*{N-lim}_{n\to\infty} \int_{a_n}^\infty x^\lambda \varphi(x)\,dx.$$

More generally, if we increase the number of negligible functions in N to also include linear sums of the functions $a_n^\lambda \ln^p a_n$ with $\lambda \leqslant 0$ and $p = 1, 2, \ldots$ we can prove that if $-r-1 < \lambda < -r$ and $p = 0, 1, 2, \ldots$

$$(x_+^\lambda \ln^p x_+, \varphi) = \int_0^\infty x^\lambda \ln^p x \Big[\varphi(x) - \sum_{s=0}^{r-1} \frac{x^s}{s!}\varphi^{(s)}(0)\Big]\,dx$$
$$= \operatorname*{N-lim}_{n\to\infty} \int_{a_n}^\infty x^\lambda \ln^p x\, \varphi(x)\,dx.$$

The above set of negligible functions arise naturally in the following discussion of the product of distributions and so we will be using the above neutrix throughout this paper.

2. Definition of the product. The product of two distributions f and g was defined in [2] as the limit of the sequence $\{f_n g_n\}$, provided this sequence is regular, where

$$f_n = f * \delta_n, \qquad g_n = g * \delta_n,$$

for $n = 1, 2, \ldots$ and $\{\delta_n\}$ is a sequence of infinitely differentiable functions satisfying the following properties:

(1) $\delta_n(x) = 0$ for $|x| \geqslant a_n \to 0$,

(2) $\delta_n(x) \geqslant 0$,

(3) $\delta_n(x) = \delta_n(-x)$,

(4) $\displaystyle\int_{-a_n}^{a_n} \delta_n(x)\,dx = 1$.

It is obvious that the sequence $\{\delta_n\}$ converges to the Dirac delta-function $\delta(x)$.

Mikusiński, see [5], had earlier used this definition for the particular product $x^{-1}\delta(x)$.

It was then proved that

$$x_+^{\lambda}x_-^{-1-\lambda} = -\tfrac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\delta(x) \tag{2.1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and

$$x_+^{r}\delta^{(r)}(x) = \tfrac{1}{2}(-1)^r r!\,\delta(x), \tag{2.2}$$

$$x_-^{r}\delta^{(r)}(x) = \tfrac{1}{2}r!\,\delta(x) \tag{2.3}$$

for $r = 1, 2, \ldots$

In [3] it was proved that in general, with the above definition, the product $x_+^{\lambda}x_-^{-r-\lambda}$ did not exist but the product $x_+^{-r-1/2}x_-^{-r-1/2}$ did exist and

$$x_+^{-r-1/2}x_-^{-r-1/2} = \frac{(-1)^r\pi\delta^{(2r)}(x)}{2(2r)!} \tag{2.4}$$

for $r = 0, 1, 2, \ldots$

We now give a definition of the product of two distributions for which further products of distributions can be defined.

DEFINITION. Let f and g be arbitrary distributions and let

$$f_n = f*\delta_n, \qquad g_n = g*\delta_n.$$

We will say that the *product of f and g exists* and is *equal to the distribution h* provided that

$$\operatorname*{N-lim}_{n\to\infty}(f_n g_n, \varphi) = (h, \varphi)$$

for all test functions φ in K, where N is the particular neutrix given in the introduction.

It is obvious that if the product fg exists by the former definition then it will exist by the new definition and will define the same distribution.

Using this new definition of the product we have:

THEOREM. *Let f and g be distributions and suppose the products fg and fg' exist. Then the product $f'g$ exists and*

$$f'g = (fg)' - fg'.$$

Proof. Since f_n and g_n are infinitely differentiable functions, we have

$$f_n' g_n = (f_n g_n)' - f_n g_n'$$

and so for arbitrary test function φ in K

$$(f_n' g_n, \varphi) = \big((f_n g_n)' - f_n g_n', \varphi\big).$$

It follows that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}(f_n' g_n, \varphi) &= \operatorname*{N-lim}_{n\to\infty}\big((f_n g_n)', \varphi\big) - \operatorname*{N-lim}_{n\to\infty}(f_n g_n', \varphi)\\ &= \big((fg)', \varphi\big) - (fg', \varphi)\end{aligned}$$

and the result follows.

