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On an integral of Marcinkiewicz

by

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Abstract. The integral of Marcinkiewicz which plays an important role in the theory of singular integrals is generalized and the norm inequalities are extended to the case of weighted L^p and exponential norms.

Let P be a closed subset of $\boldsymbol{R}^n$ and let $\delta(y)$ denote the distance from y to P, and $|x-y|$ the distance from x to y. Marcinkiewicz introduced the integrals (see, for example, [5], Chapter IV, and [2]).

$$(Jf)(x) = \int_P \frac{\delta(y)^\lambda}{|x-y|^{n+\lambda}} f(y)\,dy, \qquad \lambda > 0$$

$$(If)(x) = \int_{\delta(y)\leqslant\delta_0<1} \frac{[\log\delta(y)^{-1}]^{-1}}{|x-y|^n} f(y)\,dy,$$

which he used to obtain some remarkable results in Fourier series. On the other hand, a useful modification of these integrals was considered by Carleson [1] and Zygmund [4]; namely, in Zygmund's version

$$(J^*f)(x) = \int_{\boldsymbol{R}^n} \frac{\delta(y)^\lambda}{[|x-y|+\delta(y)]^{n+\lambda}} f(y)\,dy,$$

$$(I^*f)(x) = \int_{\delta(y)\leqslant\delta_0<1} \frac{[\log\delta(y)^{-1}]^{-1}}{[|x-y|+\delta(y)]^n} f(y)\,dy.$$

Clearly, $(J^*f)(x) = (Jf)(x)$ and $(I^*f)(x) = (If)(x)$ for $x \in P$. In this form these integrals play also an important role in the theory of singular and hypersingular integrals. The purpose of this note is to generalize them and obtain weighted L^p-norm inequalities and exponential integrability results.

Let $\varphi(\varrho, t) \geqslant 0$ be a function defined in $t \geqslant 0$, $\varrho > 0$, such that

(i) *for each* t, $t \geqslant 0$, $(\varrho+t)^{-n}\varphi(\varrho, t)$ *is a non-increasing function of* ϱ, *which tends to zero as* $\varrho\to\infty$;

(ii) *there is a constant c such that*

$$\int_0^\infty \varrho^{n-1}(\varrho+t)^{-n}\varphi(\varrho,t)\,d\varrho \leqslant c$$

for all t, $t \geqslant 0$.

Let $\chi(y) \geqslant 0$ be an arbitrary measurable function and, assuming that $\varphi(|x-y|, \chi(y))$ is measurable, consider the integral

$$(2) \qquad (Kf)(x) = \int_{R^n} \frac{1}{[|x-y|+\chi(y)]^n}\,\varphi(|x-y|, \chi(y))f(y)\,dy.$$

Setting

$$\varphi(\varrho, t) = \frac{t^\lambda}{(\varrho+t)^\lambda}, \quad \chi(y) = \delta(y),$$

we find that $Kf = J^*f$.

On the other hand, if f has support in a sphere B of radius r and $\varphi(\varrho, t) = [\log t^{-1}]^{-1}$ for $t \leqslant \delta_0 < 1$ and $\varrho \leqslant 2r$ (clearly, there is a function φ satisfying these conditions and (i) and (ii)) $\chi(y) = \delta(y)$, we will have

$$(Kf)(x) = (I^*f)(x) \quad \text{for} \quad x \in B.$$

Let $(Mg)(x)$ be the Hardy–Littlewood maximal function associated with the function g, that is,

$$(Mg)(x) = \sup_{\varrho>0}\frac{1}{\omega\varrho^n}\int_{|x-y|\leqslant\varrho}|g(y)|\,dy,$$

where ω denotes the volume of the unit sphere. Then we have

THEOREM 1. *Let $f, g \geqslant 0$. Then*

$$\int_{R^n}(Kf)(x)g(x)\,dx \leqslant cn\omega\int_{R^n}f(x)(Mg)(x)\,dx,$$

where c is the constant in (ii).

