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Inequalities for the maximal function relative to a metric

by

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Abstract. Weighted L^p-norm inequalities for the maximal function relative to a family of spheres defined by a pseudo-metric are obtained.

The purpose of this note is to obtain weighted L^p-norm inequalities for the maximal function defined by the spheres of a certain pseudo-metric. These inequalities generalize those known to hold in Euclidean space with the ordinary metric (see [2]), and other metrics considered by D. Kurtz [3] but they do not cover his results about maximal functions defined by certain families of rectangles.

Let X be a metric space with a measure μ and assume that the space of continuous functions with bounded support is contained and is dense in the space of integrable functions. Further, suppose that there is given a real-valued function $\varrho(x, y)$ in $X\times X$ (it need not be the distance function) with the following properties

(i) $\varrho(x, x) = 0$;

(ii) $\varrho(x, y) = \varrho(y, x) > 0$ if $x \neq y$;

(iii) *there is a constant c such that $\varrho(x, z) \leqslant c[\varrho(x, y)+\varrho(y, z)]$ for all x, y, and z;*

(iv) *given a neighborhood N of a point x there is an ε, $\varepsilon > 0$, such that the sphere $B_\varepsilon(x) = \{y|\ \varrho(x, y)\leqslant \varepsilon\}$ with center at x is contained in N;*

(v) *the spheres $B_r(x) = \{y|\ \varrho(x, y)\leqslant r\}$ are measurable, the measure $|B_r(x)|$ of $B_r(x)$ is a continuous function of r for each x, and there is a constant c, $c > 1$, such that*

$$|B_{2r}(x)| \leqslant c\,|B_r(x)| < \infty$$

for all r and x. For convenience we shall assume that the constant here coincides with the one in (iii).

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Given a function which is integrable on all the sets $B_r(x)$, we define the *maximal function* Mf of f as

(1) $$(Mf)(x) = \sup_r \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,d\mu.$$

On the other hand, we say that the weight function $w(x)$, $w(x) > 0$, *belongs to the class* A_p if

(2) $$\Big[\int_B w(x)\,d\mu\Big]\Big[\int_B w(x)^{-\frac{1}{p-1}}\,d\mu\Big]^{p-1} \leqslant c_w|B|^p, \quad 1 < p < \infty,$$
$$\Big[\int_B w(x)\,d\mu\Big] \leqslant c_w|B|\operatorname*{ess\,inf}_{x\in B} w(x), \quad p = 1,$$

for all spheres B. Then the results of B. Muckenhoupt [2] for Euclidean space with the ordinary metric hold, namely

THEOREM 1. *If $w \in A_p$, $p \geqslant 1$, then*

$$\Big[\frac{1}{|B|}\int_B w^r\,d\mu\Big]^{1/r} \leqslant c_1\frac{1}{|B|}\int_B w\,d\mu,$$

with $r > 1$, and c_1 and r depend on p, the constant c_w in (2) *and the constant c in* (iii) *and* (v).

THEOREM 2. *If $w\in A_p$, $p > 1$, then $w \in A_r$ for all r, $r > p_0$, where $p_0 < p$ and p_0 depends on p, c_w and the constant c in* (iii) *and* (v).

THEOREM 3. *If $w\in A_p$, $p > 1$, then for $p \leqslant q \leqslant \infty$*

$$\int (Mf)(x)^q d\nu \leqslant c_3\int |f(x)|^q d\nu, \quad d\nu = w(x)\,d\mu,$$

where c_3 depends on c, c_w, p and q. If $p = 1$ and $w \in A_p$, the same result holds for $1 < q \leqslant \infty$, and for $\lambda > 0$

$$|\{x\mid\ (Mf)(x) > \lambda\}|_\nu \leqslant \frac{c_3}{\lambda}\int |f(x)|\,d\nu, \quad d\nu = w(x)\,d\mu,$$

where the left-hand side represents the ν-measure of the set indicated.

