défini par

$$T_\varepsilon f = \sum m(\varepsilon\lambda) H_\lambda f.$$

Supposons que T *s'étende en un opérateur de* $L^p(U/K_1)$ (2), *et que la norme d'opérateur de* T *reste uniformément bornée quand* ε *tend vers* 0. *Alors l'opérateur*

$$T: L^2(\mathfrak{p}_*) \to L^2(\mathfrak{p}_*), \quad f \mapsto Tf(x) = \int m(\xi)\hat{f}(\xi) e^{i\langle x,\xi\rangle} d\xi$$

s'étend en un opérateur borné de $L^p(\mathfrak{p}_*)$ *dans lui-même.*

Ce théorème s'applique en particulier aux fonctions radiales, puisque K^* respecte le produit scalaire, et permet d'obtenir des résultats négatifs pour la convergence des sommes de Riesz (cf. [2]).

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(2) p est un nombré fixé, $1 \leqslant p < +\infty$.

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On Λ-bounded variation

by

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Abstract. If $\Lambda = \{\lambda_n\}$ is an increasing sequence of positive numbers such that $\sum 1/\lambda_n$ diverges, the functions of *Λ-bounded variation* (ΛBV) are those f for which $\sum_1^\infty |f(I_n)|/\lambda_n < \infty$ for every sequence $\{I_n\}$ of non-overlapping intervals. These were introduced previously and various results on the summability and convergence of the Fourier series of functions of this class were proven. Here equivalent definitions are given and the point and interval variation functions are defined. The continuity properties of the variation functions are established. An analogue of Helly's theorem is given and it is shown that ΛBV is a Banach space with a suitable norm. Some open questions are indicated.

The concept of bounded variation has been generalized in many ways. In most instances, these new notions have been introduced because of their applicability to the study of Fourier series. The present generalization, Λ-bounded variation, is no exception to this rule.

Let us suppose that f is a real function defined on an interval I. We let $\{I_n\}$ denote a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset I$ and write $f(I_n) = f(b_n) - f(a_n)$. Throughout this paper, when we consider a collection of intervals, they will be assumed to be non-overlapping without further reference to that fact. Let Λ denote a non-decreasing sequence $\{\lambda_n\}$ of positive real numbers such that $\sum 1/\lambda_n$ diverges. The following definition was introduced by us previously [9].

DEFINITION. A function f is said to be of *Λ-bounded variation* (ΛBV) if, for every $\{I_n\}$, we have

$$\sum |f(I_n)|/\lambda_n < \infty.$$

The special case $\Lambda = \{n\}$ gives rise to the class of functions of *harmonic bounded variation* (HBV). The notion of HBV has its genesis in the work of Goffman and Waterman on everywhere convergence of Fourier series and everywhere convergence of Fourier series for every change of variable [3]–[5].

In our previous paper [9] we have shown that the functions of class HBV satisfy the Lebesgue test for convergence of their Fourier series and that if $\Lambda\mathrm{BV} \supset \mathrm{HBV}$ properly, then there is a continuous function in $\Lambda\mathrm{BV}$ whose Fourier series diverges at a point. In a recent paper [10] we have shown that the Fourier series of functions of class $\{n^{\beta+1}\}\mathrm{BV}$, $-1 \leqslant \beta < 0$, are (C, β) bounded and (C, α) summable for $\alpha > \beta$. If $\Lambda\mathrm{BV} \supset \{n^{\beta+1}\}\mathrm{BV}$ properly, then there is a continuous function in $\Lambda\mathrm{BV}$ whose Fourier series is not (C, β) bounded at some point. We also showed that the results on convergence and summability can be obtained from a theorem on the continuity of the variation.

In [9] we stated several results concerning functions of class $\Lambda\mathrm{BV}$ without proof. Here we intend to review the properties of functions of this class and to supply the arguments to justify our previous assertions. In §1 we shall give some equivalent definitions of $\Lambda\mathrm{BV}$ and discuss its relation to other generalizations of bounded variation. In §2 we discuss the variation functions of an interval and a point and their continuity properties. §3 contains proofs of an analogue of the Helly theorem and of the completeness of $\Lambda\mathrm{BV}$ under a suitable norm. In §4 we note some open questions.