3. The product $(x_+^{\lambda}\ln^p x_+)(x_-^{-r-\lambda}\ln^q x_-)$. The following lemma holds, see [3]:

LEMMA. *If $-\infty < t < \infty$, then*

$$\int_t^{a_n} s^p\delta_n^{(r)}(s)\,ds = -t^p\delta_n^{(r-1)}(t) + pt^{p-1}\delta_n^{(r-2)}(t) - \ldots + (-1)^{p-1}p!\,\delta_n^{(r-p-1)}(t)$$

for $r > p$ and

$$\begin{aligned}&\int_t^{a_n} s^r\delta_n^{(r)}(s)\,ds\\ &= -t^r\delta_n^{(r-1)}(t) + rt^{r-1}\delta_n^{(r-2)}(t) - \ldots + (-1)^r r!\,t\,\delta_n(t) + (-1)^r r!\,[1 - H_n(t)],\end{aligned}$$

where H denotes Heaviside's function.

We now consider the product $(x_+^{\lambda}\ln^p x_+)(x_-^{-r-\lambda}\ln^q x_-)$ for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $p, q = 0, 1, 2, \ldots$ We will first of all suppose that $-1 < \lambda < 0$. We have

$$(x_+^{\lambda}\ln^p x_+)_n = \int_{-a_n}^{x}(x-t)^{\lambda}\ln^p(x-t)\,\delta_n(t)\,dt$$

and

$$\frac{\Gamma(r+\lambda)}{\Gamma(1+\lambda)}(x_-^{-r-\lambda}\ln^q x_-)_n = \int_x^{a_n}(s-x)^{-1-\lambda}\ln^q(s-x)\,\delta_n^{(r-1)}(s)\,ds.$$

It follows that $(x_+^{\lambda}\ln^p x_+)_n(x_-^{-r-\lambda}\ln^q x_-)_n$ has its support contained in the interval $(-a_n, a_n)$ and

$$\begin{aligned}&\frac{\Gamma(r+\lambda)}{\Gamma(1+\lambda)}\int_{-a_n}^{a_n}(x_+^{\lambda}\ln^p x_+)_n(x_-^{-r-\lambda}\ln^q x_-)_n x^m\,dx\\ &= \int_{-a_n}^{a_n}\delta_n(t)\int_t^{a_n}\delta_n^{(r-1)}(s)\int_t^{s}x^m(x-t)^{\lambda}\ln^p(x-t)(s-x)^{-1-\lambda}\ln^q(s-x)\,dx\,ds\,dt\\ &= \int_{-a_n}^{a_n}\delta_n(t)\int_t^{a_n}\delta_n^{(r-1)}(s)\int_0^{1}[t(1-v)+sv]^m v^{\lambda}\ln^p[v(s-t)](1-v)^{-1-\lambda}\times\\ &\qquad\times\ln^q[(s-t)(1-v)]\,dv\,ds\,dt,\end{aligned}$$

where $x = t(1-v)+sv$. It follows that this integral is a linear sum of functions of the form

$$a_n^{-r+m+1}\ln^i a_n$$

which are negligible in N, for $m = 0, 1, \ldots, r-2$. Hence

$$\underset{n\to\infty}{\text{N-lim}} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n x^m\, dx = 0$$

for $m = 0, 1, \ldots, r-2$.

In the particular case $m = r-1$ it follows that the above integral is a linear sum of functions of the form

$$\ln^i a_n$$

for $i = 1, 2, \ldots, p+q$, which are negligible in N, plus the intergal

$$\int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} \delta_n^{(r-1)}(s) \int_0^1 [t(1-v)+sv]^{r-1} v^\lambda \ln^p v (1-v)^{-1-\lambda}\ln^q(1-v)\, dv ds dt$$
$$= \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} s^{r-1}\, \delta_n^{(r-1)}(s) \int_0^1 v^{\lambda+r-1}\ln^p v(1-v)^{-1-\lambda}\ln^q(1-v)\, dv ds dt,$$

all other integrals in the sum, on expanding $[t(1-v)+sv]^{r-1}$, being zero. We therefore have

$$\underset{n\to\infty}{\text{N-lim}} \frac{\Gamma(r+\lambda)}{\Gamma(1+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n x^{r-1}\, dx$$
$$= -\tfrac{1}{2}(-1)^{r-1}(r-1)!\, B(\lambda+r, p;\, -\lambda, q),$$

where

$$B(\lambda, p;\, \mu, q) = \int_0^1 v^{\lambda-1}\ln^p v(1-v)^{\mu-1}\ln^q(1-v)\, dv.$$

When $m = r$, it is easily seen that

$$\lim_{n\to\infty} \int_{-a_n}^{a_n} |(x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n x^r|\, dx = 0.$$