Except in the case of Theorem 1, the above statements can be proved by slightly modifying the arguments which have been used in the Euclidean case. Theorem 1, however, which is the key to the other two and is considerably more difficult, requires a different treatment. In spite of the close similarity of some of the following arguments to well-known ones, we give them in full detail for the sake of completeness.

LEMMA 1. *There is a constant γ such that*

$$|B_{ar}(x)| \leqslant ca^\gamma|B_r(x)|, \quad a \geqslant 1,$$

where c is the constant in (v).

Proof. Let $2^{k-1} \leqslant a < 2^k$, $k \geqslant 1$. Then $k \leqslant 1+\log_2 a$ and from (v) we obtain

$$|B_{ar}(x)| \leqslant |B_{2^k r}(x)| \leqslant c^k|B_r(x)| \leqslant c^{1+\log_2 a}|B_r(x)| = ca^\gamma|B_r(x)|$$

with $\gamma = \log_2 c$.

LEMMA 2. *For each x, $|B_r(x)|$ is a continuous non-decreasing function of r, and $|B_r(x)| > 0$ for $r > 0$, unless μ vanishes identically.*

Proof. We only have to show that $|B_r(x)| > 0$ for $r > 0$. Suppose on the contrary that $|B_r(x)| = 0$. Then from Lemma 1 it follows that $B_{ar}(x) = 0$ for all a, $a \geqslant 1$, and this clearly implies that μ vanishes identically.

LEMMA 3. *Let $\mathfrak{F}$ be a family of spheres with bounded radii. Then there exists a countable subfamily of disjoint spheres $B_{r_i}(x_i)$ such that each sphere in $\mathfrak{F}$ is contained in one of the spheres $B_{br_i}(x_i)$, where $b = 3c^2$ and c is the constant in* (iii).

Proof. Let M be a bound for the radii of the spheres in $\mathfrak{F}$, and let $a < 1$ be such that

$$3c^2 = c^2\Big(1+\frac{1}{a}\Big)+\frac{c}{a}.$$

Since $c > 1$, such a exists. Now for each integer k, $k > 0$, we construct inductively a family of spheres with the following properties:

1. $B_{r_{i,k}}(x_{i,k}) \in \mathfrak{F}$, $a^kM < r_{i,k} \leqslant a^{k-1}M$;
2. *the $B_{r_{i,h}}(x_{i,h})$ are disjoint for $h \leqslant k$;*
3. *for each k the family is maximal with respect to properties* 1 *and* 2.

Evidently such a family exists. Let now $B_r(x) \in \mathfrak{F}$. If $a^kM < r \leqslant a^{k-1}M$, then $B_r(x)$ intersects one of the spheres $B_{r_{i,h}}(x_{i,h})$, $h \leqslant k$. But then $r_{i,h} > ar$ and therefore, if $z\in B_r(x)$ and $y \in B_{r_{i,h}}(x_{i,h})\cap B_r(x)$, on account of (iii) we have

$$\varrho(z, x_{i,h}) \leqslant cr + c\varrho(x, x_{i,h}) \leqslant cr + c[cr + c\varrho(y, x_{i,h})]$$
$$\leqslant cr + c^2(r + r_{i,h}) \leqslant c\frac{r_{i,h}}{a} + c^2\Big(\frac{r_{i,h}}{a}+r_{i,h}\Big)$$
$$= 3c^2 r_{i,h} = br_{i,h};$$

that is, $z\in B_{br_{i,h}}(x_{i,h})$.