Proof. Replacing Kf by its expression in (2) we have

$$(3) \quad \int_{R^n}(Kf)(x)g(x)\,dx = \int_{R^n}f(y)\int_{R^n}\frac{1}{[|x-y|+\chi(y)]^n}\varphi(|x-y|, \chi(y))\,g(x)\,dx$$

$$\leqslant \int_{R^n}f(y)\sup_t\int_{R^n}\frac{1}{[|x-y|+t]^n}\varphi(|x-y|, t)\,g(x)\,dx.$$

Let

$$G(y, \varrho) = \int_{|x-y|\leqslant\varrho}g(x)\,dx.$$

Then

$$G(y, \varrho) \leqslant \omega\varrho^n(Mg)(y),$$

and

$$\int_{R^n}\frac{1}{[|x-y|+t]^n}\,\varphi(|x-y|, t)g(x)\,dx = \int_{\varrho=0}^\infty\frac{\varphi(\varrho,t)}{(\varrho+t)^n}\,dG(y,\varrho)$$

$$= -\int_{\varrho=0}^\infty G(y,\varrho)\,d\frac{\varphi(\varrho,t)}{(\varrho+t)^n} \leqslant -\omega(Mg)(y)\int_{\varrho=0}^\infty\varrho^n\,d\frac{\varphi(\varrho,t)}{(\varrho+t)^n}$$

$$= n\omega(Mg)(y)\int_0^\infty\frac{\varrho^{n-1}}{(\varrho+t)^n}\varphi(\varrho,t)\,d\varrho \leqslant nc\omega(Mg)(y).$$

Given the monotone character of the functions $G(y, \varrho)$ and $(\varrho+t)^{-n}\varphi(\varrho, t)$ and the fact that they tend to zero as $\varrho\to 0$ and $\varrho\to\infty$, respectively, the integrations by parts above are legitimate and substituting in (3) we obtain the desired result.

THEOREM 2. *Let $w(x) > 0$ be a function in A_p* (see [3]), *that is, such that*

$$(4) \qquad \left[\frac{1}{|B|}\int_B w(x)\,dx\right]\left[\frac{1}{|B|}\int_B w(x)^{-q/p}\,dx\right]^{p/q} \leqslant c_w, \qquad 1 < p = \frac{q}{q-1}$$

or

$$\frac{1}{|B|}\int_B w(x)\,dx \leqslant c_w \operatorname*{ess\,inf}_{x\in B} w(x), \qquad p = 1,$$

where B denotes an arbitrary sphere in $\boldsymbol{R}^n$ and $|B|$ its measure. Then for $r \geqslant p$ we have

$$(5) \qquad \int_{R^n}|(Kf)(x)|^r\,dx \leqslant (c_1 r)^r\int_{R^n}|f(x)|^r w(x)\,dx,$$

where c_1 is a constant depending on p, c_w and φ.

Proof. We start by observing that, as is well known and readily verifiable, if $w \in A_p$, $p \geqslant 1$, then $w \in A_r$, $p \leqslant r < \infty$. Furthermore, as is again well known and readily verifiable, $w \in A_p$, $p > 1$, if and only if $w^{-q/p} \in A_q$, $q = p/(p-1)$.

Consider the inequality

$$\int_{R^n}|(Kf)(x)|^r w(x)\,dx \leqslant c_2^r\int_{R^n}|f(x)|^r w(x)\,dx, \qquad r \geqslant 1.$$

By setting $f = gw^{-1}$ we see that this inequality is equivalent to

$$\int_{R^n}|w^{1/r}Kw^{-1/r}g|^r\,dx \leqslant c_2^r\int_{R^n}|g|^r\,dx,$$

and according to Theorem 1

$$\|w^{1/r}Kw^{-1/r}g\|_r \leqslant nc\omega \sup_{\|h\|_s=1} \int_{\boldsymbol{R}^n} |g|\,|w^{-1/r}Mw^{1/r}h|\,dx, \quad s = r/(r-1),$$

where the norms here are ordinary L^p-norms.

Thus in order to prove our theorem it will suffice to show that the operator $w^{-1/r}Mw^{1/r}$ is bounded in L^s with norm less than or equal to $c_2 r$.