Hence if φ is an arbitrary test function in K we have

$$\underset{n\to\infty}{\text{N-lim}} \frac{\Gamma(r+\lambda)}{\Gamma(1+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n \varphi(x)\, dx$$
$$= -\tfrac{1}{2}(-1)^r \varphi^{(r-1)}(0)\, B(\lambda+r, p;\, -\lambda, q).$$

Thus the product $(x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda}\ln^q x_-)$ exists and

$$(x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda}\ln^q x_-) = \frac{\Gamma(1+\lambda)B(\lambda+r, p;\, -\lambda, q)}{2\Gamma(r+\lambda)}\, \delta^{(r-1)}(x)$$

for $-1<\lambda<0$ and $r = 1, 2, \ldots$

We will now suppose that $k-1<\lambda<k$. Then

$$(x_+^\lambda \ln^p x_+)_n = \int_{-a_n}^{x} (x-t)^\lambda \ln^p(x-t)\, \delta_n(t)\, dt$$

and

$$\frac{\Gamma(r+\lambda)}{\Gamma(1-k+\lambda)} (x_-^\lambda \ln^p x_-)_n = \int_x^{a_n} (s-x)^{k-1-\lambda}\ln^q(s-x)\, \delta_n^{(k+r-1)}(s)\, ds.$$

It follows that $(x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n$ has its support contained in the interval $(-a_n, a_n)$ and

$$\frac{\Gamma(r+\lambda)}{\Gamma(1-k+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n x^m\, dx$$
$$= \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} \delta_n^{(k+r-1)}(s) \int_t^s x^m (x-t)^\lambda \ln^p(x-t)(s-x)^{k-1-\lambda}\ln^q(s-x)\, dx ds dt$$
$$= \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} \delta_n^{(k+r-1)}(s) \int_0^1 [t(1-v)+sv]^m (s-t)^{kv\lambda}\ln^p[v(s-t)](1-v)^{k-1-\lambda}\times$$
$$\times \ln^q[(s-t)(1-v)]\, dv\, ds\, dt$$

which is negligible in N for $m = 0, 1, \ldots, r-2$.

When $m = r-1$ the integral is a linear sum of negligible functions in N plus the integral

$$\int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} \delta_n^{(k+r-1)}(s) \int_0^1 [t(1-v)+sv]^{r-1}(s-t)^k v^\lambda \ln^p v(1-v)^{k-1-\lambda}\times$$
$$\times \ln^q(1-v)\, dv\, ds\, dt$$
$$= \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} s^{k+r-1}\, \delta_n^{(k+r-1)}(s) \int_0^1 v^{\lambda+r+k-1}\ln^p v(1-v)^{k-1-\lambda}\ln^q(1-v)\, dv\, ds\, dt,$$

all other integrals in the sum being zero. We therefore have

$$\underset{n\to\infty}{\text{N-lim}} \frac{\Gamma(r+\lambda)}{\Gamma(1-k+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda}\ln^q x_-)_n x^{r-1}\, dx$$
$$= -\tfrac{1}{2}(-1)^{k+r}(k+r-1)!\, B(\lambda+r+k, p;\, k-\lambda, q).$$

When $m = r$ we again have

$$\lim_{n\to\infty} \int_{-a_n}^{a_n} |(x_+^{\lambda} \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n x^r|\, dx = 0.$$

Hence for arbitrary test function φ

$$\underset{n\to\infty}{\text{N-lim}} \frac{\Gamma(r+\lambda)}{\Gamma(1-k+\lambda)} \int_{-a_n}^{a_n} (x_+^{\lambda} \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n \varphi(x)\, dx$$
$$= -\frac{(-1)^{k+r}(k+r-1)!}{2(r-1)!} B(\lambda+r+k, p;\, k-\lambda, q)\varphi^{(r-1)}(0).$$

Thus the product $(x_+^{\lambda} \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-)$ exists and

$$(x_+^{\lambda} \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-)$$
$$= \frac{(-1)^k \Gamma(1-k+\lambda)(k+r-1)!}{2\Gamma(r+\lambda)(r-1)!} B(\lambda+r+k, p;\, k-\lambda, q)\, \delta^{(r-1)}(x)$$

for $k-1 < \lambda < k$, $k = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$