LEMMA 4. *Let $w \in A_p$, $1 \leqslant p < \infty$, and let E be a subset of the sphere B. Then*

$$\frac{|E|_\nu}{|B|_\nu} \geqslant c_w^{-1}\left(\frac{|E|}{|B|}\right)^p, \quad d\nu = w\,d\mu.$$

Proof. If $p > 1$, then on account of (2) we have

$$|E| = \int_E w^{1/p} w^{-1/p}\,d\mu \leqslant \Big[\int_E w\,d\mu\Big]^{1/p}\Big[\int_B w^{-1/(p-1)}\,d\mu\Big]^{(p-1)/p}$$
$$= |E|_\nu^{1/p}\Big[\int_B w^{-1/(p-1)}\,d\mu\Big]^{(p-1)/p} \leqslant c_w^{1/p}|E|_\nu^{1/p}|B|\Big[\int_B w\,dx\Big]^{-1/p}$$
$$= c_w^{1/p}|B|\,|E|_\nu^{1/p}|B|_\nu^{-1/p}$$

which is the desired inequality.

If $p = 1$, then since

$$\operatorname*{ess\,inf}_{x\in B} w(x) \leqslant \frac{1}{|E|}\int_E w(x)\,d\mu = \frac{|E|_\nu}{|E|},$$

the second inequality in (2) gives

$$|B|_\nu \leqslant c_w|B|\frac{|E|_\nu}{|E|}.$$

LEMMA 5. *Suppose $E \subset B$ and $|E| \leqslant \delta|B|$. Then*

$$|E|_\nu \leqslant [1 - c_w^{-1}(1-\delta)^p]\,|B|_\nu.$$

Proof. Applying the preceding lemma to $E' = B - E$ we obtain

$$\frac{|E|_\nu}{|B|_\nu} = 1 - \frac{|E'|_\nu}{|B|_\nu} \leqslant 1 - c_w^{-1}(1-\delta)^p.$$

LEMMA 6. *Let $w \in A_p$, $1 \leqslant p < \infty$, and $w^{1/p}f \in L^p$, $\lambda > 0$; then*

$$|\{x|\ (Mf)(x) > \lambda\}|_\nu \leqslant \frac{c_3}{\lambda^p}\int |f|^p\,d\nu, \quad d\nu = w(x)\,d\mu,$$

where c_3 depends on c, c_w and p.

Proof. For each n, $n > 0$, we define

$$(M_nf)(x) = \sup_{r\leqslant n}\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,d\mu,$$

and we shall show that the inequality above holds with M replaced by M_n with c_3 independent of n. Once this is established, the lemma will follow by letting n tend to infinity.

Let $B = B_r(x)$ be a sphere such that $r \leqslant n$ and

$$\lambda|B| < \int_B |f|\,d\mu. \tag{3}$$

Then, according to (2), if $p > 1$ we have

$$\lambda|B| < \int_B |f|\,d\mu = \int_B |f|w^{1/p}w^{-1/p}\,d\mu \leqslant \Big[\int_B |f|^p\,d\nu\Big]^{1/p}\Big[\int_B w^{-\frac{1}{p-1}}\,d\mu\Big]^{\frac{p-1}{p}}$$
$$\leqslant \Big[\int_B |f|^p\,d\nu\Big]^{1/p} c_w^{1/p}|B|\,|B|_\nu^{-1/p},$$

whence it follows that

$$\lambda^p|B|_\nu \leqslant c_w\int_B |f|^p\,d\nu, \tag{4}$$

and for $p = 1$

$$\lambda|B| < \int_B |f|\,d\mu \leqslant [\operatorname*{ess\,inf}_{x\in B} w(x)]^{-1}\int_B |f|w\,d\mu \leqslant c_w|B|\Big[\int_B |f|w\,d\mu\Big]\Big[\int_B w\,d\mu\Big]^{-1}$$
$$= c_w|B|\,|B|_\nu^{-1}\int_B |f|\,d\nu,$$

so that (4) also holds in this case.

