STUDIA MATHEMATICA, T. LVII. (1976)

Unconditionally converging and Dunford–Pettis operators on $C_X(S)$

by

CHARLES SWARTZ (Las Cruces, N. Mex.)

Abstract. Let S be a compact Hausdorff space and X, Y B-spaces. We give characterizations of the unconditionally converging and Dunford–Pettis operators $T: C_X(S) \to Y$, where $C_X(S)$ is the B-space of continuous X-valued functions equipped with the sup-norm. These results are used to show that $C_X(S)$ has the Dunford–Pettis property if X has the Dunford–Pettis property.

In [6], I. Dobrakov posed the problem of characterizing the unconditionally converging operators on the B-space $C_X(S)$ of X-valued continous functions defined on a compact Hausdorff space S, where X is a B-space. If Y is a B-space, Dobrakov observed that if $T: C_X(S) \to Y$ is an unconditionally converging operator and $Tf = \int_S f\,dm$, $f \in C_X(S)$, where $m: B(S) \to L(X, Y)$ is an operator-valued measure on the Borel sets of S ([4], §1), then (i) $m(B): X \to Y$ is an unconditionally converging operator for each Borel set B and (ii) the operator semi-variation of m is continuous at $\emptyset$ (see also [15], Th. 5). Dobrakov conjectured that (i) and (ii) actually imply that T is an unconditionally converging operator. In this note we show that this is indeed the case.

Our methods also allow us to solve another problem posed by Dobrakov in [8], §6, concerning the Dunford–Pettis property. Namely, if X has the Dunford–Pettis property (DP property), then does $C_X(S)$ also have the DP property? (An operator from a B-space Z into a B-space Y is a Dunford–Pettis operator (DP operator) if it carries sequences which converge weakly to 0 into sequences which converge to 0 in norm; a B-space Z has the DP property if every weak compact operator on Z into another B-space is a DP operator ([13], Prop. 4; [10]).) By using the same method of proof used to characterize unconditionally converging operators, we give a characterization of DP operators on $C_X(S)$ and this characterization allows us to show $C_X(S)$ has the DP property iff X has the DP property.

1. Unconditionally converging operators. Throughout this note S will denote a compact Hausdorff space with Borel sets $B(S)$, X, Y and Z

will denote B-spaces, and $C_X(S)$ will denote the B-space of all continuous functions from S to X equipped with the sup-norm. If $T: C_X(S) \to Y$ is a bounded linear operator, then T has a representation $Tf = \int_S f\,dm$, where $m: B(S) \to L(X, Y'')$ is a finitely additive operator-valued set function with finite operator semi-variation ([4]; m has other properties which we do not list). The set function m is called the *representing "measure"* for the operator T.

If $m: B(S) \to L(X, Y)$ is finitely additive and has bounded operator semi-variation $\tilde{m}$ ([5], I, 4.1), then $\tilde{m}$ is continuous at $\emptyset$ if $B_n \downarrow \emptyset$, $B_n \in B(S)$, implies $\tilde{m}(B_n) \to 0$. This is equivalent to the existence of a finite positive measure λ on $B(S)$ such that $\lim_{\lambda(E)\to 0} \tilde{m}(E) = 0$ ([7]; see [3], Th. 6 for other equivalent formulations). Such a λ is said to be a *control measure* for m.

Recall that a bounded linear operator $T: Z \to Y$ is *unconditionally converging* (u.c.) if it carries weak unconditional Cauchy series (w.u.c. series) into unconditionally converging (u.c.) series. (A series $\sum x_n$ in X is w.u.c. if $\sum |\langle x', x_n\rangle| < \infty$ for each $x' \in X'$ and $\sum x_n$ is u.c. if every rearrangement is convergent in X [12].)

THEOREM 1. *A bounded linear operator $T: C_X(S) \to Y$ is u.c. iff*

(i) *for each Borel set B $m(B): X \to Y$ is u.c. and*

(ii) *$\tilde{m}$ is continuous at $\emptyset$.*

Proof. For the necessity of (i) and (ii) see [8], Theorem 3 or [15], Theorem 5.