By replacing x by $-x$ we have

$$(x_-^{\lambda} \ln^p x_-)(x_+^{-r-\lambda} \ln^q x_+)$$
$$= -\frac{(-1)^{k+r} \Gamma(1-k+\lambda)(k+r-1)!}{2\Gamma(r+\lambda)(r-1)!} B(\lambda+r+k, p;\, k-\lambda, q)\, \delta^{(r-1)}(x)$$

for $k-1 < \lambda < k$, $k = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, or equivalently

$$(x_+^{\lambda} \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-)$$
$$= -\frac{(-1)^{k+r} \Gamma(1-k-r-\lambda)(k+r-1)!}{2\Gamma(-\lambda)(r-1)!} B(\lambda+r+k, p;\, k-\lambda, q)\, \delta^{(r-1)}(x)$$

for $-k-r < \lambda < -k-r+1$, $k = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$

Finally, let us suppose that $-k-1 < \lambda < -k$, for $k = 1, 2, \ldots, r-2$. We then have

$$\frac{(-1)^k \Gamma(-\lambda)}{\Gamma(-k-\lambda)} (x_+^{\lambda} \ln^p x_+)_n = \int_{-a_n}^{x} (x-t)^{k+\lambda} \ln^p (x-t)\, \delta_n^{(k)}(t)\, dt$$

and

$$\frac{\Gamma(r+\lambda)}{\Gamma(k+1+\lambda)} (x_-^{-r-\lambda} \ln^p x_-)_n = \int_{x}^{a_n} (s-x)^{-k-1-\lambda} \ln^q (s-x)\, \delta_n^{(r-k-1)}(s)\, dt.$$

It follows that $(x_+^{\lambda} \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n$ has its support contained in the interval $(-a_n, a_n)$ and

$$\frac{(-1)^k \Gamma(-\lambda)\Gamma(r+\lambda)}{\Gamma(-k-\lambda)\Gamma(k+1+\lambda)} \int_{-a_n}^{a_n} (x_+^{\lambda} \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n x^m\, dx$$
$$= \int_{-a_n}^{a_n} \delta_n^{(k)}(t) \int_{t}^{a_n} \delta_n^{(r-k-1)}(s) \int_{t}^{s} x^m (x-t)^{k+\lambda} \ln^p (x-t)(s-x)^{-k-1-\lambda} \times$$
$$\times \ln^q (s-x)\, dx\, ds\, dt$$
$$= \int_{-a_n}^{a_n} \delta_n^{(k)}(t) \int_{t}^{a_n} \delta_n^{(r-k-1)}(s) \int_{0}^{1} [t(1-v)+sv]^m v^{k+\lambda} \ln^p [v(s-t)] \times$$
$$\times (1-v)^{-k-1-\lambda} \ln^q [(s-x)(1-v)]\, dv\, ds\, dt$$

which is negligible in N for $m = 0, 1, \ldots, r-2$.

When $m = r-1$ integral is a linear sum of negligible functions in N plus the integral

$$\int_{-a_n}^{a_n} \delta_n^{(k)}(t) \int_{t}^{a_n} \delta_n^{(r-k-1)}(s) \times$$
$$\times \int_{0}^{1} [t(1-v)+sv]^{r-1} v^{k+\lambda} \ln^p v (1-v)^{-k-1-\lambda} \ln^q (1-v)\, dv\, ds\, dt.$$

Expanding $[t(1-v)+sv]^{r-1}$ and using the lemma (changing the order of integration if necessary), it is seen that all the integrals in the sum are zero except the integral

$$rc_k \int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t) \int_{t}^{a_n} s^{r-k-1} \delta^{(r-k-1)}(s) \int_{0}^{1} v^{r+\lambda-1} \ln^p v (1-v)^{-1-\lambda} \times$$
$$\times \ln^q (1-v)\, dv\, ds\, dt.$$

Now

$$\int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t) \int_{t}^{a_n} s^{r-k-1} \delta^{(r-k-1)}(s)\, ds\, dt$$
$$= \int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t) \{-t^{r-k-1} \delta_n^{(r-k-1)}(t) + \ldots - (-1)^{r-k}(r-k-1)!\, t\delta_n(t) -$$
$$-(-1)^{r-k}(r-k-1)!\,[1-H_n(t)]\}\, dt$$
$$= (-1)^{r-k-1}(r-k-1)! \int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t)[1-H_n(t)]\, dt,$$

the other integrals in the sum being zero. It was proved in [3] that

$$\int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t)[1-H_n(t)]\, dt = \tfrac{1}{2}(-1)^k k!$$

so that

$$ {}_{r-1}c_k \int_{-a_n}^{a_n} t^k \delta_n^{(k)}(t) \int_t^{a_n} s^{r-k-1} \delta^{(r-k-1)}(s)\, ds\, dt = \tfrac{1}{2}(-1)^{r-1}(r-1)!. $$