Let now $\mathfrak{F}$ be the family of spheres satisfying (3). As we have just shown, they also satisfy (4). Evidently, the union of all spheres in $\mathfrak{F}$ contains the set

$$\{x|\ (M_nf)(x) > \lambda\}.$$

According to Lemma 3, there exists a disjoint family of spheres $B_{r_i}(x_i)$ in $\mathfrak{F}$ such that

$$\bigcup B_{br_i}(x_i) \supset \bigcup_{B\in\mathfrak{F}} B \supset \{x|\ (M_nf)(x) > \lambda\}.$$

By Lemmas 1 and 4 we have

$$\frac{|B_{br_i}(x_i)|_\nu}{|B_{r_i}(x_i)|_\nu} \leqslant c_w\left(\frac{|B_{br_i}(x_i)|}{|B_{r_i}(x_i)|}\right)^b \leqslant c_w c^p b^{\gamma p}$$

so that

$$|B_{br_i}(x_i)|_\nu \leqslant c_w c^p b^{\gamma p}|B_{r_i}(x_i)|.$$

Thus since the $B_{r_i}(x_i)$ are disjoint and satisfy (4), we obtain

$$|\{x|\ (M_nf)(x) > \lambda\}|_\nu \leqslant \sum |B_{br_i}(x_i)|_\nu \leqslant c_w c^p b^{\gamma p}\sum |B_{r_i}(x_i)|_\nu$$
$$\leqslant \frac{1}{\lambda^p}c_w^2 c^p b^{\gamma p}\sum \int_{B_{r_i}(x_i)} |f|^p\,d\nu \leqslant \frac{c_3}{\lambda^p}\int |f|^p\,d\nu.$$

LEMMA 7. *Let f be integrable on every sphere. Then*

$$\lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)} f\,dv = f(x)$$

almost everywhere. In particular, $|f| \leqslant Mf$ almost everywhere.

Proof. In view of (iv) our assertion holds for f continuous. Since continuous functions with bounded support are assumed to be dense in L^1, for integrable f our lemma follows from Lemma 6 in the well-known fashion and clearly, if the lemma holds for f integrable, then it holds in general.

Proof of Theorem 1. Let $B = B_{r_0}(x_0)$ and

$$\lambda = \frac{1}{|B|}\int_B w\,d\mu.$$

We shall construct a sequence $E_1 \supset E_2 \supset \ldots \supset E_m \supset \ldots$ of subsets of $B_{(b+c)r_0}(x_0)$ such that $|\bigcap_1^\infty E_m| = 0$, $w \leqslant \lambda a^m$ almost everywhere in B and outside E_m, and

$$|E_{m+1}|_v \leqslant \delta |E_m|_v, \qquad \delta < 1, \tag{5}$$

where a and δ depend only on c_w, c and p. Once these sets have been constructed, taking ε so that $a^\varepsilon\delta < 1$ we will have

$$\begin{aligned}\int_B w^{1+\varepsilon}d\mu &\leqslant \int_{B\cap E_1'} w^{1+\varepsilon}d\mu + \sum_1^\infty \int_{(E_j - E_{j+1})\cap B} w^{1+\varepsilon}d\mu\\ &\leqslant (a\lambda)^\varepsilon \int_B w\,d\mu + \sum_1^\infty (a^{j+1}\lambda)^\varepsilon \int_{E_j} w\,d\mu\\ &\leqslant (a\lambda)^\varepsilon |B|_v + \lambda^\varepsilon |E_1|_v \sum_1^\infty (a^{j+1})^\varepsilon \delta^{j-1}.\end{aligned}$$

But $E_1 \subset B_{(b+c)r_0}(x_0)$ and therefore, according to Lemmas 1 and 4, we have $|E_1|_v \leqslant c_w[c(b+c)^\gamma]^p |B|_v$. Furthermore, since $\lambda = |B|_v|B|^{-1}$, substituting above we obtain

$$\frac{1}{|B|}\int_B w^{1+\varepsilon}d\mu \leqslant c_1(|B|_v|B|^{-1})^{1+\varepsilon},$$

which is the desired result.