1. Equivalent definitions, relations to other generalizations of BV. It is easily seen that the functions of class $\Lambda\mathrm{BV}$ are bounded and have discontinuities which are simple and, therefore, at most countable. The first of our results gives some equivalent definitions of $\Lambda\mathrm{BV}$.

THEOREM 1. *The following are equivalent:*

(i) *f is a $\Lambda\mathrm{BV}$ function;*

(ii) *there is an $M < \infty$ such that, for every $\{I_n\}$,*

$$\sum_1^\infty |f(I_n)|/\lambda_n < M;$$

(iii) *there is an $M < \infty$ such that, for any finite collection $\{I_n\}$, $n = 1, \ldots, N$,*

$$\sum |f(I_n)|/\lambda_n < M.$$

We require the following lemma.

LEMMA 1. *If $\sup\{\sum |f(I_n)|/\lambda_n\colon I_n \subset I\} = \infty$, then there exists a point $x_0 \in I$ such that, for every interval J with $x_0 \in J^0$ (relative to I),*

$$\sup\left\{\sum |fI_n|/\lambda_n\colon I_n \subset J\right\} = \infty.$$

Proof of Lemma 1. If there were an $M < \infty$ such that $\sum |f(I_n)|/\lambda_n < M$ whenever all I_n were in the lower half of I or all were in the upper half of I, then we would have $\sup\{\sum |f(I_n)|/\lambda_n\colon I_n \subset I\} \leqslant 2M$. Hence the set of such sums is unbounded in one half of the interval. Continuing this process, we obtain a nested sequence of intervals J_n converging to a point x_0. If $x_0 \in J^0$, then $J \supset J_n$ for large n, from which the conclusion follows. ■

Proof of Theorem 1. It is clear that (ii) $\Leftrightarrow$ (iii) and (ii) $\Rightarrow$ (i). Let us now assume (i) and not (ii). Without loss of generality, we may assume that $\lambda_1 = 1$. Let $B = \sup f - \inf f$. There is a collection of intervals $\{J_n\}$ such that

$$\sum_1^\infty |f(J_n)|/\lambda_n > 1 + 2B$$

and, therefore, there is an N such that

$$\sum_1^N |f(J_n)|/\lambda_n > 1 + 2B.$$

Omitting the at most two values of n for which $x_0 \in J_n$, where x_0 is the point obtained in Lemma 1, we have

$$\sum_1^{N'} |f(J_{\nu_n})|/\lambda_{\nu_n} > 1,$$

where $\nu_n < \nu_{n+1}$, $N \geqslant N' \geqslant N-2$. Letting $J_{\nu_n} = I_n$ and $N' = n_1$, we have

$$\sum_1^{n_1} |f(I_n)|/\lambda_n > 1,$$

since $\lambda^n \nearrow$.

Suppose we have shown that there exists a collection $\{I_n\}$, $n = 1, \ldots, n_k$, none of the I_n containing x_0, such that

$$\sum_1^{n_k} |f(I_n)|/\lambda_n > k.$$

Then there is an interval J such that $x_0 \in J^0$ (relative to I) and $J \cap \bigcup_1^{n_k} I_n = \emptyset$.

In J there is a collection of intervals J_n, $n = 1, \ldots, N$, such that

$$\sum_1^N |f(J_n)|/\lambda_n > B(n_k+2)+1.$$

Thus $N > n_k+2$, since $\sum_1^r |f(J_n)|/\lambda_n \leqslant Br$. Then

$$\sum_{n_k+1}^N |f(J_n)|/\lambda_n > 2B+1,$$

and if we eliminate the at most two terms for which $x_0 \in J_n$, we obtain J_{ν_n}, $n = 1, \ldots, N'$, $\nu_n < \nu_{n+1}$, $N - n_k - 2 \leqslant N' \leqslant N - n_k$, the ν_n denoting the indices of the remaining terms. Then

$$\sum_1^{N'} |f(J_{\nu_n})|/\lambda_{\nu_n} > 1.$$

Setting $I_{n_k+j} = J_{\nu_j}$ and $n_{k+1} = n_k + N'$, we have, since $\lambda_n \nearrow$,

$$\sum_{n_k+1}^{n_{k+1}} |f(I_n)|/\lambda_n > 1$$

and, therefore,

$$\sum_1^{n_{k+1}} |f(I_n)|/\lambda_n > k+1.$$

Thus we may construct a collection $\{I_n\}$ such that

$$\sum_1^{\infty} |f(I_n)|/\lambda_n = \infty$$

contrary to (i). ■

Let us now suppose that Φ is a convex function, $\Phi(x) = o(x)$ as $x \to 0+$, $\Phi(x)/x \to \infty$ as $x \to \infty$, and $\Phi(0) = 0$. Let

$$\Psi(x) = \sup\{xy - \Phi(y) \colon y \geqslant 0\}.$$

Then

$$xy \leqslant \Phi(x) + \Psi(y),$$

which is Young's inequality.