We therefore have

$$ \underset{n\to\infty}{\text{N-lim}} \frac{(-1)^k \Gamma(-\lambda)\Gamma(r+\lambda)}{\Gamma(-k-\lambda)\Gamma(k+1+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n x^{r-1}\, dx $$
$$ = -\tfrac{1}{2}(-1)^r (r-1)!\, B(r+\lambda, p;\ -\lambda, q). $$

When $m = r$ we have

$$ \lim_{n\to\infty} \int_{-a_n}^{a_n} |(x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n x^r|\, dx = 0. $$

Hence for arbitrary test function φ

$$ \underset{n\to\infty}{\text{N-lim}} \frac{(-1)^k \Gamma(-\lambda)\Gamma(r+\lambda)}{\Gamma(-k-\lambda)\Gamma(k+1+\lambda)} \int_{-a_n}^{a_n} (x_+^\lambda \ln^p x_+)_n (x_-^{-r-\lambda} \ln^q x_-)_n \varphi(x)\, dx $$
$$ = -\tfrac{1}{2}(-1)^r B(r+\lambda, p;\ -\lambda, q)\varphi^{(r-1)}(0). $$

Thus the product $(x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-)$ exists and

$$ (x_+^\lambda \ln^p x_+)(x_-^{-r-\lambda} \ln^q x_-) = \frac{(-1)^k \Gamma(-k-\lambda)\Gamma(k+1+\lambda)}{2\Gamma(-\lambda)\Gamma(r+\lambda)} B(r+\lambda, p;\ -\lambda, q)\, \delta^{(r-1)}(x) $$

for $-k-1 < \lambda < -k$, $k = 1, 2, \ldots, r-2$ and $r = 2, 3, \ldots$

In the particular case $p = q = 0$ we have the simplified result

$$ x_+^\lambda x_-^{-r-\lambda} = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!} \delta^{(r-1)}(x) $$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$, which generalizes equation (2.1).

4. The product $x_+^r \delta^{(r+p)}(x)$**.** The support of $(x_+^r)_n \delta_n^{(r+p)}(x)$ is obviously contained in the interval $(-a_n, a_n)$ and

$$ \int_{-a_n}^{a_n} (x_+^r)_n \delta_n^{(r+p)}(x) x^m\, dx = \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} x^m (x-t)^r \delta_n^{(r+p)}(x)\, dx\, dt. $$

This integral is a linear sum of functions of the form

$$ a_n^{-p+m} $$

which are negligible in N, for $m = 0, 1, \ldots, p-1$. Hence

$$ \underset{n\to\infty}{\text{N-lim}} \int_{-a_n}^{a_n} (x_+^r)_n \delta_n^{(r+p)}(x) x^m\, dx = 0. $$

for $m = 0, 1, \ldots, p-1$.

When $m = p$ we have

$$ \int_{-a_n}^{a_n} (x_+^r)_n \delta_n^{(r+p)}(x) x^p\, dx = \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} x^p (x-t)^r \delta_n^{(r+p)}(x)\, dx\, dt $$
$$ = \int_{-a_n}^{a_n} \delta_n(t) \int_t^{a_n} x^{r+p} \delta_n^{(r+p)}(x)\, dx\, dt, $$

all other integrals in the sum being zero. On using the lemma we now have

$$ \int_{-a_n}^{a_n} (x_+^r)_n \delta_n^{(r+p)}(x) x^p\, dx = (-1)^{r+p}(r+p)! \int_{-a_n}^{a_n} \delta_n(t)[1 - H_n(t)]\, dt $$
$$ = \tfrac{1}{2}(-1)^{r+p}(r+p)!. $$

When $m = p+1$ we obviously have

$$ \lim_{n\to\infty} \int_{-a_n}^{a_n} |(x_+^r)_n \delta_n^{(r+p)}(x) x^{p+1}|\, dx = 0. $$

It follows that if φ is an arbitrary test function in K

$$ \underset{n\to\infty}{\text{N-lim}} \int_{-a_n}^{a_n} (x_+^r)_n \delta_n^{(r+p)}(x) \varphi(x)\, dx = \frac{(-1)^{r+p}\,(r+p)!}{2p!} \varphi^{(p)}(0). $$