Now suppose (i) and (ii) hold. We first make two simplifications. First note that we may assume that X is separable; for if $\sum f_n$ is w.u.c. in $C_X(S)$, let X_0 be the closed linear span of $\{f_n(t): n \geq 1, t \in S\}$. Then $f_n \in C_{X_0}(S)$ and if we define $T_0: C_{X_0}(S) \to Y$ by $T_0 f = Tf$, then the representing measure for T_0 still satisfies (i) and (ii) and if T_0 is u.c., $\sum Tf_n$ is u.c. in Y.

Next observe that we may assume S is metrizable. For let $\sum f_n$ be w.u.c. in $C_X(S)$. Define an equivalence relation $\sim$ on S by $s \sim t$ if $f_n(s) = f_n(t)$ for all n. Let S_0 be the set of equivalence classes under $\sim$ and let $\pi: S \to S_0$ be the natural map from s onto its equivalence class $\hat{s} = \pi(s)$ with respect to $\sim$. (The technique used here is that of [9], VI, 7.6.) Define a metric d on S_0 by

$$d(\hat{s}, \hat{t}) = \sum_1^\infty |f_n(s) - f_n(t)|/2^n$$

and note that since each f_n is continuous on S, π is continuous and therefore S_0 is a compact metric space. Define a bounded linear operator T_0: $C_X(S_0) \to Y$ by $T_0 \varphi = T(\varphi \circ \pi)$. Then the representing measure m_0 for T_0 is just the image of the measure m by the map π ([5], III, 20.1) so that if m satisfies (i) and (ii), then m_0 likewise satisfies (i) and (ii). Now if T_0 is u.c. and if we define $\varphi_n \in C_X(S_0)$ by $\varphi_n(\hat{s}) = f_n(s)$ (note φ_n is well-defined), then $\sum \varphi_n$ is w.u.c. ($\|\sum_\sigma \varphi_n\| = \|\sum_\sigma f_n\|$ for any finite subset σ of the positive integers N [12]) and thus $\sum T_0 \varphi_n = \sum Tf_n$ is u.c. in Y.

Thus we may assume that X is separable and S is metrizable. Let $\sum f_n$ be w.u.c. in $C_X(S)$ with $\|\sum_\sigma f_n\| \leq M$ for every finite $\sigma \subseteq N$. Let $l^1_\omega(X)$ denote the B-space of all X-valued sequences $\{x_n\}$ such that $\sum x_n$ is w.u.c. equipped with the norm $\varepsilon\{x_n\} = \sup\{\|\sum_\sigma x_n\|: \sigma \subseteq N \text{ finite}\}$ ([14], 1.2; the norm defined here is equivalent to the norm employed by Pietsch [12]). Define $F: S \to l^1_\omega(X)$ by $F(t) = \{f_n(t)\}$. We claim that F is strongly measurable with respect to λ where λ is a control measure for m (recall the remarks preceeding Theorem 1). For this let τ be the topology of pointwise convergence on $l^1_\omega(X)$, i.e., the relative product topology $\prod_{n=1}^\infty X$. Now $l^1_\omega(X)$ is separable with respect to τ since X is separable, τ is weaker than the norm topology, and F is τ-continuous and thus measurable. Therefore, we may apply the remark on page 55 of [17] and conclude that F is strongly measurable with respect to λ and the norm topology.

Let $\delta > 0$. There is a (countable) partition $\{E_i: 1 \leq i < \infty\}$ of S by Borel sets such that

$$\varepsilon\Big\{F(t) - \sum_{i=1}^\infty C_{E_i}(t) F(s_i)\Big\} < \delta \quad \text{for all } t \in S,$$

where s_i is a fixed point of E_i and C_E denotes the characteristic function of E ([11], Cor. 1 of 3.5.3 or [10], 8.15.2: actually this estimate only holds for λ-almost all $t \in S$ but for convenience we assume it holds throughout S). That is, we have

$$\Big\|\sum_{n\in\sigma}\Big(f_n(t) - \sum_{i=1}^\infty C_{E_i}(t) f_n(s_i)\Big)\Big\| < \delta \quad \text{for } t \in S \text{ and } \sigma \subseteq N \text{ finite}. \tag{1}$$