To construct the sets E_m we proceed as follows. Given a point x in B consider the ratio

$$\frac{|B_r(x)|_v}{|B_r(x)|}.$$

This is a continuous function of r for $r > 0$. Furthermore, as we shall see, if a, $a > 1$, is sufficiently large as compared with c_w and c, then

$$\frac{|B_{r_0}(x)|_v}{|B_{r_0}(x)|} < a\lambda, \tag{6}$$

so that for each m, $m \geqslant 1$, there is a largest value of r, $r \leqslant r_0$, such that

$$\frac{|B_r(x)|_v}{|B_r(x)|} = a^m\lambda,$$

or else, $(M_{r_0}w)(x) \leqslant a^m\lambda$. Let us denote these spheres by $B^{(m)}$ and by $B^{(m,i)}$ the spheres in a subfamily as in Lemma 3. Let $\tilde B^{(m,i)}$ be the spheres concentric with the $B^{(m,i)}$ and b times their radii. Then if

$$E_m = \bigcup_i \tilde B^{(m,i)},$$

according to Lemma 3, E_m constains all spheres $B^{(m)}$ and therefore $(Mw)(x) \leqslant a^m\lambda$ in B and outside E_m and, by Lemma 7, $w \leqslant a^m\lambda$ almost everywhere in B and outside E_m.

Let us prove (6) before proceeding to show that the sets E_m have the properties stated above. As is readily verified we have

$$B_{r_0}(x) \subset B_{2cr_0}(x_0), \qquad B_{r_0}(x_0) \subset B_{2cr_0}(x),$$

and therefore from Lemmas 1 and 4 we obtain

$$|B_{r_0}(x_0)| \leqslant |B_{2cr_0}(x)| \leqslant c(2c)^\gamma |B_{r_0}(x)|,$$

that is,

$$|B_{r_0}(x_0)| \leqslant 2^\gamma c^{\gamma+1}|B_{r_0}(x)|,$$

and

$$\begin{aligned}|B_{r_0}(x)|_v \leqslant |B_{2cr_0}(x_0)|_v &\leqslant c_w |B_{r_0}(x_0)|_v \left[\frac{|B_{2cr_0}(x_0)|}{B_{r_0}(x_0)}\right]^p\\ &\leqslant (2^\gamma c^{\gamma+1})^p c_w |B_{r_0}(x_0)|_v,\end{aligned}$$

and from these inequalities it follows that

$$\frac{|B_{r_0}(x)|_v}{|B_{r_0}(x)|} \leqslant (2^\gamma c^{\gamma+1})^{p+1} c_w \frac{|B_{r_0}(x_0)|_v}{|B_{r_0}(x_0)|} = (2^\gamma c^{\gamma+1})^{p+1} c_w \lambda.$$

Returning to the sets E_m, let us show next that $E_m \supset E_{m+1}$. Consider one of the spheres $B^{(m+1,j)}$. Let us denote this sphere by $B_r(x)$. Then $B^{(m+1,j)} = B_r(x)$ is contained in a sphere $B^{(m)} = B_s(x)$ and

$$\frac{|B_r(x)|_v}{|B_r(x)|} = a^{m+1}\lambda, \qquad \frac{|B_s(x)|_v}{|B_s(x)|} = a^m\lambda$$

and therefore, according to Lemma 1,

$$1 \leqslant \frac{|B_s(x)|_\nu}{|B_r(x)|_\nu} = a^{-1}\frac{|B_s(x)|}{|B_r(x)|} \leqslant a^{-1}c\left(\frac{s}{r}\right)^\gamma$$

or $s \geqslant ra^{1/\gamma}c^{-1/\gamma}$, so that if $a^{1/\gamma}c^{-1/\gamma}\geqslant b$ we will have $s\geqslant br$, and therefore

$$E_m \supset B^{(m)} = B_s(x) \supset B_{br}(x) = \tilde B^{(m+1,j)},$$

and this clearly implies that $E_m\supset E_{m+1}$.