A function is said to be of Φ-BV if for some $k > 0$, the Φ-variation of kf, i.e., the supremum over all partitions $x_0 < x_1 < \ldots < x_n$ of I of

$$\sum_1^{n-1} \Phi\big(k|f(x_{i+1}) - f(x_i)|\big),$$

is finite. This class has been thoroughly studied by Musielak and Orlicz [6].

By Young's inequality,

$$k\sum_1^N |f(I_n)|/\lambda_n \leqslant \sum_1^N \Phi\big(k|f(I_n)|\big) + \sum_1^N \Psi(1/\lambda_n).$$

Thus $\sum_1^\infty \Psi(1/\lambda_n) < \infty$ and f of Φ-BV implies that f is of ΛBV. For $\lambda_n = n$, the condition $\sum \Psi(1/n) < \infty$ was shown by Salem [8] to imply that the Fourier series of a continuous function in Φ-BV converges uniformly. Goffman [3] showed that if continuity is not required, one obtains everywhere convergence. These results are contained in our results on HBV [9].

The Banach indicatrix of f, $n(y)$, is the cardinality of $\{x \colon f(x) = y\}$ if this set is finite and ∞ otherwise. Garsia and Sawyer [2] showed that if f is continuous with range $[0, 1]$ and

$$\int_0^1 \log n(y)\, dy < \infty,$$

then the Fourier series of f converges uniformly. Goffman [3] showed that if continuity is replaced by the existence of right and left-hand limits and the notion of the Banach indicatrix is suitably extended, then everywhere convergence is obtained.

Using a lemma of Goffman, we showed [9] that if $\inf f = A$ and $\sup f = B$, f has only simple discontinuities, and $L(x)$ is an increasing function such that $L(n) \sim \sum_1^n 1/\lambda_k$ as $n \to \infty$, then

$$\int_A^B L\big(n(y)\big)\, dy < \infty$$

implies that f is in class ΛBV. Thus the results on convergence and uniform convergence of Fourier series of functions for which $\int \log n(y)\, dy < \infty$ are contained in our results for functions of class HBV [9].

2. The Λ-variation functions. In this section we shall consider the behavior of the interval function

$$V(I) = V_\Lambda(f; I) = \sup\Big\{\sum |f(I_n)|/\lambda_n \colon I_n \subset I\Big\},$$

which we call the *Λ-variation of f on the interval I*. On a given interval $I = [a, b]$ we also consider the *Λ-variation function of f*

$$v(x) = v_\Lambda(f; x) = V_\Lambda(f; [a, x]).$$

We shall show that V is a subadditive interval function, that it possesses an interesting continuity property, and that the continuity properties of v are exactly those of the function from which it is derived.

THEOREM 2. *If $a < x < b$, then*

$$V([a, b]) \leqslant V([a, x]) + V([x, b]).$$

COROLLARY. *If $a < x < y$ then*

$$v(x) \leqslant v(y) \leqslant v(x) + V([x, y]).$$

Proof of Theorem 2. The corollary follows immediately from the obvious monotonicity of v and the theorem.

Now let $\{I_n\}$ be a collection of intervals in I and let $A = [a, x]$, $B = [x, b]$. Write

$$\{n_k\} = \{n \colon I_n \cap A^0 \neq \emptyset\}, \qquad \{m_k\} = \{n \colon I_n \cap B^0 \neq \emptyset\}$$

with $n_k\nearrow$, $m_k\nearrow$. If there exist k' and k'' such that $n_{k'}=m_{k''}$, denote the common value by $\bar{n}$. Then letting