Suppose first that $p = r = 1$. Then if $h \in L^\infty$ and B is any sphere in $\boldsymbol{R}^n$ we have

$$\frac{1}{|B|}\int_B |w^{-1}Mwh|\,dx \leqslant [\operatorname*{ess\,inf}_{x\in B} w(x)]^{-1}\|h\|_\infty \frac{1}{|B|}\int_B w\,dx \leqslant c_w\|h\|_\infty,$$

that is, $w^{-1}Mw$ is bounded in L^∞ and has norm less than or equal to c_w.

Now suppose that $r > 1$, $1 \leqslant p \leqslant r$. Arguing as above, it follows that $w^{-1/r}Mw^{1/r}$ is bounded in L^s with norm less than or equal to c_3 if and only if

$$\int_{\boldsymbol{R}^n} (Mf)(x)^s w^{-s/r}(x)\,dx \leqslant c_3^s \int_{\boldsymbol{R}^n} |f(x)|^s w(x)^{-s/r}\,dx,$$

that is, M is bounded with norm less than or equal to c_3 in the space L^s_ν, i.e., the space of functions integrable to the sth power with respect to the measure $d\nu = w^{-s/r}\,dx$.

Now since $w \in A_p$, as is well known (see [3], for example) there exists $\mu > 1$ such that

$$\left[\frac{1}{|B|}\int_B w(x)^\mu\,dx\right]^{1/\mu} \leqslant c_4\left[\frac{1}{|B|}\int_B w(x)\,dx\right],$$

where B is any sphere in $\boldsymbol{R}^n$ and c_4 and μ depend only on c_w, p and n.

Now let $r_1, s_1 = \dfrac{r_1}{r_1 - 1}$ be such that $\mu\dfrac{s_1}{r_1} = \dfrac{s}{r} \leqslant \dfrac{q}{p}$. Since $\mu > 1$, we have $s_1 < s$ and, if $p > 1$,

$$\begin{aligned}
&\left[\frac{1}{|B|}\int_B w^{-s/r}\,dx\right]\left[\frac{1}{|B|}\int_B w^{(-s/r)(-r_1/s_1)}\,dx\right]^{s_1/r_1} \\
&\qquad = \left[\frac{1}{|B|}\int_B w^{-s/r}\,dx\right]\left[\frac{1}{|B|}\int_B w^{\mu}\,dx\right]^{\mu s_1/r_1} \\
&\qquad \leqslant c_4^{\mu s_1/r_1}\left[\frac{1}{|B|}\int_B w^{-s/r}\,dx\right]\left[\frac{1}{|B|}\int_B w ds\right]^{\mu s_1/r_1} \\
&\qquad \leqslant c_4^{s/r}\left[\frac{1}{|B|}\int w^{-q/p}\,dx\right]^{sp/rq}\left[\frac{1}{|B|}\int w dx\right]^{s/r} \leqslant c_4^{s/r}c_w^{s/r},
\end{aligned}$$

so that $w^{-s/r}$ belongs to A_{s_1} with constant $(c_4c_w)^{s/r}$, where c_w is the constant in (4) which is independent of r. As is readily verified, the same conclusion holds for $p = 1$. Now, as is well known, this implies that the operator M is of weak type (s_1, s_1) with respect to the measure ν, or, more specifically, that

$$|\{Mf > \lambda\}|_\nu \leqslant \lambda^{-s_1}(c_4c_w)^{2s/r}3^{s_1}\int |f|^{s_1}\,d\nu, \qquad \lambda > 0,$$

where the left-hand side is the ν-measure of the set $\{Mf > \lambda\}$, and since M is also of strong type (∞, ∞) with norm 1, from Marcinkiewicz's interpolation theorem (see [5], Chapter XII) it follows that M is bounded in L^s_ν and has norm not larger than

$$2\left(\frac{s_1}{s - s_1}\right)^{1/s}(c_4c_w)^{2/r}3^{s_1/s}.$$

Now, since $r \geqslant 1$ and $s_1 < s$, the preceding expression is majorized by

$$c_5\left(\frac{s_1}{s - s_1}\right)^{1/s},$$

where c_5 depends only on c_w, p and n. On the other hand, a simple calculation gives

$$\frac{s_1}{s - s_1} \leqslant \frac{\mu(r-1)+1}{\mu - 1}$$

and since $\mu > 1$, $r > 1$ and $s > 1$, we obtain

$$\frac{s_1}{s - s_1} \leqslant \frac{\mu}{\mu - 1}r,$$

and

$$\left(\frac{s_1}{s - s_1}\right)^{1/s} \leqslant \frac{\mu}{\mu - 1}r.$$

Thus M is bounded in L^s_ν with norm less than or equal to a constant depending on c_w, p and n, times r, and the same holds for $w^{-1/r}Mw^{1/r}$ as an operator in L^s. This completes the proof of our theorem.