There remains only to prove (5), which clearly implies that $|\bigcap_m E_m| = 0$. To prove this we shall show that

(7) $$|E_{m+1}\cap B^{(m,i)}| \leqslant \delta_1|B^{m,i}|,$$

where $\delta_1<1$, provided a is sufficiently large. Once this is established from Lemma 5, it will follow that

$$|E_{m+1}\cap B^{(m,i)}|_\nu \leqslant [1-c_w^{-1}(1-\delta_1)^p]\,|B^{(m,i)}|_\nu$$

and, since the $B^{(m,i)}$ are disjoint,

(8) $$|E_{m+1}\cap\big(\bigcup_i \tilde B^{(m,i)}\big)|_\nu \leqslant [1-c_w^{-1}(1-\delta_1)^p]\,\big|\bigcup_i B^{(m,i)}\big|_\nu.$$

On the other hand, on account of Lemmas 1 and 4 we have

$$|\tilde B^{(m,i)}|_\nu \leqslant c_w|B^{(m,i)}|_\nu(cb^\gamma)^p,$$

whence it follows that

(9) $$|E_m|_\nu = \big|\bigcup B^{(m,i)}\big|_\nu \leqslant c_w(cb^\gamma)^p\big|\bigcup_i B^{(m,i)}\big|_\nu.$$

Thus, from (8) and (9), since $E_{m+1}\subset E_m$, we obtain

$$|E_{m+1}|_\nu \leqslant \big|E_{m+1}\cap\big(\bigcup_i B^{(m,i)}\big)\big|_\nu + \big|E_m - \bigcup B^{(m,i)}\big|_\nu$$
$$\leqslant \big|\bigcup B^{(m,i)}\big|_\nu - c_w^{-1}(1-\delta_1)^p\big|\bigcup B^{(m,i)}\big|_\nu + \big|E_m - \bigcup_i B^{(m,i)}\big|_\nu$$
$$= |E_m|_\nu - c_w^{-1}(1-\delta_1)^p\big|\bigcup B^{(m,i)}\big|_\nu$$
$$\leqslant |E_m|_\nu - c_w^{-2}(1-\delta_1)^p(cb^\gamma)^{-p}|E_m|_\nu = \delta|E_m|_\nu,$$

where

$$\delta = 1-c_w^{-2}(1-\delta_1)c^{-p}b^{-\gamma p},$$

which is the inequality (5).

To prove (7) let us consider a sphere $\tilde B^{(m+1,j)}$. Let us set $B^{(m,i)} = B_r(x)$, $\tilde B^{(m+1,j)} = B_{r_1}(x_1)$. If $\tilde B^{(m+1,j)}$ intersects $B^{(m,i)}$ but is not entirely contained in it, and $y\in B^{(m,i)}\cap\tilde B^{(m+1,j)}$ and $z\in \tilde B^{(m+1,j)}$ we have

$$\varrho(y,z)\leqslant 2cr_1, \qquad \varrho(x,z)\leqslant cr+2c^2r_1 = r_2$$

and therefore $\tilde B^{(m+1,j)}\subset B_{r_2}(x)\subset B_{2cr_2}(x_1)$, and

$$a^m\lambda = \frac{|B^{(m,i)}|_\nu}{|B^{(m,i)}|} \geqslant \frac{|B_{r_2}(x)|_\nu}{|B_{r_2}(x)|} \geqslant \frac{|\tilde B^{(m+1,j)}|_\nu}{|B_{r_2}(x)|} \geqslant \frac{|B^{(m+1,j)}|_\nu}{|B_{r_2}(x)|} \geqslant a^{m+1}\lambda\frac{|B^{(m+1,j)}|}{|B_{2cr_2}(x_1)|}.$$

Now this and Lemma 1 yield

$$|B_{2cr_2}(x_1)| \geqslant a|B^{(m+1,j)}| = a|B_{r_1/b}(x_1)| \geqslant ac^{-1}\left(\frac{r_1}{2cbr_2}\right)^\gamma|B_{2cr_2}(x_1)|,$$

whence it follows that

$$r_1\leqslant 2a^{-1/\gamma}c^{1/\gamma+2}b(r+2cr_1).$$

Consequently, if $4a^{-1/\gamma}c^{1/\gamma+3}b\leqslant 1/2$, we will have

$$r_1\leqslant 4a^{-1/\gamma}c^{1/\gamma+2}br.$$

Thus, as is readily verified, if a is sufficiently large and $\tilde B^{(m+1,j)}$ contains points outside $B^{(m,i)}=B_r(x)$, then $B^{(m+1,j)}$ does not intersect $B_{r/2c}(x)$.