$$I'_{\bar{n}}=I_{\bar{n}}\cap A, \qquad I''_{\bar{n}}=I_{\bar{n}}\cap B,$$

we have, observing that $m_k\geqslant k$ and $n_k\geqslant k$ for all k,

$$\sum|f(I_n)|/\lambda_n=\sum_{k\neq k'}|f(I_{n_k})|/\lambda_{n_k}+\sum_{k\neq k''}|f(I_{m_k})|/\lambda_{m_k}+|f(I_{\bar{n}})|/\lambda_{\bar{n}}$$
$$\leqslant\Big(|f(I'_{\bar{n}})|/\lambda_{k'}+\sum_{k\neq k'}|f(I_{n_k})|/\lambda_k\Big)+\Big(|f(I''_{\bar{n}})|/\lambda_{k''}+\sum_{k\neq k''}|f(I_{m_k})|/\lambda_k\Big)$$
$$\leqslant V(A)+V(B).$$

Since $\{I_n\}$ was arbitrary, we have

$$V(A\cup B)\leqslant V(A)+V(B)$$

as was to be shown. ■

The following theorem expresses a continuity property of V that we have found very useful in our study of Fourier series [10].

THEOREM 3. *Let f be of class ΛBV on $I=[a,b]$. Then*

(i) *if f is right continuous at a,*

$$V(f;\ [a,x])\to 0 \quad \text{as} \quad x\searrow a,$$

if f is left continuous at b,

$$V(f;\ [x,b])\to 0 \quad \text{as} \quad x\nearrow b,$$

(ii) *if $[x,y]\subset I^0$, then*

$$V(f;\ [x,y])\to 0$$

as x and y together approach either a or b.

Proof of Theorem 3. It is clear that (i) implies (ii), for if we suppose $[x,y]\subset I^0$ and set $g(x)=f(x)$ for $x\in(a,b)$ and $g(a)=f(a+)$, $g(b)=g(b-)$, then g is right continuous at a and left continuous at b and

$$0\leqslant V(f;\ [x,y])=V(g;[x,y])\leqslant V(g;[a,y])\to 0 \quad \text{as} \quad y\searrow a.$$

Similarly,

$$0\leqslant V(f;[x,y])\leqslant V(g;[x,b])\to 0 \quad \text{as} \quad x\nearrow b.$$

Thus we need only prove (i). We consider only the case in which f is right continuous at a.

There is a collection of intervals I_n, $n=1,\ldots,N_1$, such that $|f(I_n)|\searrow$, $I_n\subset[a,b]$, $f(I_n)\neq 0$ for each n, and

$$\sum_1^{N_1}|f(I_n)|/\lambda_n>\tfrac12 V_\Lambda(f;[a,b]).$$

Since f is right continuous at a, we may assume that $I_n\subset(a,b]$ for each n.

Choose $y_1\in(a,b)$ such that

$$[a,y_1]\cap\bigcup_1^N I_n=\emptyset, \qquad y_1<(a+b)/2,$$

and $I\subset(a,y_1]$ implies

$$|f(I)|\leqslant\min\{|f(I_n)|:\ n=1,\ldots,N_1\}.$$

Then there is a collection of intervals I_n, $n=N_1+1,\ldots,N_2$, such that $|f(I_n)|\searrow$ we have $I_n\subset(a,y_1]$ and $f(I_n)\neq 0$ for each n, and

$$\sum_{N_1+1}^{N_2}|f(I_n)|/\lambda_{n-N_1}>\tfrac12 V_\Lambda(f;[a,y_1]).$$

Continuing in this manner we can choose, for $k=1,2,\ldots$, $y_k\searrow a$ (by choosing $y_{k+1}<(a+y_k)/2$) and $I_{N_k+1},\ldots,I_{N_{k+1}}$, intervals contained in $[y_{k+1},y_k]$ with

$$\sum_{N_k+1}^{N_{k+1}}|f(I_n)|/\lambda_{n-N_k}>\tfrac12 V_\Lambda(f;[a,y_k])$$

and $|f(I_n)|\searrow$.