THEOREM 3. *Let $w > 0$ be a function in A_p* (see Theorem 2) *and let f be bounded. Then*

$$\int_{\boldsymbol{R}^n} |(Kf)(x)|\,|^p e^{\lambda|(Kf)(x)|}w(x)\,dx \leqslant c_\lambda \int |(Kf)(x)|^p w(x)\,dx,$$

for $0 \leqslant \lambda < 1/c_1e\|f\|_\infty$, where c_1 is the constant in (5), *and c_λ is a (finite) constant depending on c_w, p, λ and n.*

Proof. Expanding the exponential in power series and using Theorem 2 we have

$$\int_{R^n} |Kf|^p e^{\lambda|Kf|} w dx = \sum_0^\infty \frac{\lambda^k}{k!} \int_{R^n} |Kf|^{p+k} w\, dx$$

$$\leqslant \sum_0^\infty \frac{\lambda_k}{k!} c_1^{p+k}(p+k)^{p+k} \int_{R^n} |f|^{p+k} w\, dx$$

$$\leqslant \Big[\sum_0^\infty \frac{\lambda^k}{k!} c_1^{p+k}(p+k)^{p+k} \|f\|_\infty^k \int_{R^n} |f|^p w\, dx,$$

and using the ratio test and the fact that

$$\lim_{k\to\infty} \frac{1}{k} \frac{(p+k)^{p+k}}{(p+k-1)^{p+k-1}} = c,$$

we find that the series converges for $\lambda c_1 \|f\|_\infty c < 1$, which proves our assertion.

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Maximal smoothing operators and some Orlicz classes

by

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Abstract. The paper gives a characterization of the Orlicz classes of functions that are "near" $L_\alpha^{n/\alpha}(\boldsymbol{R}^n)$, $0 < \alpha < n$, for which the functions belonging to them have the property of possessing total differential of order α at almost all the points of $\boldsymbol{R}^n$. When α is not an integer, the finiteness of $M_\alpha^*(f)$ replaces the existence of the α-differential (see [5]).

0. Introduction, notation and definitions. In an earlier joint paper [5], one of the authors studied the differential properties of functions belonging to classes $L_\alpha^p(\boldsymbol{R}^n)$, $0 < \alpha < n$, $p > n/\alpha$. The purpose of this paper is to extend those results to Orlicz classes of functions that are "near" $L_\alpha^{n/\alpha}(\boldsymbol{R}^n)$, $0 < \alpha < n$. More precisely, we characterize those Orlicz classes that are "near" $L_\alpha^{n/\alpha}(\boldsymbol{R}^n)$, for which the functions belonging to them possess total differential of order α at almost all the points of $\boldsymbol{R}^n$. If α is not an integer, we replace the existence of the α-differential by the finiteness of $M_\alpha^*(f)$; see [5] or definition below.

Earlier results in this direction are due to A. P. Calderón [4] when $\alpha = 1$. Positive results go back to W. Stepanov [11]; see also [6], [7] and [9].

Throughout this paper we keep the notation and constructions used in [5] and our method is partially borrowed from [4] and [5].

Almost all the lemmas in this paper use results in [10] and [12], and we shall refer to them systematically.

0.1. Let $\psi(t)$ be a non-decreasing function of the variable $t \geqslant 0$, continuous and such that $\psi(0) = 0$. We say that $\psi(t)$ is *near* t^c if the following conditions are met:

(i) $\psi(t) = t^c\varphi(t)$, $t > 0$ and $\varphi(t) > 0$.

(ii) *$\varphi(t)$ is slowly varying, that is, for each positive δ, there exists a number $N > 0$ such that for $t > N$, $\varphi(t)t^\delta$ is increasing, while $\varphi(t)t^{-\delta}$ is decreasing.*

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