Thus the product $x_+^r \delta^{(r+p)}(x)$ exists and

$$ x_+^r \delta^{(r+p)}(x) = \frac{(-1)^r\,(r+p)!}{2p!} \delta^{(p)}(x) \tag{4.1} $$

for $r, p = 0, 1, 2, \ldots$

It is obvious that equation (2.2) is a particular case of this equation. Since

$$ x^r \delta^{(r+p)}(x) = [x_+^r + (-1)^r x_-^r]\, \delta^{(r+p)}(x) = \frac{(-1)^r (r+p)!}{p!} \delta^{(p)}(x) $$

for $r, p = 0, 1, 2, \ldots$, it follows that

$$ x_-^r \delta^{(r+p)}(x) = \frac{(r+p)!}{2p!} \delta^{(p)}(x) $$

for $r, p = 0, 1, 2, \ldots$

5. The product $\delta^{(r)}(x)\,\delta^{(p)}(x)$. When $r = 0$ in equation (5.1) we have

$$H(x)\,\delta^{(p)}(x) = \tfrac{1}{2}\,\delta^{(p)}(x)$$

for $p = 0, 1, 2, \ldots$ On using the theorem it follows that

$$\delta(x)\,\delta^{(p)}(x) = \tfrac{1}{2}\,\delta^{(p+1)}(x) - H(x)\,\delta^{(p+1)}(x) = 0$$

for $p = 0, 1, 2, \ldots$

We will now assume that

$$\delta^{(r)}(x)\,\delta^{(p)}(x) = 0$$

for some r and $p = 0, 1, 2, \ldots$ Then using the theorem we have

$$\delta^{(r+1)}(x)\,\delta^{(p)}(x) = 0 - \delta^{(r)}(x)\,\delta^{(p+1)}(x) = 0$$

or $p = 0, 1, 2, \ldots$ It follows by induction that

$$\delta^{(r)}(x)\,\delta^{(p)}(x) = 0$$

or $r, p = 0, 1, 2, \ldots$

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Received February 28, 1975 (968)

STUDIA MATHEMATICA, T. LVII. (1976)

An inequality for integrals

by

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Abstract. An inequality for n-fold integrals of products of functions of less than n variables is obtained and applied to obtain a Sobolev type inequaliy.

Consider the following identity

$$\int_{R^n}\Big[\prod_{j=1}^{n} f_j(x_j)\Big]dx = \prod_{j=1}^{n}\Big[\int_{-\infty}^{+\infty} f_j(x_j)\,dx_j\Big], \tag{1}$$

where R^n is the n-dimensional Euclidean space and $dx = dx_1 dx_2 \ldots dx_n$. This identity can be generalized to an inequality for integrals of products of functions of less than n variables. For example, if $f_{ij}(x_i, x_j) \geqslant 0$ then

$$\int_{R^3} f_{12}(x_1, x_2) f_{13}(x_1, x_3) f_{23}(x_2, x_3)\,dx_1 dx_2 dx_3$$
$$\leqslant \Big[\int_{R^2} f_{12}^2(x_1, x_2)\,dx_1 dx_2\Big]^{1/2}\Big[\int_{R^2} f_{13}^2(x_1, x_3)\,dx_1 dx_3\Big]^{1/2}\Big[\int_{R^2} f_{23}^2(x_2, x_3)\,dx_2 dx_3\Big]^{1/2}.$$

In order to describe the general result of which this is a special case, consider subsets ω of the set of indices $\{1, 2, \ldots, n\}$ and denote by $|\omega|$ the number of their elements. Let x_ω denote the set of variables $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, where $\{i_1, i_2, \ldots, i_k\} = \omega$, and let f_ω denote a function depending only on x_ω. Then the inequality

$$\int_{R^n}\Big[\prod_{|\omega|=k} f_\omega(x_\omega)\Big]dx \leqslant \prod_{|\omega|=k}\Big[\int_{R^k} f_\omega^r(x_\omega)\,dx_\omega\Big]^{1/r} \tag{2}$$

holds, where r is the binomial coefficient $\binom{n-1}{k-1}$ and the products extend over all subsets ω of $\{1, 2, \ldots, n\}$ with $|\omega| = k$.

For $k = 1$, (2) is actually an equality, namely (1), and for $k = n$ the two sides of (2) become the same. Thus in order to prove our assertion we may assume that $2 \leqslant k < n$, and argue by induction on n.

* Research partly supported by NSF GP 36775