For $\sigma \subseteq N$ finite and k any positive integer, we have ([7], Th. 3)

$$\sum_{n\in\sigma} Tf_n = \sum_{n\in\sigma}\sum_{i=1}^k m(E_i) f_n(s_i) + \sum_{n\in\sigma}\int_{\bigcup_{i=1}^k E_i}\Big(f_n(t) - \sum_{i=1}^k C_{E_i}(t) f_n(s_i)\Big)\,dm(t) + \sum_{n\in\sigma}\int_{\bigcup_{i=k+1}^\infty E_i} f_n\,dm. \tag{2}$$

To show $\sum Tf_n$ is u.c. it suffices to show that there is a finite $\sigma_0 \subseteq N$ such that $\|\sum_\sigma Tf_n\|$ is small for $\sigma \cap \sigma_0 = \emptyset$, $\sigma \subseteq N$ finite. In view of (2), we can accomplish this by estimating each term on the right-hand side of (2).

For the last term in (2), we have

$$\Big\|\sum_{n\in\sigma}\int_{\bigcup_{i=k+1}^{\infty}E_i} f_n\,dm\Big\|\leqslant M\tilde m\Big(\bigcup_{i=k+1}^{\infty}E_i\Big)$$

and by (ii) there exists a k such this term is less than δ. Choose such a k and fix it for the remainder of the proof. For the middle term in (2), we have

$$\Big\|\int_{\bigcup_{i=1}^{k}E_i}\sum_{n\in\sigma}\Big(f_n(t)-\sum_{i=1}^{k}C_{E_i}(t)f_n(s_i)\Big)\,dm(t)\Big\|<\delta\tilde m(S)$$

from (1). For the first term on the right-hand side of (2), note that for each i, $\|\sum_{n\in\sigma}f_n(s_i)\|\leqslant M$ so $\sum_n f_n(s_i)$ is w.u.c. in X. By (i), with k fixed, there exists a finite $\sigma_0\subseteq N$ such that $\sigma\cap\sigma_0=\emptyset$, $\sigma\subseteq N$ finite, implies

$$\Big\|\sum_{i=1}^{k}\sum_{n\in\sigma}m(E_i)f_n(s_i)\Big\|<\delta.$$

For such σ, from (2), $\|\sum_{n\in\sigma}Tf_n\|<\delta\big(2+\tilde m(S)\big)$, i.e., $\sum Tf_n$ is u.c. in Y.

Remark 2. Partial results pertaining to this problem were given in [2], [15], Theorem 6, and [16].

2. The Dunford–Pettis property. By using the method of proof of Theorem 1, we can also give a characterization of DP operators on $C_X(S)$. This characterization can then be used to show $C_X(S)$ has the DP property iff X has the DP property thus answering the question posed by Dobrakov in [8], § 6.

THEOREM 3. *A bounded linear operator $T\colon C_X(S)\to Y$ is a* DP *operator iff*

(i) *for each Borel set B $m(B)\colon X\to Y$ is a* DP *operator,*

(ii) *$\tilde m$ is continuous at $\emptyset$.*

Proof. If T is a DP operator, it is shown in [1] or [16] that (i) and (ii) hold.

Suppose (i) and (ii) hold. As in Theorem 1 we may assume that X is separable and S is metrizable. Let $f_n\to 0$ weakly in $C_X(S)$; then $\{f_n\}$ is norm bounded so there is an M such that $\|f_n(t)\|\leqslant M$ for all n, t. Define $F\colon S\to l^\infty(X)$, the B-space of all bounded X-valued sequences equipped with the sup-norm, by $F(t)=\{f_n(t)\}$. Again we claim that F is strongly measurable with respect to λ, where λ is a control measure for m. For let τ be the topology of pointwise convergence on $l^\infty(X)$, i.e., the relative product topology $\prod_1^\infty X$. Since X is separable, $l^\infty(X)$ is separable with respect to τ, τ is weaker than the norm topology and F is τ-continuous. Hence by the remark on p. 55 of [17], F is strongly measurable.