Let $|f(I_n)|=a_n$, $1/\lambda_n=b_n$. Then $a_n\searrow 0$, $b_n=O(1)$. Now $v(f;[a,b])\geqslant\sum a_nb_n$ implies that given $\varepsilon>0$, there is an N such that

$$\sum_{N+1}^{\infty}a_nb_n<\varepsilon$$

and, therefore, since for $j>0$, $a_{n+j}b_n\leqslant a_nb_n$,

$$\sum_{N+1}^{\infty}a_{n+j}b_n<\varepsilon.$$

Now there is a $J(\varepsilon)$ such that

$$\sum_1^N a_{n+j}b_n\leqslant a_{j+1}\sum_1^N b_n<\varepsilon \quad \text{if} \quad j>J(\varepsilon).$$

Writing $j=N_k$, we see that

$$\tfrac12 V(f;[a,y_k])<\sum_1^{\infty}a_{n+N_k}b_n<2\varepsilon$$

if k is sufficiently large. Hence

$$V(f;[a,y_k])\to 0 \quad \text{as} \quad k\to\infty$$

implying that

$$V(f;[a,y])\to 0 \quad \text{as} \quad y\searrow a$$

since $V(f;[a,y])$ is a monotone function of y. ∎

In order to prove that the continuity of v implies that of f, we require a surprisingly difficult lemma.

LEMMA 2. *Let f be of class ΛBV on I. If $[x,y]\subset I$ and $|f(x)-f(y)|\geqslant\delta>0$, then*

$$v(y)-v(x)\geqslant \delta/\lambda_{k_0}$$

where

$$k_0=\inf\Big\{k:\ \sum_1^k 1/\lambda_n>2v(x)/\delta\Big\}.$$

Proof of Lemma 2. Given $\eta>0$, there exist I_n, $n=1,\dots,N_1$, in $[a,x]$, such that $|f(I_n)|\searrow$ and

$$v(x)\leqslant\sum_1^N |f(I_n)|/\lambda_n+\eta.$$

Let $m=\inf(\{n:\ |f(I_n)|<\delta/2\}\cup\{N+1\})$. We shall show that

$$(*)\qquad v(y)-v(x)\geqslant\delta/2\lambda_m-\eta.$$

Put $|f(I_n)|=a_n$, $T=\sum_1^N a_n/\lambda_n$. Let S denote the sum obtained by adjoining δ to the collection $\{a_n\}$ and forming a sum of $N+1$ terms as indicated below:

1°. If $a_n\geqslant\delta/2$, $n=1,\dots,k$, but $a_{k+1}<\delta/2$, set

$$S=a_1/\lambda_1+\dots+a_k/\lambda_k+\delta/\lambda_{k+1}+a_{k+1}/\lambda_{k+2}+\dots+a_N/\lambda_{N+1};$$

then

$$S-T=(\delta-a_{k+1})/\lambda_{k+1}+(a_{k+1}-a_{k+2})/\lambda_{k+2}+\dots+a_N/\lambda_{N+1}>\delta/2\lambda_{k+1}.$$

2°. If $a_n\geqslant\delta/2$ for all n, set

$$S=a_1/\lambda_1+\dots+a_N/\lambda_N+\delta/\lambda_{N+1};$$

then

$$S-T=\delta/\lambda_{N+1}.$$

3°. If $\delta/2>a_n$ for all n, set

$$S=\delta/\lambda_1+a_1/\lambda_2+\dots+a_N/\lambda_{N+1};$$

then

$$S-T=(\delta-a_1)/\lambda_1+(a_1-a_2)/\lambda_2+\dots+a_N/\lambda_{N+1}>\delta/2\lambda_1.$$

Now $v(y)\geqslant S$. Hence

$$v(y)-v(x)\geqslant v(y)-T-\eta\geqslant S-T-\eta$$

and so

$$v(y)-v(x)\geqslant\begin{cases}\delta/2\lambda_{k+1}-\eta & \text{if } a_n\geqslant\delta/2 \text{ for } n\leqslant k,\ a_{k+1}<\delta/2,\\ \delta/2\lambda_{N+1}-\eta & \text{if } a_n\geqslant\delta/2 \text{ for all } n,\\ \delta/2\lambda_1-\eta & \text{if } a_n<\delta/2 \text{ for all } n,\end{cases}$$

which is (*).

Now $k_0\geqslant m$ since

(a) if $m=1$, $k_0\geqslant 1$;

(b) if $m=N+1$, then $v(x)\geqslant\sum_1^N|f(I_n)|/\lambda_n\geqslant(\delta/2)\sum_1^N 1/\lambda_n$ and, therefore, $k_0\geqslant N+1$;

(c) if $1<m<N+1$, then $v(x)\geqslant(\delta/2)\sum_1^{m-1}1/\lambda_n$ and, therefore, $k_0>m-1$.

From (*) we then have

$$v(y)-v(x)\geqslant\delta/2\lambda_{k_0}-\eta,$$

which implies the desired result since k_0 is independent of η. ∎

Our principal result on the continuity of the variation is the following.

THEOREM 4. *Let f be of class ΛBV on I. Then v is right (left) continuous at any point of I if and only if f is right (left) continuous at that point.*

Proof. We shall consider only the behavior at the right of a point. The arguments for the left are analogous.

Suppose $I=[a,b]$, $a\leqslant x<b$, and f is right continuous at x. If $x<y\leqslant b$, then from the corollary to Theorem 2 we have

$$0\leqslant v(y)-v(x)\leqslant V([x,y]).$$

From Theorem 3 we have that $V([x,y])\to 0$ as $y\searrow x$ and, therefore, v is continuous on the right at x.

Suppose f is not right continuous at x. Then there is a $\delta>0$ such that for $y>x$ but sufficiently close to x, $|f(x)-f(y)|\geqslant\delta$. Applying Lemma 2, we see then that

$$v(y)-v(x)\geqslant\delta/2\lambda_{k_0}$$

for such y and, therefore, v is discontinuous on the right at x. ∎

3. The space ΛBV. The functions of class ΛBV on an interval I form a Banach space if we define the norm of a function f as

$$\|f\|=|f(a)|+V(f;I),$$

where a is an arbitrary but fixed point of I. To see that the space with this norm is complete, let $\{f_n\}$ be a Cauchy sequence. Then

$$|f_n(a)-f_m(a)|+\big|f_n(a)-f_m(a)-\big(f_n(x)-f_m(x)\big)\big|/\lambda_1=o(1)$$

uniformly in x as $n, m\to\infty$, implying that $\{f_n\}$ converges uniformly. Denote the pointwise limit function by f. Now

$$|V(f_n)-V(f_m)|\leqslant V(f_n-f_m),$$

implying that $\lim V(f_n)$ exists. For any finite collection of intervals I_k, $k=1,\ldots,K$, we have

$$\sum_1^k|f(I_k)|/\lambda_k\leqslant\sum_1^k|f_n(I_k)|/\lambda_k+o(1)\leqslant V(f_n)+o(1)$$

as $n\to\infty$, implying that f is in ΛBV and

$$V(f)\leqslant\lim V(f_n).$$

Now given $\varepsilon>0$, there is an N such that for $m, n>N$

$$\sum|f_n(I_k)-f_m(I_k)|/\lambda_k<\varepsilon$$

for any collection of intervals I_k in I. Letting $m\to\infty$ and taking the supremum over all such collections, we have

$$V(f-f_n)\leqslant\varepsilon.$$

Hence $\|f-f_n\|\to 0$ as $n\to\infty$.

It is interesting to note that, since convergence in norm implies uniform convergence, the continuous functions of class ΛBV are a closed subspace of ΛBV.

If we suppose that $\lambda_n\nearrow\infty$, then BV is a proper subspace of ΛBV. With this condition on $\{\lambda_n\}$, S. J. Perlman [7] has proven the following remarkable theorem.

BV *is the intersection of all* ΛBV *spaces, but not the intersection of any countable collection. The space of functions with only simple discontinuities is the union of all* ΛBV *spaces, but not the union of any countable collection.*

The next result is the analogue of the Helly theorem for ΛBV.

THEOREM 5. *If* $\{f_n\}$ *is a sequence in* ΛBV *with* $\|f_n\|\leqslant M$, *then there exists a subsequence* $\{f_{n_k}\}$ *converging pointwise to a function* f *in* ΛBV *with* $\|f\|\leqslant M$.

Proof of Theorem 5. Let $v_n(x)=v(f_n;x)$. Then $v_n(x)\leqslant M$ for every x and n, and by applying the Helly Selection Principle we obtain $\{v_{n_k}\}$ converging pointwise to an increasing function v. By the diagonal method we may find $\{n_{k_j}\}$ such that $\{f_{n_{k_j}}(x)\}$ converges at the endpoints of I, at a, and at each rational x. Let $f^j=f_{n_{k_j}}$, $v^j=v_{n_{k_j}}$.

For $\varepsilon>0$, set $m(\varepsilon)=\inf\{m\colon \sum_1^m 1/\lambda_n>2M/\varepsilon\}$. Suppose now that v is continuous at an irrational $x_0\in I^0$. There is a rational $y>x_0$ such that

$$0\leqslant v(y)-v(x_0)<\eta=\varepsilon/6\lambda_{m(\varepsilon)}.$$

There is an integer J such that for $j>J$,

$$|v^j(y)-v(y)|<\eta,\qquad |v^j(x_0)-v(x_0)|<\eta.$$

Then for $j>J$,

$$0\leqslant v^j(y)-v^j(x_0)<3\eta=\varepsilon/2\lambda_{m(\varepsilon)}.$$

By Lemma 2 we have, for $j>J$,

$$|f^j(y)-f^j(x_0)|<\varepsilon.$$

Since $\{f^j(y)\}$ converges, there is an integer J' such that, for $j, k>J'$, we have

$$|f^j(y)-f^k(y)|<\varepsilon.$$

Thus, for $j, k>\max(J, J')$, we have

$$|f^j(x_0)-f^k(x_0)|<3\varepsilon,$$

implying that $\{f^j(x_0)\}$ converges.

Thus $\{f^j(x)\}$ converges except perhaps on an at most countable set, the points of discontinuity of v, but, by the diagonal method, we can choose from $\{f^j\}$ a subsequence convergent on this set and, therefore, on all of I to a function f.

Denote this subsequence by $\{f_{(i)}\}$ and the corresponding subsequence of $\{v^j\}$ by $\{v_{(i)}\}$. Given $\varepsilon>0$, for a fixed x there is an integer J such that $i>J$ implies $v(x)>v_{(i)}(x)-\varepsilon$. Let I_n, $n=1,\ldots,N$, be intervals in I to the left of x. Choose $k>J$ so large that

$$\Big|\sum_1^N|f_k(I_n)|/\lambda_n-\sum_1^N|f(I_n)|/\lambda_n\Big|<\varepsilon.$$

Then

$$v(x)>v_{(k)}(x)-\varepsilon\geqslant\sum_1^N|f_{(k)}(I_n)|/\lambda_n-\varepsilon>\sum_1^N|f(I_n)|/\lambda_n-2\varepsilon.$$

Hence f is in ΛBV and

$$v(x) \geqslant v(f; x).$$

Since $v_{(i)}(x) \leqslant M - |f_{(i)}(a)|$ for each $x \in I$ and each i, we have

$$M - f(a) \geqslant v(x) \geqslant v(f; x)$$

for each x in I and, therefore,

$$M \geqslant \|f\|. \blacksquare$$

4. Some open questions. In our paper on the summability of Fourier series [10] we introduced the notion of continuity in Λ-variation. Let $\Lambda^m = \{\lambda_{n+m}\}$, $m = 1, 2, \ldots$ A function f in ΛBV on I is said to be *continuous in Λ-variation* if

$$V_{\Lambda^m}(f; I) \to 0 \quad \text{as} \quad m \to \infty.$$

Clearly, if $\Lambda BV = BV$, this implies that f is constant. Thus we consider only $\lambda_n \nearrow \infty$.

Q.1. *How can we characterize the functions which are continuous in Λ-variation?*

In two papers with Goffman ([4], [5]), we characterized GW, the class of functions f such that the Fourier series of $f \circ h$ is everywhere convergent for each homeomorphism h of $[0, 2\pi]$ with itself. With Baernstein [1], we characterized UGW, the class of continuous functions for which uniform convergence is preserved by such compositions. Let (reg) denote a restriction to *regulated* functions, those with only simple discontinuities, and (cont) denote a restriction to continuous functions. Then we have, clearly,

$$GW_{(\mathrm{reg})} \supseteq HBV$$

and

$$UGW \supseteq HBV_{(\mathrm{cont})}.$$

Q.2. *Are these inclusions proper?*

Proceeding as in the class BV, we can define positive and negative Λ-variations by setting

$$v_\Lambda^+(f; x) = \sup\Big\{\sum f(I_n)/\lambda_n \colon I_n \subset [a, x],\ f(I_n) > 0\Big\},$$

$$v_\Lambda^-(f; x) = \sup\Big\{\sum |f(I_n)|/\lambda_n \colon I_n \subset [a, x],\ f(I_n) < 0\Big\},$$

when f is of class ΛBV on $I = [a, b]$.

This leads at once to the following question.

Q.3. *To what extent do the positive and negative Λ-variations of a function characterize the function?*

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