Next, let us consider the spheres $\tilde B^{(m+1,j)}$ which are intirely contained in $B^{(m,i)}$. For such spheres, we have

$$\sum|\tilde B^{(m+1,j)}| \leqslant cb^\gamma\sum|B^{(m+1)j)}| = cb^\gamma a^{-m-1}\lambda^{-1}\sum|B^{(m+1,j)}|_\nu$$
$$\leqslant cb^\gamma a^{-m-1}\lambda^{-1}|B^{(m,i)}|_\nu = cb^\gamma a^{-1}|B^{(m,i)}|$$
$$= cb^\gamma a^{-1}|B_r(x)|\leqslant 2^\gamma c^{2+\gamma}b^\gamma a^{-1}|B_{r/2c}(x)|,$$

so that if, again, a is sufficiently large, then

$$\sum|\tilde B^{(m+1,j)}|\leqslant \tfrac12|B_{r/2c}(x)|,$$

where the sum is extended over all spheres $B^{(m+1,j)}$ entirely contained in $B^{(m,i)}=B_r(x)$. Since the other spheres $\tilde B^{(m+1,j)}$ do not intersect $B_{r/2c}(x)$, we find that

$$|E_{m+1}\cap B^{(m,i)}| = \big|\bigcup_j \tilde B^{(m+1,j)}\cap B^{(m,i)}\big|$$
$$\leqslant |B^{(m,i)} - B_{r/2c}(x)| + \tfrac12|B_{r/2c}(x)| = |B^{(m,i)}| - \tfrac12|B_{r/2c}(x)|$$
$$\leqslant [1-(2c)^{-\gamma-1}]\,|B^{(m,i)}|.$$

Thus (7) holds with $\delta_1 = 1-(2c)^{-\gamma-1}$ provided a is sufficiently large as compared with c. This completes the proof of the theorem.

Proof of Theorem 2. If $w\in A_p$, $p>1$, then, as is readily verified, $w^{-1/(p-1)}\in A_q$, $(q-1)(p-1)=1$, and therefore according to Theorem 1 we have

$$\left[\frac{1}{|B|}\int_B w^{-1/(r-1)}\,d\mu\right]^{r-1}\leqslant c_1\left[\frac{1}{|B|}\int_B w^{-1/(p-1)}\,d\mu\right]^{p-1}$$

for some $r < p$ and $c_1 < \infty$, and by Hölder's inequality,

$$\left[\frac{1}{|B|}\int w^{-1/(r-1)}\,d\mu\right]^{r-1} \leqslant \left[\frac{1}{|B|}\int w^{-1/(p-1)}\,d\mu\right]^{p-1}$$

for all r, $r \geqslant p$. Thus substituting in the first inequality in (2) we find that $w \in A_r$ for some $r < p$, and all $r \geqslant p$. Thus the interval of values of r for which $w \in A_r$ is an open half-line containing p, as we wished to show.

Proof of Theorem 3. If $w \in A_p$, $p > 1$, then $w \in A_r$ for some $r < p$, and according to Lemma 6, the maximal operator $M\colon f \to (Mf)$ is of weak type (r, r) with respect to the measure $d\nu = w\,d\mu$. Since M is obviously also of type (∞, ∞), by the Marcinkiewicz interpolation theorem, it follows that M is of strong type (p, p) with respect to the measure ν, which is the desired result. Since the case $p = 1$ is covered by Lemma 6, this establishes Theorem 3.

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