Let $\delta>0$. There exists a partition $\{E_i\}_1^\infty$, of S by Borel sets such that

(3) $$\Big\|f_n(t)-\sum_{i=1}^{\infty}C_{E_i}(t)f_n(s_i)\Big\|<\delta\quad\text{for all } t\in S,\ n\geqslant 1,\ \text{where } s_i\in E_i.$$

(Again we assume (3) holds everywhere neglecting the λ-null set.) Now for any positive integer k ([7], Th. 3),

(4) $$Tf_n=\sum_{i=1}^{k}m(E_i)f_n(s_i)+\int_{\bigcup_{i=1}^{k}E_i}\Big(f_n(t)-\sum_{i=1}^{k}C_{E_i}(t)f_n(s_i)\Big)\,dm(t)+\int_{\bigcup_{i=k+1}^{\infty}E_i}f_n\,dm.$$

To show $\|Tf_n\|\to 0$, we estimate each term on the right-hand side of (4). For the last term in (4),

$$\Big\|\int_{\bigcup_{i=k+1}^{\infty}E_i}f_n\,dm\Big\|\leqslant M\tilde m\Big(\bigcup_{i=k+1}^{\infty}E_i\Big)$$

and by (ii) there exists a k such that this term is less than δ. Fix such a k. For the middle term in (4), from (3) the norm of this term is less than $\delta\tilde m(S)$. To treat the first term on the right-hand side of (4) note that for each i the linear map $f\to f(s_i)$ from $C_X(S)$ to X is norm-continuous and therefore weak-continuous ([9], V, 3.15) so $\lim_n f_n(s_i)=0$ (weak limit). Thus by (i), for $1\leqslant i\leqslant k$, $\lim_n m(E_i)f_n(s_i)=0$ (norm limit); from (4) and this fact, there exists an N such that $n\geqslant N$ implies $\|Tf_n\|<\delta\big(2+\tilde m(S)\big)$.

We may now treat the problem posed by Dobrakov in [8], § 6.

THEOREM 4. *Let T be locally compact, Hausdorff, and let $C_0(T,X)$ be the X-valued continuous functions on T which vanish at ∞ equipped with the sup-norm. Then $C_0(T,X)$ has the* DP *property iff X has the* DP *property.*

Proof. Assume that $C_0(T,X)$ has the DP property. Fix $t\in T$ and pick $\varphi\in C_0(T)$ such that $\varphi(t)=1$ and $\|\varphi\|=1$. Define $U\colon C_0(T,X)\to X$ by $Uf=f(t)$. Suppose Y is an arbitrary B-space and $V\colon X\to Y$ is a weakly compact operator. Then VU is weakly compact and therefore DP. But if $x_n\to 0$ weakly in X, $\varphi x_n\to 0$ weakly in $C_0(T,X)$ ([8], Th. 9) so that $VU(\varphi x_n)=Vx_n\to 0$ in norm. That is, V is a DP operator and X has the DP property.

Assume that X has the DP property. First note that if T is compact,

$C_X(T)$ has the DP property from Theorem 3 and Theorem 8 of [15]. If T is locally compact, let T^* be the one-point compactification of T with ∞ denoting the point at infinity. Then $C_0(T, X)$ is isometrically isomorphic to the closed subspace Γ of $C_X(T^*)$ consisting of those functions which vanish at ∞. But Γ is complemented in $C_X(T^*)$ via the projection P: $f \to f - f(\infty)$ and $C_X(T^*)$ has the DP property, so Γ, and hence $C_0(T, X)$, has the DP property ([10], 9.4.3).

Remark 5. Partial solutions to this problem were given in [8], [2], and [16]; for the scalar version see [9], VI, 7.4.

It also follows from Theorem 4 that if Z is a complemented subspace of a space $C(S)$, then $Z \otimes_\varepsilon X$ ([14], 7.1.1) has the DP property when X has the DP property for $Z \otimes_\varepsilon X$ is then a complemented subspace of $C(S) \otimes_\varepsilon X = C_X(S)$. This suggests the conjecture that if X and Y have the DP property, then $X \otimes_\varepsilon Y$ also has the DP property.

References

[1] G. Alexander, *Linear operators on the space of vector-valued continuous functions*, thesis, New Mexico State University, 1972.
[2] G. Alexander and C. Swartz, *Linear operators on c_X*, Czech. Math. Journ. 23 (1973), pp. 231–234.
[3] J. Batt, *Applications of the Orlicz–Pettis Theorem to operator-valued measures and compact and weakly compact transformations on spaces of continuous functions*, Rev. Roum. Math. 14 (1969), pp. 907–935.
[4] J. Batt and E. Berg, *Linear bounded transformations on the space of continuous functions*, Journ. Functional Anal. 4 (1969), pp. 215–239.
[5] N. Dinculeanu, *Vector Measures*, Berlin 1966.
[6] I. Dobrakov, *A representation theorem for unconditionally converging linear operators on $C(T, X)$*, Studia Math. 38 (1970), pp. 460–461.
[7] — *On integration in Banach spaces I*, Czech. Math. Journ. 20 (1970), pp. 511–536.
[8] — *On representation of linear operators on $C_0(T, X)$*, ibid. 21 (1971), pp. 13–30.
[9] N. Dunford and J. T. Schwartz, *Linear Operators*, New York 1966.
[10] R. E. Edwards, *Functional Analysis*, New York 1965.
[11] E. Hille and R. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. 1957.
[12] C. McArthur, *On relationships amongst certain spaces of sequences in an arbitrary Banach space*, Canad. J. Math. 8 (1956), pp. 192–197.
[13] A. Pełczyński, *On weakly compact polynomial operators on B-spaces with Dunford–Pettis property*, Bull. Pol. Acad. Sci. 11 (1963), pp. 371–378.
[14] A. Pietsch, *Nucleare Localkonvexe Räume*, Berlin 1965.
[15] C. Swartz, *Unconditionally converging operators on the space of continuous functions*, Rev. Roum Math. 17 (1972), pp. 1695–1702.
[16] — *Linear operators on $C_X(\Omega)$ for Ω dispersed*, preprint.
[17] G. E. F. Thomas, *The Lebesgue–Nikodym Theorem for Vector Valued Radon Measures*, Amer. Math. Soc. Mem., 139, 1974.

Received February 3, 1975 (945)

STUDIA MATHEMATICA, T. LVII. (1976)

On the Vitali covering properties of a differentiation basis

by

ANTONIO CORDOBA (Princeton, N. J.)

Abstract. A functional analysis technique is introduced to relate differentiation and covering properties of a basis.

A. Let $\mathscr{B}$ be a Busemann–Feller differentiation basis in $\boldsymbol{R}^n$. That is, for each $x \in \boldsymbol{R}^n$ we have a collection of bounded open sets $\mathscr{B}(x)$ containing x, such that there exists at least one sequence $\{R_k\} \subset \mathscr{B}(x)$ with diameter $(R_k) \to 0$, and if $x \in R \in \mathscr{B}$, then $R \in \mathscr{B}(x)$.

Given a mesurable set E in $\boldsymbol{R}^n$, we say that $V \subset \mathscr{B}$ is a *$\mathscr{B}$-Vitali covering of E* if for every $x \in E$ there is a sequence $\{R_k\} \subset V$ such that $R_k \in \mathscr{B}(x)$ for each k and $R_k \to x$ as $k \to \infty$.

DEFINITION 1. The differentiation basis $\mathscr{B}$ has the *covering property V_q* if there exists a constant C such that for every measurable bounded set E, every $\mathscr{B}$-Vitali covering V of E and any $\varepsilon > 0$, one can select a sequence $\{R_k\} \subset V$ with the properties:

(i) $\left|E - \bigcup R_k\right| = 0$, $\left|\bigcup R_k - E\right| \leqslant \varepsilon$,

(ii) $\left\|\sum \chi_{R_k}\right\|_q \leqslant C\,|E|^{1/q}$.

Given a locally integrable function f, we define the upper derivative $\overline{D}(\int f, x)$ with respect to $\mathscr{B}$ as follows:

$$\overline{D}\left(\int f, x\right) = \sup \limsup_{k\to\infty} \frac{1}{|R_k|} \int_{R_k} f(y)\,dy,$$

where the "sup" is taken over all the sequences $\{R_k\} \subset \mathscr{B}(x)$ such that $R_k \to x$ as $k \to \infty$. The lower derivative $\underline{D}(\int f, x)$ is defined by setting inf lim inf above.

DEFINITION 2. We say that $\mathscr{B}$ *differentiates* $\int f$ if

$$\overline{D}\left(\int f, x\right) = \underline{D}\left(\int f, x\right) = f(x) \quad \text{at almost every point } x \in \boldsymbol{R}^n.$$

The purpose of this paper is to relate the following two properties of a differentiation basis: